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# Implicit Hilfer-Katugampula-Type Fractional Pantograph Differential Equations with Nonlocal Katugampola Fractional Integral Condition

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**Abstract** A class of implicit Hilfer-Katugampola-type fractional pantograph differential equation with nonlocal katugampola fractional integral conditions is considered in this paper. By using Schaefer's fixed point theorem and Banach contraction principle, the existence and uniqueness solutions for the considered problem are proved. Ulam-Hyers stability for the considered problem is established. Finally, an example is presented to illustrate our main results.

## 1 Introduction.

Fractional differential equations have been widely applied in the field of science and engineering. Recently, Hilfer fractional differential equations have attracted the attention of many authors ([1]-[8]). Nowadays, the generalized fractional derivative introduced by U.N. Katugampola ([9], [10]) is unified with Hilfer fractional derivative by Oliveira and E. Capelas de Oliveira in ([11]) is named as Hilfer-Katugampola fractional derivative. This formulation interpolates the well-known fractional derivatives of Hilfer, Hilfer-Hadamard, Riemann-Liouville, Hadamard, Caputo, Caputo-Hadamard, Liouville, Weyl, Caputo-type fractional derivatives. Thus, the Hilfer-Katugampola fractional derivative proposed in this paper is a generalization of the classical fractional derivatives. Few authors studied Hilfer-Katugampola fractional differential equations ([11]-[13]).

The pantograph is a device in electric trains to collect electric currents from the overload lines and play an important role in physics, pure and applied mathematics, such as control systems, number theory, quantum mechanics and electrodynamics. This equation is a special class of delay differential equation arising in deterministic situations and was modeled by Ockendon and Tayler ([14]). Motivated by their importance, a lot of scientists generalized these equations into various types and introduced the solvability aspect of such problems both theoretically and numerically ([15]-[20]).

To the best of our Knowledge, there are no results of implicit pantograph Hilfer-Katugampola fractional differential equation with nonlocal Katugampola fractional integral conditions. Motivated by the above discussion, the aim of this paper is to study the implicit pantograph Hilfer-Katugampola-type fractional differential equations with nonlocal katugampola fractional integral conditions of the form:

$$\begin{aligned} {}^{\rho}D_{0+}^{\alpha,\beta}u(t) &= f(t,u(t),u(\lambda t),{}^{\rho}D_{0+}^{\alpha,\beta}u(\lambda t)), \qquad t\in J:=(0,T], \quad 0<\lambda<1 \\ \left({}^{\rho}\mathcal{J}_{0+}^{1-\nu}u\right)(0) &= \sum_{i=1}^{m}\zeta_i {}^{\rho}\mathcal{J}_{0+}^{r}u(\tau_i), \quad \nu=\alpha+\beta(1-\alpha). \end{aligned} \tag{1.1}$$

where ${}^{\rho}D_{0+}^{\alpha,\beta}$ is Hilfer-Katugampola fractional derivative of order $\alpha$ and type $\beta(0\le\beta\le 1)$, ${}^{\rho}\mathcal{J}_{0+}^{r}$, ${}^{\rho}\mathcal{J}_{0+}^{1-\nu}$ are Katugampola fractional integral of order $1-\nu(\nu=\alpha+\beta(1-\alpha))$, $0<\alpha<1$, $r,\rho>0$ and $\zeta_i\in\mathbb{R}$, $\tau_i, i=1,2,3,\cdots,m$ are prefixed points satisfying $0<\tau_1\le\tau_2\le\cdots\le$

$\tau_m < T$. Here we let $\mathfrak{X}$ is a Banach space, $f : J \times \mathfrak{X} \times \mathfrak{X} \times \mathfrak{X} \to \mathfrak{X}$ is a given continuous function. In addition, the nonlocal Katugampola fractional integral condition $\left({}^\rho\mathcal{J}_{0+}^{1-\nu}u\right)(0) = \sum_{i=1}^{m} \zeta_i {}^\rho\mathcal{J}_{0+}^{r}u(\tau_i)$ generalized the following initial condition:

- If $r \to 0$, the initial condition reduces to multi-point nonlocal condition.

and in physics problems yields better effect than the initial conditions $\left({}^\rho\mathcal{J}_{0+}^{1-\nu}u\right)(0) = u_0$, (see, ([21])).

The outline of this paper is organized as follows: In section 2 recalls some basic definitions and lemmas. In section 3, the existence and uniqueness of solutions for equation (1.1) are established. stability analysis results are discussed in section 4. The last section contains an example to illustrate our main results.

## 2 Preliminary.

Let $\rho > 0$ and $0 \le \nu < 1$.

(i) Let $\mathcal{C}(J, \mathfrak{X})$ be a Banach space of all continuous functions $u$ from $J$ into $\mathfrak{X}$.

(ii) Let $\mathcal{C}_{1-\nu,\rho}(J, \mathfrak{X})$ is a weighted space defined by

$$\mathcal{C}_{1-\nu,\rho}(J, \mathfrak{X}) := \left\{ u : J \to \mathfrak{X} : \left(\frac{t^\rho}{\rho}\right)^{1-\nu} u(t) \in \mathcal{C}(J, \mathfrak{X}) \right\}.$$

with the norm

$$\|u\|_{\mathcal{C}_{1-\nu,\rho}} := \left\| \left(\frac{t^\rho}{\rho}\right)^{1-\nu} u(t) \right\|_{\mathcal{C}} = \sup_{t\in J} \left| \left(\frac{t^\rho}{\rho}\right)^{1-\nu} u(t) \right|.$$

**Definition 2.1.** ([9]) Let $\alpha, c \in \mathbb{R}$ with $\alpha > 0$ and $\varphi \in X_c^p(a, b)$, where $\varphi \in X_c^p(a, b)$ consists of those complex-valued Lebesgue measurable functions. The generalized left-sided fractional integral is defined by

$$\left({}^\rho\mathcal{J}_{0+}^{\alpha}\varphi\right)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t \frac{s^{\rho-1}\varphi(s)}{(t^\rho - s^\rho)^{1-\alpha}} ds, \quad t > 0.$$

where $\Gamma(\cdot)$ is the gamma function and the corresponding generalized (Katugampola) left-sided fractional derivative ${}^\rho D_{0+}^{\alpha}$ is defined by

$$\left({}^\rho D_{0+}^{\alpha}\varphi\right)(t) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \left(t^{1-\rho}\frac{d}{dt}\right)^n \int_0^t \frac{s^{\rho-1}\varphi(s)}{(t^\rho - s^\rho)^{\alpha-n+1}} ds, \quad t > 0.$$

**Lemma 2.2.** *([22]) Let $\mathfrak{X}$ be a Banach space and $\varpi : \mathfrak{X} \to \mathfrak{X}$ is a continuous and compact mapping. If,*

$$E = \{u \in \mathfrak{X} : u = \gamma\varpi(u) \quad for \quad some \quad \gamma \in [0, 1]\}$$

*is a bounded set, then $\varpi$ has a fixed point.*

**Lemma 2.3.** *([11]) For the generalized (Katugampola) left-sided fractional integral and derivative, the following properties are satisfied:*

*(i) The semigroup property*

$$\left({}^\rho\mathcal{J}_{0+}^{\alpha}\, {}^\rho\mathcal{J}_{0+}^{\beta}\varphi\right)(t) = \left({}^\rho\mathcal{J}_{0+}^{\alpha+\beta}\varphi\right)(t).$$

*(ii) For $\alpha > 0$, ${}^\rho\mathcal{J}_{0+}^{\alpha}$ maps $\mathcal{C}(J, \mathfrak{X})$ into $\mathcal{C}(J, \mathfrak{X})$.*

*(iii) Let $\alpha > 0$, and $0 \le \nu < 1$, then ${}^\rho\mathcal{J}_{0+}^{\alpha}$ is bounded from $\mathcal{C}_{\nu,\rho}(J, \mathfrak{X})$ into $\mathcal{C}_{\nu,\rho}(J, \mathfrak{X})$.*

*(iv)* *Let* $\alpha > 0$, *and* $0 \leq \nu < 1$, *and* $\varphi \in \mathcal{C}_\nu(J, \mathfrak{X})$. *Then*

$$\left({}^\rho D^\alpha_{0+}\, {}^\rho \mathcal{J}^\alpha_{0+} \varphi\right)(t) = \varphi(t)$$

*(v)* *Let* $0 < \alpha < 1$, *and* $0 \leq \nu < 1$. *If* $\varphi \in \mathcal{C}_\nu(J, \mathfrak{X})$ *and* ${}^\rho \mathcal{J}^{1-\alpha}_{0+} \varphi \in \mathcal{C}^1_\nu(J, \mathfrak{X})$, *then*

$$\left({}^\rho \mathcal{J}^\alpha_{0+}\, {}^\rho D^\alpha_{0+} \varphi\right)(t) = \varphi(t) - \frac{{}^\rho \mathcal{J}^{1-\alpha}_{0+} \varphi(0)}{\Gamma(\alpha)} \left(\frac{t^\rho}{\rho}\right)^{\alpha-1}.$$

**Definition 2.4** ([11],[12]). The Hilfer-Katugampola fractional derivative with respect to $t$, with $\rho > 0$, is defined by

$$\left({}^\rho D^{\alpha,\beta}_{0\pm} u\right)(t) = \left(\pm {}^\rho \mathcal{J}^{\beta(1-\alpha)}_{0\pm} \delta_\rho\, {}^\rho \mathcal{J}^{(1-\beta)(1-\alpha)}_{0\pm}\right)(t) \tag{2.1}$$

where $\delta_\rho = \left(t^{\rho-1} \frac{d}{dt}\right)$.

(i) The operator ${}^\rho D^{\alpha,\beta}_{0+}$ can be written as

$${}^\rho D^{\alpha,\beta}_{0+} = {}^\rho \mathcal{J}^{\beta(1-\alpha)}_{0+} \delta_\rho\, {}^\rho \mathcal{J}^{1-\nu}_{0+} = {}^\rho \mathcal{J}^{\beta(1-\alpha)}_{0+}\, {}^\rho D^\nu_{0+}$$

where $\nu = \alpha + \beta(1 - \alpha)$.

In order to solve our problem, the following spaces are presented:

$$\mathcal{C}^{\alpha,\beta}_{1-\nu,\rho}(J, \mathfrak{X}) := \{u(t) \in \mathcal{C}_{1-\nu,\rho}(J, \mathfrak{X}), {}^\rho D^{\alpha,\beta}_{0+} u(\cdot) \in \mathcal{C}_{1-\nu,\rho}(J, \mathfrak{X})\}.$$

and

$$\mathcal{C}^\nu_{1-\nu,\rho}(J, \mathfrak{X}) := \{u(t) \in \mathcal{C}_{1-\nu,\rho}(J, \mathfrak{X}), {}^\rho D^\nu_{0+} u(\cdot) \in \mathcal{C}_{1-\nu,\rho}(J, \mathfrak{X})\}.$$

It is obvious that

$$\mathcal{C}^\nu_{1-\nu,\rho}(J, \mathfrak{X}) \subset \mathcal{C}^{\alpha,\beta}_{1-\nu,\rho}(J, \mathfrak{X}).$$

**Lemma 2.5.** *([11]) Let* $0 < \alpha < 1, 0 \leq \beta \leq 1$, $0 \leq \nu < 1$ *and* $\nu = \alpha + \beta(1 - \alpha)$. *If* $u \in \mathcal{C}_{\nu,\rho}(J, \mathfrak{X})$, *then*

$$\begin{aligned} {}^\rho \mathcal{J}^\nu_{0+}\, {}^\rho D^\nu_{0+} u(\cdot) &= {}^\rho \mathcal{J}^\alpha_{0+}\, {}^\rho D^{\alpha,\beta}_{0+} u(\cdot). \\ {}^\rho D^\nu_{0+}\, {}^\rho \mathcal{J}^\alpha_{0+} u(\cdot) &= {}^\rho D^{\beta(1-\alpha)}_{0+} u(\cdot). \end{aligned}$$

**Lemma 2.6.** *([11]) Assume that* $\alpha > 0$, $0 \leq \nu < 1$, $\rho > 0$ *and* $u \in \mathcal{C}_{\nu,\rho}(J, \mathfrak{X}))$. *If* $\nu < \alpha$, *then*

$$\left({}^\rho \mathcal{J}^\alpha_{0+} u\right)(0) = \lim_{t \to 0^+} \left({}^\rho \mathcal{J}^\alpha_{0+} u\right)(t) = 0.$$

**Lemma 2.7** ([11],[12]). *Let* $0 < \alpha < 1, 0 \leq \beta \leq 1$, $0 \leq \nu < 1$, $\nu = \alpha + \beta(1 - \alpha)$, *and assume that* $f(t, u(t)) \in \mathcal{C}_{1-\nu,\rho}(J, \mathfrak{X})$ *for any* $u \in \mathcal{C}_{1-\nu,\rho}(J, \mathfrak{X})$. *A function* $u \in \mathcal{C}^\nu_{1-\nu,\rho}(J, \mathfrak{X})$ *is a solution of fractional IVP*

$$\begin{aligned} {}^\rho D^{\alpha,\beta}_{0+} u(t) &= f(t, u(t)), \quad 0 < \alpha < 1, 0 \leq \beta \leq 1, \\ \left({}^\rho \mathcal{J}^{1-\nu}_{0+} u\right)(0) &= c, \quad \gamma = \alpha + \beta(1 - \alpha), c \in \mathbb{R}. \end{aligned}$$

*if and only if* $u$ *satisfies the following second kind Volterra fractional integral equation:*

$$u(t) = \frac{c}{\Gamma(\nu)} \left(\frac{t^\rho}{\rho}\right)^{\nu-1} + \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} f(s, u(s)) ds, \quad t > 0.$$

**Lemma 2.8.** *([11]) Let* $t > 0$ , ${}^\rho \mathcal{J}^\alpha_{0+}$ *and* ${}^\rho D^\alpha_{0+}$ *as defined in Definitions (2.1, 2.4). Then for* $\alpha \geq 0$ , $\beta > 0$, *we have*

$$\left({}^\rho \mathcal{J}^\alpha_{0+} \left(\frac{t^\rho}{\rho}\right)^{\beta-1}\right)(t) = \frac{\Gamma(\beta)}{\Gamma(\alpha+\beta)} \left(\frac{t^\rho}{\rho}\right)^{\alpha+\beta-1}$$

*and for* $0 < \alpha < 1$,

$$\left({}^\rho D^\alpha_{0+} \left(\frac{t^\rho}{\rho}\right)^{\alpha-1}\right)(t) = 0.$$

**Lemma 2.9.** *Let* $0 < \alpha < 1, 0 \leq \beta \leq 1, 0 \leq \nu < 1$, *and assume that* $f(t, u(t), u(\lambda t), {}^{\rho}D_{0+}^{\alpha,\beta}u(\lambda t)) \in \mathcal{C}_{1-\nu,\rho}(J, \mathfrak{X})$ *for any* $u \in \mathcal{C}_{1-\nu,\rho}(J, \mathfrak{X})$. *If* $u \in \mathcal{C}_{1-\nu,\rho}^{\nu}(J, \mathfrak{X})$ *then* $u$ *satisfies the problem (1.1) if and only if* $u$ *satisfies the following mixed-type Volterra fractional integral equation:*

$$\begin{aligned} u(t) &= \frac{\Psi\Gamma(\nu+r)}{\Gamma(\nu)\Gamma(\alpha+r)}\left(\frac{t^{\rho}}{\rho}\right)^{\nu-1}\sum_{i=1}^{m}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^{\rho}-s^{\rho}}{\rho}\right)^{\alpha+r-1}s^{\rho-1}\mathbb{K}_u(s)ds \\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^{t}\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}s^{\rho-1}\mathbb{K}_u(s)ds. \end{aligned} \tag{2.2}$$

*Where*

$$\Psi := \frac{1}{\Gamma(\nu+r)-\sum_{i=1}^{m}\zeta_i\left(\frac{\tau_i^{\rho}}{\rho}\right)^{r+\nu-1}}, \quad \Gamma(\nu+r) \neq \sum_{i=1}^{m}\zeta_i\left(\frac{\tau_i^{\rho}}{\rho}\right)^{r+\nu-1}.$$

*and for simplicity, we take*

$$\mathbb{K}_u(t) := {}^{\rho}D_{0+}^{\alpha,\beta}u(t) = f(t, u(t), u(\lambda t), \mathbb{K}_u(t)).$$

*Proof.* Firstly, we will prove the necessary condition. According to Lemma 2.3 and Lemma 2.7, a solution of the problem (1.1) can be expressed by

$$\begin{aligned} u(t) &= \frac{\left({}^{\rho}\mathcal{J}_{0+}^{1-\nu}u\right)(0)}{\Gamma(\nu)}\left(\frac{t^{\rho}}{\rho}\right)^{\nu-1} \\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^{t}\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}s^{\rho-1}f(s, u(s), u(\lambda s), \mathbb{K}_u(s))ds \end{aligned} \tag{2.3}$$

Next, we substitute $t = \tau_i$ and multiply both sides by $\zeta_i$, we can write

$$\begin{aligned} \zeta_i u(\tau_i) &= \frac{\left({}^{\rho}\mathcal{J}_{0+}^{1-\nu}u\right)(0)}{\Gamma(\nu)}\zeta_i\left(\frac{\tau_i^{\rho}}{\rho}\right)^{\nu-1} \\ &\quad+\frac{1}{\Gamma(\alpha)}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}s^{\rho-1}f(s, u(s), u(\lambda s), \mathbb{K}_u(s))ds \end{aligned} \tag{2.4}$$

Now, applying ${}^{\rho}\mathcal{J}_{0+}^{r}$ to both sides of Equation (2.4) and using Lemma 2.3 and Lemma 2.8, we have

$$\begin{aligned} {}^{\rho}\mathcal{J}_{0+}^{r}\zeta_i u(\tau_i) &= \frac{\left({}^{\rho}\mathcal{J}_{0+}^{1-\nu}u\right)(0)}{\Gamma(\nu+r)}\zeta_i\left(\frac{\tau_i^{\rho}}{\rho}\right)^{r+\nu-1} \\ &\quad+\frac{1}{\Gamma(\alpha+r)}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^{\rho}-s^{\rho}}{\rho}\right)^{\alpha+r-1}s^{\rho-1}\mathbb{K}_u(s)ds. \end{aligned}$$

which implies that

$$\begin{aligned} \sum_{i=1}^{m}{}^{\rho}\mathcal{J}_{0+}^{r}\zeta_i u(\tau_i) &= \frac{\left({}^{\rho}\mathcal{J}_{0+}^{1-\nu}u\right)(0)}{\Gamma(\nu+r)}\sum_{i=1}^{m}\zeta_i\left(\frac{\tau_i^{\rho}}{\rho}\right)^{r+\nu-1} \\ &\quad+\frac{1}{\Gamma(\alpha+r)}\sum_{i=1}^{m}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^{\rho}-s^{\rho}}{\rho}\right)^{\alpha+r-1}s^{\rho-1}\mathbb{K}_u(s)ds. \end{aligned}$$

thus, we get

$$\begin{aligned} \left({}^{\rho}\mathcal{J}_{0+}^{1-\nu}u\right)(0) &= \frac{\left({}^{\rho}\mathcal{J}_{0+}^{1-\nu}u\right)(0)}{\Gamma(\nu+r)}\sum_{i=1}^{m}\zeta_i\left(\frac{\tau_i^{\rho}}{\rho}\right)^{r+\nu-1} \\ &\quad+\frac{1}{\Gamma(\alpha+r)}\sum_{i=1}^{m}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^{\rho}-s^{\rho}}{\rho}\right)^{\alpha+r-1}s^{\rho-1}\mathbb{K}_u(s)ds. \end{aligned}$$

which implies that

$$\begin{aligned}\frac{1}{\Gamma(\alpha+r)}\sum_{i=1}^{m}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^{\rho}-s^{\rho}}{\rho}\right)^{\alpha+r-1}s^{\rho-1}\mathbb{K}_u(s)ds.\\ &=\left(1-\frac{1}{\Gamma(\nu+r)}\sum_{i=1}^{m}\zeta_i\left(\frac{\tau_i^{\rho}}{\rho}\right)^{r+\nu-1}\right)\left({}^{\rho}\mathcal{J}_{0+}^{1-\nu}u\right)(0)\\ &=\frac{1}{\Psi\Gamma(\nu+r)}\end{aligned}$$

thus,

$$\left({}^{\rho}\mathcal{J}_{0+}^{1-\nu}u\right)(0)=\frac{\Psi\Gamma(\nu+r)}{\Gamma(\nu)\Gamma(\alpha+r)}\left(\frac{t^{\rho}}{\rho}\right)^{\nu-1}\sum_{i=1}^{m}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^{\rho}-s^{\rho}}{\rho}\right)^{\alpha+r-1}s^{\rho-1}\mathbb{K}_u(s)ds \tag{2.5}$$

Substituting (2.5) in (2.3), we obtain (2.2). Secondly, let $u\in\mathcal{C}_{1-\nu,\rho}^{\nu}(J,\mathfrak{X})$ satisfies the mixed-type integral equation (2.2), then we prove that $u$ satisfies (1.1). By applying ${}^{\rho}D_{0+}^{\nu}$ on both sides of (2.2) and using Lemma 2.3 and Lemma 2.8, we get

$$\begin{aligned}{}^{\rho}D_{0+}^{\nu}u(t) &= {}^{\rho}D_{0+}^{\nu}\left(\frac{\Psi\Gamma(\nu+r)}{\Gamma(\nu)\Gamma(\alpha+r)}\left(\frac{t^{\rho}}{\rho}\right)^{\nu-1}\sum_{i=1}^{m}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^{\rho}-s^{\rho}}{\rho}\right)^{\alpha+r-1}s^{\rho-1}\mathbb{K}_u(s)ds\right)\\ &\quad+{}^{\rho}D_{0+}^{\nu}\left(\frac{1}{\Gamma(\alpha)}\int_0^{t}\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}s^{\rho-1}\mathbb{K}_u(s)ds\right)\\ &={}^{\rho}D_{0+}^{\beta(1-\alpha)}f(t,u(t),u(\lambda t),{}^{\rho}D_{0+}^{\alpha,\beta}u(\lambda t)).\end{aligned} \tag{2.6}$$

Since ${}^{\rho}D_{0+}^{\alpha,\beta}u\in\mathcal{C}_{1-\nu,\rho}(J,\mathfrak{X})$, then by definition of $\mathcal{C}_{1-\nu,\rho}^{\nu}(J,\mathfrak{X})$ and by using Equation (2.6), we have

$$ {}^{\rho}D_{0+}^{\beta(1-\alpha)}f=\delta_{\rho}{}^{\rho}\mathcal{J}_{0+}^{1-\beta(1-\alpha)}f\in\mathcal{C}_{1-\nu,\rho}(J,\mathfrak{X}). \tag{2.7}$$

For every $f\in\mathcal{C}_{1-\nu,\rho}(J,\mathfrak{X})$ it is clear that ${}^{\rho}\mathcal{J}_{0+}^{1-\beta(1-\alpha)}f\in\mathcal{C}_{1-\nu,\rho}(J,\mathfrak{X})$, which implies that ${}^{\rho}\mathcal{J}_{0+}^{1-\beta(1-\alpha)}f\in\mathcal{C}_{1-\nu,\rho}^{1}(J,\mathfrak{X})$. Therefore $f$ and ${}^{\rho}\mathcal{J}_{0+}^{1-\beta(1-\alpha)}f$ satisfy conditions of Lemma 2.3. Applying ${}^{\rho}\mathcal{J}_{0+}^{\beta(1-\alpha)}$ on both sides of Equation (2.6), we can obtain

$$\begin{aligned}{}^{\rho}\mathcal{J}_{0+}^{\beta(1-\alpha)}{}^{\rho}D_{0+}^{\nu}f &= {}^{\rho}\mathcal{J}_{0+}^{\beta(1-\alpha)}{}^{\rho}D_{0+}^{\beta(1-\alpha)}\mathbb{K}_u(t)\\ &= \mathbb{K}_u(t)-\frac{\left({}^{\rho}\mathcal{J}_{0+}^{1-\beta(1-\alpha)}\mathbb{K}_u\right)(0)}{\Gamma(\beta(1-\alpha))}\left(\frac{t^{\rho}}{\rho}\right)^{\beta(1-\alpha)-1}\\ &= f(t,u(t),u(\lambda t),{}^{\rho}D_{0+}^{\alpha,\beta}u(\lambda t)).\end{aligned}$$

By applying ${}^{\rho}\mathcal{J}_{0+}^{1-\nu}$ on both sides of the Equation (2.2), we get

$$\begin{aligned}\left({}^{\rho}\mathcal{J}_{0+}^{1-\nu}u\right)(t) &= {}^{\rho}\mathcal{J}_{0+}^{1-\nu}\left(\frac{\Psi\Gamma(\nu+r)}{\Gamma(\nu)\Gamma(\alpha+r)}\left(\frac{t^{\rho}}{\rho}\right)^{\nu-1}\sum_{i=1}^{m}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^{\rho}-s^{\rho}}{\rho}\right)^{\alpha+r-1}s^{\rho-1}\mathbb{K}_u(s)ds\right)\\ &\quad+{}^{\rho}\mathcal{J}_{0+}^{1-\nu}\left(\frac{1}{\Gamma(\alpha)}\int_0^{t}\left(\frac{t^{\rho}-s^{\rho}}{\rho}\right)^{\alpha-1}s^{\rho-1}\mathbb{K}_u(s)ds\right).\end{aligned}$$

since, $1-\nu<1-\beta(1-\alpha)$, then

$$\begin{aligned}\left({}^{\rho}\mathcal{J}_{0+}^{1-\nu}u\right)(t) &= \frac{\Psi\Gamma(\nu+r)}{\Gamma(\alpha+r)}\sum_{i=1}^{m}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^{\rho}-s^{\rho}}{\rho}\right)^{\alpha+r-1}s^{\rho-1}\mathbb{K}_u(s)ds\\ &\quad+{}^{\rho}\mathcal{J}_{0+}^{1-\beta(1-\alpha)}\mathbb{K}_u(t).\end{aligned}$$

taking the limit as $t\to 0^{+}$ yields

$$\left({}^{\rho}\mathcal{J}_{0+}^{1-\nu}u\right)(0)=\frac{\Psi\Gamma(\nu+r)}{\Gamma(\alpha+r)}\sum_{i=1}^{m}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^{\rho}-s^{\rho}}{\rho}\right)^{\alpha+r-1}s^{\rho-1}\mathbb{K}_u(s)ds$$

Using the substitution $t = \tau_i$ and multiplying through by $\zeta_i$ in (2.2), we have

$$\begin{aligned}\zeta_i u(\tau_i) &= \frac{\Psi\Gamma(\nu+r)}{\Gamma(\nu)\Gamma(\alpha+r)}\zeta_i\left(\frac{\tau_i^\rho}{\rho}\right)^{\nu-1}\sum_{i=1}^{m}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^\rho-s^\rho}{\rho}\right)^{\alpha+r-1}s^{\rho-1}\mathbb{K}_u(s)ds\\ &+\frac{1}{\Gamma(\alpha)}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\mathbb{K}_u(s)ds. \end{aligned}\tag{2.8}$$

Applying ${}^\rho\mathcal{J}_{0+}^r$ to both sides of (2.8), we get

$$\begin{aligned}{}^\rho\mathcal{J}_{0+}^r\zeta_i u(\tau_i) &= \frac{\Psi}{\Gamma(\alpha+r)}\zeta_i\left(\frac{\tau_i^\rho}{\rho}\right)^{r+\nu-1}\sum_{i=1}^{m}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^\rho-s^\rho}{\rho}\right)^{\alpha+r-1}s^{\rho-1}\mathbb{K}_u(s)ds\\ &+\frac{1}{\Gamma(\alpha+r)}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^\rho-s^\rho}{\rho}\right)^{r+\alpha-1}s^{\rho-1}\mathbb{K}_u(s)ds.\end{aligned}$$

which implies

$$\begin{aligned}\sum_{i=1}^{m}{}^\rho\mathcal{J}_{0+}^r\zeta_i u(\tau_i) &= \frac{\Psi}{\Gamma(\alpha+r)}\sum_{i=1}^{m}\zeta_i\left(\frac{\tau_i^\rho}{\rho}\right)^{r+\nu-1}\sum_{i=1}^{m}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^\rho-s^\rho}{\rho}\right)^{\alpha+r-1}s^{\rho-1}\mathbb{K}_u(s)ds\\ &+\frac{1}{\Gamma(\alpha+r)}\sum_{i=1}^{m}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^\rho-s^\rho}{\rho}\right)^{r+\alpha-1}s^{\rho-1}\mathbb{K}_u(s)ds.\\ &=\left(\Psi\sum_{i=1}^{m}\zeta_i\left(\frac{\tau_i^\rho}{\rho}\right)^{r+\nu-1}+1\right)\frac{1}{\Gamma(\alpha+r)}\sum_{i=1}^{m}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^\rho-s^\rho}{\rho}\right)^{\alpha+r-1}s^{\rho-1}\mathbb{K}_u(s)ds.\end{aligned}$$

and then we can easily see that

$$\left({}^\rho\mathcal{J}_{0+}^{1-\nu}u\right)(0)=\sum_{i=1}^{m}{}^\rho\mathcal{J}_{0+}^r\zeta_i u(\tau_i).$$

□

## 3 Existence and Uniqueness.

In order to prove our main results, we need the following assumptions:

H1. Let $f: J\times\mathfrak{X}\times\mathfrak{X}\times\mathfrak{X}\to\mathfrak{X}$ be a function such that $f\in\mathcal{C}_{1-\nu,\rho}(J,\mathfrak{X})$ for any $u\in\mathcal{C}_{1-\nu,\rho}(J,\mathfrak{X})$, and for $t\in J$, $u,v,\omega\in\mathfrak{X}$, there exist $\varrho,p,q,\chi\in\mathcal{C}_{1-\nu,\rho}(J,\mathfrak{X})$ such that

$$\|f(t,u,v,\omega)\|\le\varrho(t)+p(t)\|u(t)\|+q(t)\|v(t)\|+\chi(t)\|\omega(t)\|.$$

with

$$\varrho^*=\sup_{t\in J}\varrho(t),\quad p^*=\sup_{t\in J}p(t),\quad q^*=\sup_{t\in J}q(t),\quad \mathit{and}\quad \chi^*=\sup_{t\in J}\chi(t)<1.$$

H2. There exist constants $K>0, 0<K^*<1$ such that

$$\|f(t,u_1,v_1,\omega_1)-f(t,u_2,v_2,\omega_2)\|\le K(\|u_1-u_2\|+\|v_1-v_2\|)+K_1^*\|\omega_1-\omega_2\|.$$

for any $u_1,v_1,u_2,v_2,\omega_1,\omega_2\in\mathfrak{X}$.

H3. Suppose that the constant $\Lambda$ satisfies the following estimate

$$\Lambda:=\frac{2K}{1-K^*}\left(\frac{|\Psi|\Gamma(\nu+r)B(\nu,\alpha+r)}{\Gamma(\nu)\Gamma(\alpha+r)}\sum_{i=1}^{m}\zeta_i\left(\frac{\tau_i^\rho}{\rho}\right)^{\alpha+r+\nu-1}+\frac{B(\nu,\alpha)}{\Gamma(\alpha)}\left(\frac{T^\rho}{\rho}\right)^{\alpha}\right)<1.$$

**Theorem 3.1.** *Assume that* $[H1]$ *is satisfied. Then problem (1.1) has at least one solution in* $\mathcal{C}_{1-\nu,\rho}^{\nu}(J,\mathfrak{X})\subset\mathcal{C}_{1-\nu,\rho}^{\alpha,\beta}(J,\mathfrak{X})$.

*Proof.* Consider the well-defined operator $N : \mathcal{C}_{1-\nu,\rho}(J,\mathfrak{X}) \to \mathcal{C}_{1-\nu,\rho}(J,\mathfrak{X})$ defined by

$$\begin{aligned} N(u)(t) &= \frac{\Psi\Gamma(\nu+r)}{\Gamma(\nu)\Gamma(\alpha+r)}\left(\frac{t^\rho}{\rho}\right)^{\nu-1}\sum_{i=1}^m \zeta_i \int_0^{\tau_i}\left(\frac{\tau_i^\rho - s^\rho}{\rho}\right)^{\alpha+r-1} s^{\rho-1}\mathbb{K}_u(s)ds \\ &\quad + \frac{1}{\Gamma(\alpha)}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\mathbb{K}_u(s)ds. \end{aligned} \tag{3.1}$$

We will present the proof into several steps.

Step.1: The operator $N$ is continuous.
Let $u_m$ be a sequence such that $u_m \to u$ in $\mathcal{C}_{1-\nu,\rho}(J,\mathfrak{X})$. Then for each $t \in J$, we have

$$\left|((Nu_m)(t) - (Nu)(t))\,(t^\rho/\rho)^{1-\nu}\right| \le \mathbb{V}_1 + \mathbb{V}_2,$$

where

$$\mathbb{V}_1 = \frac{\mid \Psi \mid \Gamma(\nu+r)}{\Gamma(\nu)\Gamma(\alpha+r)}\sum_{i=1}^m \zeta_i \int_0^{\tau_i}\left(\frac{\tau_i^\rho - s^\rho}{\rho}\right)^{\alpha+r-1} s^{\rho-1} \mid \mathbb{K}_{u_m}(s) - \mathbb{K}_u(s) \mid ds.$$

and

$$\mathbb{V}_2 = \frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho}{\rho}\right)^{1-\nu}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} \mid \mathbb{K}_{u_m}(s) - \mathbb{K}_u(s) \mid ds.$$

hence,

$$\mathbb{V}_1 \le \frac{\mid \Psi \mid \Gamma(\nu+r)}{\Gamma(\nu)\Gamma(\alpha+r)}\left\|\mathbb{K}_{u_m}(\cdot) - \mathbb{K}_u(\cdot)\right\|_{\mathcal{C}_{1-\nu,\rho}}\sum_{i=1}^m \zeta_i \int_0^{\tau_i}\left(\frac{\tau_i^\rho - s^\rho}{\rho}\right)^{\alpha+r-1} s^{\rho-1}\left(\frac{s^\rho}{\rho}\right)^{\nu-1} ds.$$

and

$$\mathbb{V}_2 \le \frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho}{\rho}\right)^{1-\nu}\left\|\mathbb{K}_{u_m}(\cdot) - \mathbb{K}_u(\cdot)\right\|_{\mathcal{C}_{1-\nu,\rho}}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\left(\frac{s^\rho}{\rho}\right)^{\nu-1} ds.$$

therefore,

$$\left|((Nu_m)(t) - (Nu)(t))\,(t^\rho/\rho)^{1-\nu}\right| \le \Phi_1 \left\|\mathbb{K}_{u_m}(\cdot) - \mathbb{K}_u(\cdot)\right\|_{\mathcal{C}_{1-\nu,\rho}}$$

where $B(\cdot,\cdot)$ is the Beta function and

$$\Phi_1 = \left(\frac{\mid \Psi \mid \Gamma(\nu+r)B(\nu,\alpha+r)}{\Gamma(\nu)\Gamma(\alpha+r)}\sum_{i=1}^m \zeta_i \left(\frac{\tau_i^\rho}{\rho}\right)^{\alpha+r+\nu-1} + \frac{B(\nu,\alpha)}{\Gamma(\alpha)}\left(\frac{T^\rho}{\rho}\right)^{\alpha}\right)$$

Since $f$ is continuous, this implies that $\mathbb{K}_u(\cdot)$ is also continuous. Then we get

$$\|Nu_m - Nu\|_{\mathcal{C}_{1-\nu,\rho}} \to 0 \qquad , m \to \infty.$$

Step.2: The operator $N$ maps bounded sets into bounded sets in $\mathcal{C}_{1-\nu,\rho}(J,\mathfrak{X})$.
Assume that $\Theta_\kappa = \{u \in \mathcal{C}_{1-\nu,\rho}(J,\mathfrak{X}) : \|u\|_{\mathcal{C}_{1-\nu,\rho}}\}$. Actually, it is suffices to show that for any $\kappa > 0$, there exists $\vartheta > 0$ such that for any $u \in \Theta_\kappa$, we have $\|Nu\|_{\mathcal{C}_{1-\nu,\rho}} \le \vartheta$

$$\begin{aligned} \left|((Nu)(t))\,(t^\rho/\rho)^{1-\nu}\right| &\le \frac{\mid \Psi \mid \Gamma(\nu+r)}{\Gamma(\nu)\Gamma(\alpha+r)}\sum_{i=1}^m \zeta_i \int_0^{\tau_i}\left(\frac{\tau_i^\rho - s^\rho}{\rho}\right)^{\alpha+r-1} s^{\rho-1} \mid \mathbb{K}_u(s) \mid ds. \\ &\quad + \frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho}{\rho}\right)^{1-\nu}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} \mid \mathbb{K}_u(s) \mid ds. \end{aligned}$$

For simplicity, we put

$$\begin{aligned} \mathcal{A} &= \frac{\mid \Psi \mid \Gamma(\nu+r)}{\Gamma(\nu)\Gamma(\alpha+r)}\sum_{i=1}^m \zeta_i \int_0^{\tau_i}\left(\frac{\tau_i^\rho - s^\rho}{\rho}\right)^{\alpha+r-1} s^{\rho-1} \mid \mathbb{K}_u(s) \mid ds, \\ \mathcal{B} &= \frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho}{\rho}\right)^{1-\nu}\int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} \mid \mathbb{K}_u(s) \mid ds. \end{aligned}$$

Now, by assumption $[H1]$, we get

$$\begin{aligned} \mid \mathbb{K}_u(t) \mid &= \mid f(t, u(t), u(\lambda t), \mathbb{K}_u(t)) \mid \\ &\le \varrho(t) + p(t) \mid u \mid + q(t) \mid u \mid + \chi(t) \mid \mathbb{K}_u(t) \mid \\ &\le \frac{\varrho^* + (p^* + q^*) \mid u \mid}{1 - \chi^*} \end{aligned}$$

then,

$$\begin{aligned}\mathcal{A} &\leq \frac{|\Psi|\Gamma(\nu+r)}{\Gamma(\nu)\Gamma(\alpha+r)}\sum_{i=1}^{m}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^\rho-s^\rho}{\rho}\right)^{\alpha+r-1}s^{\rho-1}\left(\frac{\varrho^*+(p^*+q^*)\mid u\mid}{1-\chi^*}\right)ds,\\ &= \frac{|\Psi|\Gamma(\nu+r)}{\Gamma(\nu)(1-\chi^*)}\left(\frac{\varrho^*\left(\frac{\tau_i^\rho}{\rho}\right)^{\alpha+r}}{\Gamma(\alpha+r+1)}+\frac{(p^*+q^*)\left(\frac{\tau_i^\rho}{\rho}\right)^{\alpha+r+\nu-1}B(\nu,\alpha+r)\|u\|_{\mathcal{C}_{1-\nu,\rho}}}{\Gamma(\alpha+r)}\right).\end{aligned}$$

and

$$\mathcal{B}\leq\frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho}{\rho}\right)^{1-\nu}\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\left(\frac{\varrho^*+(p^*+q^*)\mid u\mid}{1-\chi^*}\right)ds.$$

thus,

$$\begin{aligned}\left|((Nu)(t))\left(t^\rho/\rho\right)^{1-\nu}\right| &\leq \frac{\varrho^*}{1-\chi^*}\left(\frac{|\Psi|\Gamma(\nu+r)}{\Gamma(\nu)\Gamma(\alpha+r+1)}\sum_{i=1}^{m}\zeta_i\left(\frac{\tau_i^\rho}{\rho}\right)^{\alpha+r}+\frac{1}{\Gamma(\alpha+1)}\left(\frac{T^\rho}{\rho}\right)^{\alpha-\nu+1}\right)\\ &\quad+\frac{(p^*+q^*)}{1-\chi^*}\Phi_1\|u\|_{\mathcal{C}_{1-\nu,\rho}}:=\vartheta.\end{aligned}$$

Step.3: The operator $N$ maps bounded sets into equicontinuous set of $\mathcal{C}_{1-\nu,\rho}(J,\mathfrak{X})$.
Let $t_1,t_2\in J$ such that $t_1\geq t_2$. Let $u\in\Theta_\kappa$, then

$$\begin{aligned}\left|((Nu)(t_1)-(Nu)(t_2))\left(t^\rho/\rho\right)^{1-\nu}\right| &\leq \Bigg|\frac{1}{\Gamma(\alpha)}\left(\frac{t_1^\rho}{\rho}\right)^{1-\nu}\int_0^{t_1}\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\mathbb{K}_u(s)ds\\ &\quad-\frac{1}{\Gamma(\alpha)}\left(\frac{t_2^\rho}{\rho}\right)^{1-\nu}\int_0^{t_2}\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\mathbb{K}_u(s)ds\Bigg|\end{aligned}$$

,

$$\begin{aligned}\left|((Nu)(t_1)-(Nu)(t_2))\left(t^\rho/\rho\right)^{1-\nu}\right| &\leq \left|\frac{1}{\Gamma(\alpha)}\int_0^{t_1}\Phi_2 s^{\rho-1}\mathbb{K}_u(s)ds\right|\\ &\quad+\left|\frac{1}{\Gamma(\alpha)}\left(\frac{t_2^\rho}{\rho}\right)^{1-\nu}\int_{t_1}^{t_2}\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\mathbb{K}_u(s)ds\right|\end{aligned}$$

where

$$\Phi_2=\left(\left(\frac{t_1^\rho}{\rho}\right)^{1-\nu}\left(\frac{t_1^\rho-s^\rho}{\rho}\right)^{\alpha-1}-\left(\frac{t_2^\rho}{\rho}\right)^{1-\nu}\left(\frac{t_2^\rho-s^\rho}{\rho}\right)^{\alpha-1}\right)$$

The right-hand side of the previous inequality tends to zero, as $t_1\to t_2$. Consequently, from Step.1-Step.3, together with Arzela-Ascoli theorem, we can deduce that

$$N:\mathcal{C}_{1-\nu,\rho}(J,\mathfrak{X})\to\mathcal{C}_{1-\nu,\rho}(J,\mathfrak{X})$$

is a completely continuous.

Step.4: The set

$$\Omega=\{u\in\mathcal{C}_{1-\nu,\rho}(J,\mathfrak{X}):u=\varpi(Nu),0\leq\varpi\leq1\}$$

is a bounded set.
Let $u\in\Omega$, $u=\varpi(Nu)$ for some $0\leq\varpi\leq1$. Then, for each $t\in J$, we have

$$\begin{aligned}u(t) &= \varpi(\frac{\Psi\Gamma(\nu+r)}{\Gamma(\nu)\Gamma(\alpha+r)}\left(\frac{t^\rho}{\rho}\right)^{\nu-1}\sum_{i=1}^{m}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^\rho-s^\rho}{\rho}\right)^{\alpha+r-1}s^{\rho-1}\mathbb{K}_u(s)ds\\ &\quad+\frac{1}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\mathbb{K}_u(s)ds.)\end{aligned}$$

It follows from assumption $[H1]$, that

$$\begin{aligned}\left|u(t)\left(\frac{t^\rho}{\rho}\right)^{1-\nu}\right| &\leq \left|(Nu)(t)\left(\frac{t^\rho}{\rho}\right)^{1-\nu}\right|\\ &\leq \frac{\varrho^*}{1-\chi^*}\left(\frac{|\Psi|\Gamma(\nu+r)}{\Gamma(\nu)\Gamma(\alpha+r+1)}\sum_{i=1}^{m}\zeta_i\left(\frac{\tau_i^\rho}{\rho}\right)^{\alpha+r}+\frac{1}{\Gamma(\alpha+1)}\left(\frac{T^\rho}{\rho}\right)^{\alpha-\nu+1}\right)\\ &\quad+\frac{(p^*+q^*)}{1-\chi^*}\Phi_1\|u\|_{\mathcal{C}_{1-\nu,\rho}}\\ &\quad<\infty.\end{aligned}$$

Therefore, the set $\Omega$ is bounded.

Consequently, by Schaefer's fixed point theorem we conclude that $N$ has a fixed point which is a solution of problem (1.1). $\square$

**Theorem 3.2.** *Assume that $[H1]-[H3]$ hold. Then the problem (1.1) has a unique solution.*

*Proof.* Consider the well-defined operator $N : \mathcal{C}_{1-\nu,\rho}(J,\mathfrak{X}) \to \mathcal{C}_{1-\nu,\rho}(J,\mathfrak{X})$, defined in (3.1). Clearly, by Lemma 2.9 and Theorem 3.1, the fixed points of the operator $N$ are solutions of the problem (1.1). Now it remains to show that the solution is unique. Let $u, v \in \mathcal{C}_{1-\nu,\rho}(J,\mathfrak{X})$, then we get

$$\begin{aligned}\left|((Nu)(t)-(Nv)(t))\,(t^\rho/\rho)^{1-\nu}\right| &\le \frac{|\,\Psi\,|\,\Gamma(\nu+r)}{\Gamma(\nu)\Gamma(\alpha+r)}\sum_{i=1}^{m}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^\rho-s^\rho}{\rho}\right)^{\alpha+r-1} s^{\rho-1}\,|\,\mathbb{K}_u(s)-\mathbb{K}_v(s)\,|\,ds.\\ &+\frac{1}{\Gamma(\alpha)}\left(\frac{t^\rho}{\rho}\right)^{1-\nu}\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\,|\,\mathbb{K}_u(s)-\mathbb{K}_v(s)\,|\,ds. \quad (3.2)\end{aligned}$$

, and by using assumption $[H2]$, we can estimate

$$|\,\mathbb{K}_u(s)-\mathbb{K}_v(s)\,| \le \frac{2K\,|\,u-v\,|}{1-K^*}. \tag{3.3}$$

By substituting (3.3) into inequalities (3.2) and using $[H2]$ we get

$$\left|(Nu)(t)-(Nv)(t)\,(t^\rho/\rho)^{1-\nu}\right| \le \frac{2K}{1-K^*}\Phi_1\,\|u-v\|_{\mathcal{C}_{1-\nu,\rho}}.$$

then,

$$\left|(Nu)(t)-(Nv)(t)\,(t^\rho/\rho)^{1-\nu}\right| \le \Lambda\,\|u-v\|_{\mathcal{C}_{1-\nu,\rho}}.$$

It follows from assumption $[H3]$, that the operator $N$ is a contraction map. By well known Banach contraction principle, we can deduce that $N$ has a unique fixed point which is the solution of the problem (1.1). $\square$

## 4 Ulam's Stability Results.

In this section, We will prove the Ulam stability result for the problem (1.1). Now, we will give definitions of Ulam-Hyers stable (U.H.S.) for the implicit pantograph Hilfer-Katugampola-type fractional differential equations (1.1).

For $\epsilon > 0$ and $\psi : J \to [0,\infty)$ is a continuous function, we consider the following inequality:

$$|\,{}^\rho D_{0+}^{\alpha,\beta} Z(t) - f(t, Z(t), Z(\lambda t), {}^\rho D_{0+}^{\alpha,\beta} Z(\lambda t))\,| \le \epsilon. \tag{4.1}$$

**Definition 4.1.** The problem (1.1) is said to be U.H.S. if there exists the real number $C_u > 0$ such that for all $\epsilon > 0$ and for each solution $Z \in \mathcal{C}^{\nu}_{1-\nu,\rho}(J,\mathfrak{X})$ of the inequality (4.1) there exists the solution $u \in \mathcal{C}^{\nu}_{1-\nu,\rho}(J,\mathfrak{X})$ of the problem (1.1) with

$$|\,Z(t)-u(t)\,| \le C_u\epsilon, \qquad t \in J.$$

**Remark 4.2.** The function $Z \in \mathcal{C}^{\nu}_{1-\nu,\rho}(J,\mathfrak{X})$ is a solution of the inequality (4.1), if and only if, there exists the function $g \in \mathcal{C}^{\nu}_{1-\nu,\rho}(J,\mathfrak{X}))$ such that

(a). $|\,g(t)\,| \le \epsilon, \qquad t \in J.$

(b). ${}^\rho D_{0+}^{\alpha,\beta} Z(t) = f(t, Z(t), Z(\lambda t), {}^\rho D_{0+}^{\alpha,\beta} Z(\lambda t)) + g(t), \qquad t \in J.$

**Lemma 4.3.** *Let $\rho > 0$, $0 < \alpha < 1$, and $0 \le \beta \le 1$. If the function $Z \in \mathcal{C}^{\nu}_{1-\nu,\rho}(J,\mathfrak{X})$ is the solution of the inequality (4.1), then $Z$ is the solution of the following integral inequality*

$$\left|Z(t)-\Xi_Z-\frac{1}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\mathbb{K}_Z(s)ds\right| \le \Upsilon\epsilon$$

*where*

$$\begin{aligned}\Xi_Z &= \frac{\Psi\Gamma(\nu+r)}{\Gamma(\nu)\Gamma(\alpha+r)}\left(\frac{t^\rho}{\rho}\right)^{\nu-1}\sum_{i=1}^{m}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^\rho-s^\rho}{\rho}\right)^{\alpha+r-1}s^{\rho-1}\mathbb{K}_Z(s)ds\\ \Upsilon &= \frac{|\,\Psi\,|\,\Gamma(\nu+r)\left(\frac{T^\rho}{\rho}\right)^{\alpha+r+\nu-1} m\zeta}{\Gamma(\nu)\Gamma(\alpha+r+1)}+\frac{1}{\Gamma(\alpha+1)}\left(\frac{T^\rho}{\rho}\right)^{\alpha}.\end{aligned}$$

*Proof.* In view of Remark (4.2), we have

$$\begin{aligned} {}^{\rho}D_{0+}^{\alpha,\beta}Z(t) &= f(t,Z(t),Z(\lambda t),{}^{\rho}D_{0+}^{\alpha,\beta}Z(\lambda t))+g(t),\\ &= \mathbb{K}_Z(t)+g(t). \end{aligned}$$

Then

$$\begin{aligned} Z(t) &= \frac{\Psi\Gamma(\nu+r)}{\Gamma(\nu)\Gamma(\alpha+r)}\left(\frac{t^\rho}{\rho}\right)^{\nu-1}\sum_{i=1}^{m}\zeta_i\left(\int_0^{\tau_i}\left(\frac{\tau_i^\rho-s^\rho}{\rho}\right)^{\alpha+r-1}s^{\rho-1}\mathbb{K}_Z(s)ds+\int_0^{\tau_i}\left(\frac{\tau_i^\rho-s^\rho}{\rho}\right)^{\alpha+r-1}s^{\rho-1}g(s)ds\right)\\ &\quad+\frac{1}{\Gamma(\alpha)}\left(\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\mathbb{K}_Z(s)ds+\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}g(s)ds\right). \end{aligned}$$

From this we get the following:

$$\begin{aligned} &\left|Z(t)-\Xi_Z-\frac{1}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\mathbb{K}_Z(s)ds\right|\\ &=\left|\frac{\Psi\Gamma(\nu+r)}{\Gamma(\nu)\Gamma(\alpha+r)}\left(\frac{t^\rho}{\rho}\right)^{\nu-1}\sum_{i=1}^{m}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^\rho-s^\rho}{\rho}\right)^{\alpha+r-1}s^{\rho-1}g(s)ds+\frac{1}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}g(s)ds\right|\\ &\le\frac{|\Psi|\Gamma(\nu+r)}{\Gamma(\nu)\Gamma(\alpha+r)}\left(\frac{t^\rho}{\rho}\right)^{\nu-1}\sum_{i=1}^{m}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^\rho-s^\rho}{\rho}\right)^{\alpha+r-1}s^{\rho-1}|g(s)|ds+\frac{1}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}|g(s)|ds\\ &\le\frac{|\Psi|\Gamma(\nu+r)}{\Gamma(\nu)\Gamma(\alpha+r)}\left(\frac{t^\rho}{\rho}\right)^{\nu-1}\sum_{i=1}^{m}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^\rho-s^\rho}{\rho}\right)^{\alpha+r-1}s^{\rho-1}\epsilon ds+\frac{1}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\epsilon ds\\ &\le\left(\frac{|\Psi|\Gamma(\nu+r)\left(\frac{T^\rho}{\rho}\right)^{\alpha+r+\nu-1}m\zeta}{\Gamma(\nu)\Gamma(\alpha+r+1)}+\frac{1}{\Gamma(\alpha+1)}\left(\frac{T^\rho}{\rho}\right)^{\alpha}\right)\epsilon\\ &=\Upsilon\epsilon \end{aligned}$$

□

**Lemma 4.4.** *([23]) Assume that $\phi:[0,T]\to[0,\infty)$ is a real function and $\psi(\cdot)$ is a non-negative locally integrable function on $[0,T]$. Let there exists $k>0,\theta>0$, and $0<\alpha<1$, such that*

$$\phi(t)\le\psi(t)+\theta\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\phi(s)ds$$

*Then, there exist a constant $C=C(\alpha)$ such that for $t\in[0,T]$, we have*

$$\phi(t)\le\psi(t)+C\theta\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\psi(s)ds$$

**Theorem 4.5.** *Assume that the hypotheses $[H2],[H3]$ are satisfied. Then, the problem (1.1) is U.H.S..*

*Proof.* Let $\epsilon>0$ and let for any $t\in J$ the function $Z\in\mathcal{C}_{1-\nu,\rho}^{\nu}(J,\mathfrak{X})$ satisfies the inequality (4.1). In the light of Theorem 3.2, $u\in\mathcal{C}_{1-\nu,\rho}^{\nu}(J,\mathfrak{X})$ is the unique solution of the problem (1.1). By using Lemma 2.9, we have

$$u(t)=\Xi_u+\frac{1}{\Gamma(\alpha)}\int_0^t\left(\frac{t^\rho-s^\rho}{\rho}\right)^{\alpha-1}s^{\rho-1}\mathbb{K}_u(s)ds.$$

where

$$\Xi_u=\frac{\Psi\Gamma(\nu+r)}{\Gamma(\nu)\Gamma(\alpha+r)}\left(\frac{t^\rho}{\rho}\right)^{\nu-1}\sum_{i=1}^{m}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^\rho-s^\rho}{\rho}\right)^{\alpha+r-1}s^{\rho-1}\mathbb{K}_u(s)ds.$$

Now, if $u(\tau_i)=Z(\tau_i)$ and ${}^{\rho}\mathcal{J}_{0+}^{1-\nu}u(0)={}^{\rho}\mathcal{J}_{0+}^{1-\nu}Z(0)$, then $\Xi_u=\Xi_Z$, and that

$$\begin{aligned} |\Xi_u-\Xi_Z| &\le \frac{\Psi\Gamma(\nu+r)}{\Gamma(\nu)\Gamma(\alpha+r)}\left(\frac{t^\rho}{\rho}\right)^{\nu-1}\sum_{i=1}^{m}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^\rho-s^\rho}{\rho}\right)^{\alpha+r-1}s^{\rho-1}|\mathbb{K}_u(s)\mathbb{K}_Z(s)|ds,\\ &\le \frac{\Psi\Gamma(\nu+r)}{\Gamma(\nu)\Gamma(\alpha+r)}\left(\frac{t^\rho}{\rho}\right)^{\nu-1}\sum_{i=1}^{m}\zeta_i\int_0^{\tau_i}\left(\frac{\tau_i^\rho-s^\rho}{\rho}\right)^{\alpha+r-1}s^{\rho-1}\left(\frac{2K}{1-K^*}\right)|u(s)-Z(s)|ds,\\ &\le \frac{\Psi\Gamma(\nu+r)}{\Gamma(\nu)\Gamma(\alpha+r)}\left(\frac{t^\rho}{\rho}\right)^{\nu-1}\left(\frac{2K}{1-K^*}\right)\sum_{i=1}^{m}\zeta_i{}^{\rho}\mathcal{J}_{0+}^{\alpha+r}|u(\tau_i)-Z(\tau_i)|=0. \end{aligned}$$

Then we have,

$$u(t) = \Xi_Z + \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\mathbb{K}_u(s)ds.$$

By applying Lemma 4.3 and integration of inequality (4.1) for any $t \in J$, we have

$$\begin{aligned} |Z(t) - u(t)| \quad &\leq \quad \left| Z(t) - \Xi_Z - \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1}\mathbb{K}_Z(s)ds \right| \\ &\quad + \frac{1}{\Gamma(\alpha)} \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} \mid \mathbb{K}_Z(s) - \mathbb{K}_u(s) \mid ds \end{aligned}$$

and by using inequality (3.3), we can obtain that

$$\mid Z(t) - u(t) \mid \leq \Upsilon\epsilon + \frac{1}{\Gamma(\alpha)\left(\frac{2K}{1-K^*}\right)} \int_0^t \left(\frac{t^\rho - s^\rho}{\rho}\right)^{\alpha-1} s^{\rho-1} \mid Z(s) - u(s) \mid ds$$

Now, applying Lemma 4.4, we get

$$\mid Z(t) - u(t) \mid \leq \Upsilon\epsilon \left(1 + \left(\frac{2K}{(1-K^*)\Gamma(\alpha+1)}\right)\left(\frac{T^\rho}{\rho}\right)^\alpha\right) := C_u\epsilon$$

Therefore, the problem (1.1) is U.H.S. □

## 5 Applications.

In this section, we give an example to illustrate our main results.

**Example 5.1.** We consider the following implicit pantograph Hilfer-Katugampola-type fractional differential equation

$$\begin{aligned} {}^{1/2}D_{0+}^{\frac{2}{3},\frac{1}{2}}u(t) \quad &= \quad \frac{1}{35e^{2t+1}\left(1+ \mid u(t) \mid + \mid u(\frac{1}{2}t) \mid + \mid {}^{1/2}D_{0+}^{\frac{2}{3},\frac{1}{2}}u(\frac{1}{2}t) \mid\right)} \qquad t \in (0,1], \\ \left({}^{1/2}\mathcal{J}_{0+}^{\frac{1}{6}}u\right)(0) \quad &= \quad 3^{1/2}\mathcal{J}_{0+}^{\frac{1}{2}}u(3/2) \quad ,\nu = \alpha + \beta(1-\alpha). \end{aligned} \tag{5.1}$$

Here, ${}^\rho D_{1+}^{\alpha,\beta}$ is the Hilfer-Katugampola fractional derivative, $\rho = 1/2, \alpha = 2/3, \beta = 1/2$, $\nu = 5/6$, $\zeta_1 = 3$ and $\tau_1 = 3/2$. Let

$$f(t,u,v,\omega) = \frac{1}{35e^{2t+1}\,(1+ \mid u \mid + \mid v \mid + \mid \omega \mid)}$$

Clearly, for $u, v, \omega, \bar{u}, \bar{v}, \bar{\omega} \in \mathbb{R}_+$ and $t \in (0,1]$, the functions $f$ is continuous and the hypotheses $[H2], [H3]$ are satisfied with $K = K^* = \frac{1}{35e}$. Therefore, by simple calculations, we get $\mid \Psi \mid = 0.3935$ and $\Lambda \approx 0.097813 < 1$. Hence, from Theorem 3.2, it follows that the problem (5.1) has a unique solution. Furthermore, it implies from Theorem 4.5, that the problem (5.1) is U.H.S.

## 6 Conclusion

The existence and uniqueness theorems of solutions to a class of implicit Hilfer-Katugampola-type fractional pantograph differential equation with nonlocal katugampola fractional integral conditions have been studied. For the mentioned theorems, the obtained results have been derived by different methods of analysis like Schaefer's fixed point theorem and Banach contraction principle. Also, some convenient results about (U.H.) stability have been established. The acquired results have been justified by one pertinent example. In the future, the above results and analysis can be extended to stochastic fractional differential equation involving Hilfer-Katugampola fractional derivative.

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