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VERTEX INDUCED 2-EDGE COLORING AND VERTEX INCIDENT 2-EDGE COLORING OF SOME GRAPH PRODUCTS

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Abstract Let $G = (V, E)$ be a simple connected graph with vertex set V and edge set E. The vertex induced 2-edge coloring number $\psi'_{vi2}(G)$ is the maximum number of colors used in coloring the edges of a graph G such that for each vertex $v \in V$, at most two edges in the induced subgraph $\langle N[v]\rangle$, generated by the closed neighborhood $N[v]$, receive different colors. The vertex incident 2-edge coloring number $\psi'_{vin2}(G)$ of graph G is the maximum number of colors required to color the edges of G such that at most two edges incident to a vertex v in G receive different colors. In this paper, the vertex induced 2-edge coloring number and vertex incident 2-edge coloring number of some graph products such as Cartesian product and strong product are discussed. The $\psi'_{vi2}(G)$ and $\psi'_{vin2}(G)$ number in the rooted product of a general connected graphs with some graph classes are also discussed in this paper.

1 Introduction

Graph coloring problems usually aim at minimizing the number of colors. But there is another fast-growing area of the literature where the number of colors used in a graph coloring problem can be maximized under certain conditions as seen in the articles such as the 3-consecutive vertex coloring number [9], 3-consecutive edge coloring number [2], 3-successive c-edge coloring number[1] and worm coloring [3]. This article focuses on two types of edge coloring problems where the number of colors used is maximized under certain conditions.

Let $G = (V, E)$ be a simple connected graph with $V(G)$ and $E(G)$ as its vertex set and edge set respectively. For each vertex $v \in V(G)$, the closed neighborhood of v is given as $N[v] = \{v\} \cup \{u : uv \in E(G)\}$ and $\langle N[v]\rangle$ represents the induced subgraph generated by $N[v]$. A vertex induced 2-edge coloring of a graph G is an edge coloring of a graph G such that for each vertex $v \in V(G)$, at most two edges in $\langle N[v]\rangle$ are differently colored. The vertex induced 2-edge coloring number, denoted as $\psi'_{vi2}(G)$, is the maximum number of colors required to color the edges of a given graph. Similarly, the edges incident at a vertex $v \in V(G)$ or in other words a star at vertex v is a subgraph of graph G containing all edges incident at vertex v. Thus vertex incident 2-edge coloring of a graph G is a coloring of the edges of G such that at most two different colors are used to color the edges incident to a vertex v in G. The vertex incident 2-edge coloring number, denoted as $\psi'_{vin2}(G)$ gives the maximum number of colors required to color the edges of a given graph. This concept is the same as the edge coloring condition discussed in [8].

The above-mentioned parameters are likely to be anticipated in warfare where the number of soldiers has to be deployed. Consider $G = (V, E)$ to be a network of lanes in an affected area where the security personnel has to be deputed. $|V(G)|$ gives the total number of junctions in the area whereas $|E(G)|$ gives the total number of lanes. Assume that for some reasons at most two guards are allowed to secure the houses or civilians on the lanes at the subgraph induced by each junction $v \in V$. Under such conditions, one may be interested to know the maximum number of security personnel that can be deputed in the network G. This is $\psi'_{vi2}(G)$. Similarly, the maximum number of guards required to secure the civilians in at most two lanes at each junction is $\psi'_{vin2}(G)$ [8].

We use the following definitions and notations for the further development of this article. For more definitions of graph theory, we refer book [6].

- The Cartesian product of graphs G_1 and G_2, denoted as $G_1 \square G_2$, is the graph with vertex set $V(G_1 \square G_2) = V(G_1) \times V(G_2)$ and two vertices $u = (u_1, u_2)$ and $v = (v_1, v_2)$ are adjacent in $G_1 \square G_2$ if $u_1 = v_1$ and u_2 adjacent to v_2 in G_2 or u_1 adjacent to v_1 in G_1 and $u_2 = v_2$ (see [7]).
- The strong product of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted as $G_1 \boxtimes G_2$, is the graph with vertex set $V_1 \times V_2$ in which vertices (u_1, u_2) and (v_1, v_2) are adjacent whenever $u_1 = u_2$ and $v_1v_2 \in E_2$, or $v_1 = v_2$ and $u_1u_2 \in E_1$, or $u_1u_2 \in E_1$ and $v_1v_2 \in E_2$ (see [5]).
- Graph $G \bigodot H$ is called the rooted product of G by H if G is a labelled graph on n vertices and H is a sequence of n rooted graphs, say $H_1, H_2, \ldots, H_n$, such that the graph $G \bigodot H$ is obtained by identifying the root vertex of H_i with the i^{th} vertex of G for all $i \in \{1, 2, \ldots, n\}$ (see [4]).

Theorem 1.1 ([8]). *Let G be a connected graph, then $\psi'_{vi2}(G) \leq \psi'_{vin2}(G)$. The equality holds for triangle free graphs.*

Theorem 1.2 ([8]). *For a graph $G = (p, q)$ with $q \geq 2$, $2 \leq \psi'_{vi2}(G), \psi'_{vin2}(G) \leq q$.*

Remark 1.3. For a triangle-free graph G, $\psi'_{vi2}(G) = \psi'_{vin2}(G)$. Therefore, in most of the theorems, we find the vertex induced 2-edge coloring number unless otherwise mentioned.

In this paper, we study the vertex induced 2-edge coloring number and vertex incident 2-edge coloring number of some graph products such as Cartesian product, strong product, and rooted product.

2 Results on Cartesian and Strong Graph Products

In this section, we determine the exact values of ψ'_{vi2} and ψ'_{vin2} of some Cartesian products such as $P_n \square P_2$, $P_m \square P_n$, $C_m \square C_n$, $P_n \square K_{1,n}$. Also we find the exact values of ψ'_{vi2} and ψ'_{vin2} of the Hypercube, the strong product $P_n \boxtimes P_m$, $m, n \geq 3$ and $P_n \boxtimes P_2$.

Theorem 2.1. *If G is the Cartesian product of $P_n \square P_2$, then*

$$\psi'_{vi2}(G) = \psi'_{vin2}(G) = \begin{cases} n+2, & 2 \leq n \leq 3 \\ n+3, & n > 3 \end{cases}$$

Proof. Let $G \equiv P_n \square P_2$. Since the Cartesian product $P_n \square P_2$ is obtained by taking two copies of P_n, therefore we denote the vertex set corresponding to the first copy of P_n in G by $\{v_1, v_2, \ldots, v_n\}$ and the vertex set of the second copy by $\{v'_1, v'_2, \ldots, v'_n\}$ and $\{v_iv_{i+1}\} \cup \{v'_iv'_{i+1}\} \cup \{v_iv'_i\}$ where $1 \leq i \leq n-1$, be the edge set of the graph G.

Case-1: Assume that $n = 2$. In this case $G \equiv P_2 \square P_2$ is a cycle graph on 4 vertices and hence $\psi'_{vi2}(G) = 4 = 2 + 2$.

Case-2: Assume that $n = 3$, that is, $G \equiv P_3 \square P_2$. Consider the edges incident at the vertices $v_2 \cup v'_2$. It can be noted that these five edges can be given a maximum of three colors. If we use four colors to color the edges incident at $v_2 \cup v'_2$, then we get a contradiction to the definition of vertex induced 2-edge coloring. Now color the remaining edges $v_1v'_1$ and $v_3v'_3$ with two different colors. Thus $\psi'_{vi2}(G) = 3 + 2 = 5$.

Case-3: Let $G \equiv P_n \square P_2$ where $n > 3$. We define the α_0- sets as the set of non-adjacent vertices in the vertex set $V(G)$.

Sub-case 3.1: Assume that n is odd. In this case, we consider $\{v_1, v'_2, \ldots, v'_{n-1}, v_n\}$ as the α_0- set. The edges of the subgraph $\langle N[v_1] \cup N[v'_2] \rangle$ can be colored by four different colors. It is to be noted that, at most one new color can be given to the edges of the subgraph obtained by the addition of the new vertex v_3 from the α_0-set, where $\mid N(v_2') \cap N(v_3) \mid \neq \phi$. Continuing in this manner one can see that the edges of subgraph $\langle N[v_1] \cup N[v_2'] \cup N[v_3] \cup \ldots \cup N[v_{n-1}'] \rangle$ cab be assigned with $n+1$ distinct colors such that the edge $v_{n-1}v_{n-1}'$ is given the same color as that of edge $v_{n-2}v_{n-2}'$ while the edge $v_{n-1}'v_n'$ is colored by $(n+1)^{th}$ color. The remaining two uncolored edges can be assigned with two new colors. Noted that the above coloring pattern gives the maximum vertex induced 2-edge coloring number of G. Hence the vertex induced 2-edge coloring number of G is $n+3$.

Sub-case 3.2: Assume that n is even. In this case, we consider $\{v_1, v_2', \ldots, v_{n-1}, v_n'\}$ as the α_0- set. Then one can use at most four colors to color the edges in subgraph $\langle N[v_1] \cup N[v_2'] \rangle$. Continuing in this manner, one requires $n+1$ colors to color the edges in subgraph $\langle N[v_1] \cup N[v_2'] \ldots \cup N[v_{n-1}] \rangle$ such that the edge $v_{n-1}v_{n-1}'$ is given the same color as that of edge $v_{n-2}v_{n-2}'$ whereas the edge $v_{n-1}v_n$ is assigned with $(n+1)^{th}$ color. The remaining two uncolored edges are given two new colors. Thus the maximum vertex induced 2-edge coloring number of G is $n+3$.

Therefore, from above two sub-cases we can conclude that $\psi_{vi2}'(G) = n+1+2 = n+3$ for $n > 3$. □

Next, we consider the more general case. That is, we study the vertex induced 2-edge coloring number of the rectangular grid graph ($G_{m,n} \equiv P_m \square P_n$). The graph $G_{m,n}$ is also known as the grid graph is a graph of order mn and size $(2mn - m - n)$. We use the following lemma to prove the next proposition.

Lemma 2.2. *Consider the graph $G_{m,n} \equiv P_m \square P_n$ as the rectangular grid graph such that $m, n \geq 4$ and $m \leq n$. Let $G_{m,n}^*$ be the graph obtained by removing $(2n+2m-4)$ outer edges from the graph $G_{m,n}$ and removing the isolated vertices after the removal of the outer edges. Then*

$$\psi_{vi2}'(G_{m,n}^*) = \psi_{vin2}'(G_{m,n}^*) = \begin{cases} 2n + (m-4)\left\lfloor \frac{n}{2} \right\rfloor - 3, & \text{when } n \text{ is even} \\ 2n + (m-4)\left\lfloor \frac{n}{2} \right\rfloor + \frac{m-11}{2}, & \text{when } m \text{ \& } n \text{ are odd} \\ 2n + (m-4)\left\lfloor \frac{n}{2} \right\rfloor + \frac{m-10}{2}, & m \text{ is even \& } n \text{ is odd} \end{cases}$$

Proof. Consider the graph $G_{m,n} \equiv P_m \square P_n$ such that $m, n \geq 4$ and $m \leq n$. Let $G_{m,n}^*$ be a graph obtained from graph $G_{m,n}$ after removing $(2n+2m-4)$ edges from its boundary.

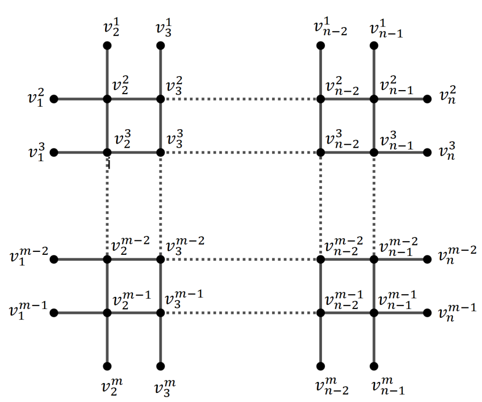

Figure 1. Vertex labeling of Graph $G_{m,n}^*$

Thus $G_{m,n}^*$ is a graph of order $(mn-4)$ and size $(2mn - 3m - 3n + 4)$. Let $V(G_{m,n}) = \{v_1^j, v_2^j, \ldots, v_n^j,\}$, where $1 \leq j \leq m$, be the vertex set of graph $G_{m,n}$. Then $V^*(G_{m,n}^*) = V(G_{m,n}) \setminus \{v_1^1, v_n^1, v_1^m, v_n^m\}$ is the vertex set of graph $G_{m,n}^*$, refer figure 1.

To maximize the number of colors we initiate the coloring procedure by giving distinct colors to the pendant edges of the graph $G_{m,n}^*$. Case-1 explains the reason for assigning $2m+2n-11$ distinct colors to the $2m+2n-8$ pendant edges of the graph $G_{m,n}^*$.

Case 1: Assume the case when $m = 4$ and $n \geq 4$. In order to maximize the number of colors in the given coloring procedure, the pendant edges of the graph $G^*_{4,n}$ can be given $2n$ distinct colors, say $\{c_1, c_2, \ldots, c_{2n}\}$, refer figure 2. The remaining uncolored edges of the graph $G^*_{4,n}$ are incident at the vertex v^i_j with degree 4, where $2 \leq j \leq n-1$ and $2 \leq i \leq m-1$.

Consider the edges incident to the vertex v^2_2. Since there are exactly two uncolored edges $v^2_2 v^3_2$ and $v^2_2 v^2_3$ incident at v^2_2, so these edges can be given either the color c_1 or the color c_{2n}. Here it can be noticed that, by coloring the edge $v^2_2 v^3_2$ with one of the above two mentioned colors, the vertex induced 2-edge coloring conditions fails at the vertex v^3_2. So the edge $v^3_1 v^3_2$ is given same color as that of edge $v^3_2 v^4_2$. Similarly, at the vertex v^3_{n-1} the vertex induced $2-$edge coloring condition fails if the edge $v^2_{n-1} v^3_{n-1}$ is colored by c_1, c_{n-2}, c_{n-1} or c_{2n}. Hence the edges $v^2_{n-1} v^2_n$ and $v^3_{n-1} v^4_{n-1}$ are given c_{n-2} and c_n colors respectively. Therefore, the maximum number of colors needed to color the pendant edges of the graph $G^*_{4,n}$ can be given $2n-3$ colors.

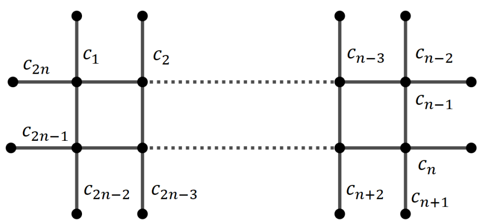

Figure 2. Coloring of Pendant edges of Graph $G^*_{4,n}$

Assume that the pendant edges of the graph $G^*_{4,n}$ are assigned $2n-3$ colors. Again, consider the edges incident at the vertices $v^2_2 \cup v^2_3$. Since there are exactly two edges that are incident at the vertex v^2_2 receives two different colors thus, the other four uncolored edges which are incident at vertices $v^2_2 \cup v^2_3$ has to be either colored by c_{2n-3} or c_1. Let the remaining edges incident at the vertices $v^2_2 \cup v^2_3$ be colored by c_{2n-3}. It can be seen that by the addition of any number of adjacent vertices of degree 4 to the vertices $v^2_2 \cup v^2_3$ there is no further increase in the vertex induced 2-edge coloring number of the graph $G^*_{4,n}$. Therefore, $\psi'_{vi2}(G^*_{4,n}) = 2n-3$.

In the cases discussed below we consider $n, m \geq 5$ and $m \leq n$.

Case-2: Assume when n is even and m is either odd or even. As discussed in Case-1, the pendant edges of graph the $G^*_{m,n}$ can be given $2(m-4)+2(n-2)+1$ colors such that the edges $v^2_{n-1} v^2_n, v^{m-1}_{n-1} v^m_{n-1}$ and $v^{m-1}_1 v^{m-1}_2$ receives the same color as that of edges $v^1_{n-1} v^2_{n-1}$, $v^{m-1}_{n-1} v^{m-1}_n$ and $v^{m-1}_2 v^m_2$ respectively. Let $\mathcal{C} = \{c_1, c_2, \ldots, c_{2m+2n-11}\}$ be the set of colors required to color the pendant edges of the graph $G^*_{m,n}$. Consider the subgraph H with the vertex set $V(H) = \{v^2_2, v^2_3, \ldots, v^2_{n-1}, v^3_{n-1}, \ldots, v^{m-1}_{n-1}, v^{m-1}_{n-2}, \ldots, v^{m-1}_2, v^{m-2}_2, \ldots, v^3_2\}$ and the edge set $E(H) = \{v^2_i v^2_{i+1}, v^j_{n-1} v^{j+1}_{n-1}, v^{m-1}_i v^{m-1}_{i+1}, v^j_2 v^{j+1}_2, v^2_k v^3_k, v^l_{n-2} v^l_{n-1}, v^{m-2}_k v^{m-1}_k, v^l_2 v^l_3\}$, where $2 \leq i \leq n-2; 2 \leq j \leq m-2; 3 \leq k \leq n-2; 3 \leq l \leq m-2$, of the graph $G^*_{m,n}$. It can be verified that these colorless edges of the subgraph H have to be either colored by c_1 or $c_{2m+2n-11}$. For if one of the edge of the subgraph H, say edge $v^2_3 v^3_3$, receives a different color $c'_n \notin \mathcal{C}$, then the vertex induced 2-edge condition fails at the vertex v^2_3 as the edges incident at vertex v^2_3 receives three different colors. Let H^k be the caterpillar subgraphs of the graph $G^*_{m,n}$ with $V(H^k) = \{v^k_3, v^k_4, \ldots, v^k_{n-2}\}$ as the vertex set and $E(H^k) = \{v^k_2 v^k_3, v^{k-1}_3 v^k_3, v^k_3 v^{k+1}_3, v^k_3 v^k_4, \ldots, v^k_{n-3} v^k_{n-2}, v^{k-1}_{n-2} v^k_{n-2}, v^k_{n-2} v^{k+1}_{n-2}, v^k_{n-2} v^k_{n-1}\}$ as the edge set such that $3 \leq k \leq m-2$. Consider the uncolored edges of the caterpillar subgraph H^3. The uncolored edges of the subgraph H^3 can be given $\lfloor \frac{n-4}{2} \rfloor$ new colors such that each alternative horizontal edges are assigned with a new edge color while the other edges of H^3 are assigned with either color c_1 or $c_{2m+2n-11}$. Similarly, the uncolored edges of each caterpillar subgraph H^k of the graph $G^*_{m,n}$ where $3 \leq k \leq m-2$ can be assigned with $\lfloor \frac{n-4}{2} \rfloor$ distinct colors. Thus, $(m-4) \times \lfloor \frac{n-4}{2} \rfloor$ new colors are required to color all the uncolored edges of the subgraph $G_{m-2,n-2}$ of the graph $G^*_{m,n}$. Therefore, $\psi'_{vi2}(G^*_{m,n}) = 2(n-2) + 2(m-4) + \lfloor \frac{n-4}{2} \rfloor (m-4) + 1 = 2n + (m-4) \lfloor \frac{n}{2} \rfloor - 3$.

Case-3: Assume when m and n are odd. As discussed in Case-1 and Case-2, the pendant edges and the colorless edges of the subgraph H of the graph $G^*_{m,n}$ can be colored with $2m + 2n - 11$ colors. Let $G_1 \equiv v^3_{n-2}v^4_{n-2} \cdots v^{m-2}_{n-2}$ be the path subgraph and H^k, for $3 \le k \le m-2$ be the caterpillar subgraphs of the graph $G^*_{m,n}$. Let $V(G_1) = \{v^3_{n-2}, v^4_{n-2}, \ldots, v^{m-2}_{n-2}\}$ be the vertex set and $E(G_1) = \{v^3_{n-2}v^4_{n-2}, v^4_{n-2}v^5_{n-2}, \ldots, v^{m-3}_{n-2}v^{m-2}_{n-2}\}$ be the edge set of the subgraph G_1. Refer Case-2 for the vertex set and the edges set of the caterpillar subgraphs H^k where $3 \le k \le m-2$. The edges of the subgraph G_1 can be given $\frac{m-5}{2}$ distinct colors in such a way that each alternative vertical edges are assigned with a new edge color while the other edges are colored with either c_1 or $c_{2m+2n-11}$. Since n is odd so the uncolored edges of the caterpillar subgraph H^3 can be given $\lfloor \frac{n-5}{2} \rfloor$ more new colors such that each alternative horizontal edges are assigned with a new edge color while the other edges of H^3 are assigned with either color c_1 or $c_{2m+2n-11}$. Similarly, the uncolored edges of each subgraph H^k, where $3 \le k \le m-2$, can be assigned with $\lfloor \frac{n-5}{2} \rfloor$ new colors. Therefore, $(m-4) \times \lfloor \frac{n-5}{2} \rfloor + \frac{m-5}{2}$ new colors are required to color the uncolored edges of the subgraph $G_{m-2,n-2}$ of the graph $G^*_{m,n}$. Hence $\psi'_{vi2}(G^*_{m,n}) = 2(n-2) + 2(m-4) + \lfloor \frac{n-5}{2} \rfloor (m-4) + \frac{m-5}{2} + 1 = 2n + (m-4) \lfloor \frac{n}{2} \rfloor + \frac{m-11}{2}$.
Case-4: Assume when n is odd and m is even. The pendant edges and the colorless edges of the subgraph H of the graph $G^*_{m,n}$ can be given $2m+2n-11$ distinct colors, refer Case-1 and Case-2. As discussed in Case-3, the edges of each caterpillar subgraph H^k, where $3 \le k \le m-2$, are assigned with $\lfloor \frac{n-5}{2} \rfloor$ new colors whereas the edges of the path subgraph G_1 of the graph $G^*_{m,n}$ can be given $\frac{m-4}{2}$ different colors. Hence $\psi'_{vi2}(G^*_{m,n}) = 2(n-2) + 2(m-4) + \lfloor \frac{n-5}{2} \rfloor (m-4) + \frac{m-4}{2} + 1 = 2n + (m-4) \lfloor \frac{n}{2} \rfloor + \frac{m-10}{2}$. □

Theorem 2.3. *Let $G_{m,n}$ be the Cartesian product $P_m \square P_n$ such that $m, n \ge 3$ and $m \le n$. Then*

$$\psi'_{vi2}(G_{m,n}) = \psi'_{vin2}(G_{m,n}) = \begin{cases} 6, & \text{when } m = n = 3 \\ 2n + (m-4)\lfloor \frac{n}{2} \rfloor + 3, & \text{when } n \text{ is even} \\ 2n + (m-4)\lfloor \frac{n}{2} \rfloor + \frac{m+1}{2}, & \text{when } m \,\&\, n \text{ are odd} \\ 2n + (m-4)\lfloor \frac{n}{2} \rfloor + \frac{m+2}{2}, & m \text{ is even } \&\, n \text{ is odd} \end{cases}$$

Proof. Let $G_{m,n} \equiv P_m \square P_n$ where $m, n \ge 3$. Without the loss of generality we assume that $m \le n$. Let $V(G_{m,n}) = \{v^j_1, v^j_2, \ldots, v^j_n,\}$, where $1 \le j \le m$, be the vertex set of graph $G_{m,n}$ (refer figure 3). In order to color all the edges which are incident to vertex with degree four we require $2(n-2) + (m-4) \times \lfloor \frac{n}{2} \rfloor + 1$ colors as proved in lemma 2.2. It is to be noted that this gives the maximum number of colors required to color the $(2mn - 3m - 3n + 4)$ edges of the graph $G_{m,n}$. The coloring of the remaining edges are discussed in the cases mentioned below.

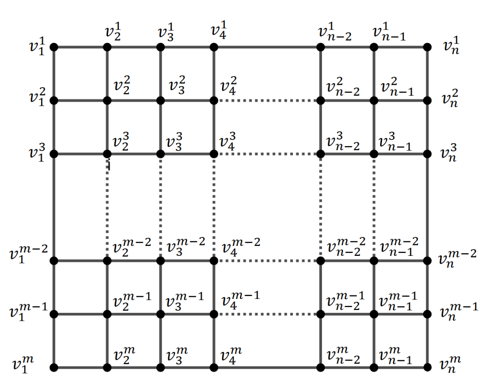

Figure 3. Vertex labeling of graph $G_{m,n}$

Case-1: Assume when $n = m = 3$. In this case there is exactly one vertex with degree four and the edges incident to this vertex can be colored by at most 2 colors. The remaining eight edges which lie on the outer edge of the grid graph $G_{3,3}$ can be given a maximum of 4 more colors. Thus, $\psi'_{vi2}(G_{3,3}) = 2 + 4 = 6$.

Case-2: Assume when n is even and m is either odd or even. As proved in lemma 2.2, $\psi'_{vi2}(G^*_{m,n}) = 2n + (m-4)\left\lfloor \frac{n}{2} \right\rfloor - 3$. In order to assign maximum color to the outer edges of the graph $G_{m,n}$, the edges $v^2_{n-1}v^2_n$, $v^{m-1}_{n-1}v^{m-1}_n$ and $v^{m-1}_1 v^{m-1}_2$ are colored by $c_{2m+2n-11}$.

Consider the path subgraphs $P_1 := v^1_2 v^1_1 v^2_1 v^3_1 \ldots v^m_1 v^m_2$, $P_2 := v^1_{n-1} v^1_n v^2_n v^3_n \ldots v^m_n v^m_{n-1}$, $P_3 := v^1_2 v^1_3 \ldots v^1_{n-2} v^1_{n-1}$ and $P_4 := v^m_2 v^m_3 \ldots v^m_{n-2} v^m_{n-1}$ of the graph $G_{m,n}$. The edges incident to the vertex with degree 2 in the path subgraph P_1 can be given 4 distinct colors, say $\{k_1, k_2, k_3, k_4\}$. Whereas the other uncolored outer edges of the path subgraph P_1 can be given color $c_{2m+2n-11}$, refer lemma 2.2. Suppose one of the edge, say edge $v^2_1 v^3_1$, in the subgraph P_1 is colored with color k_5. Then vertex induced 2-edge coloring condition fails at vertices v^2_1 and v^3_1 as three edges incident at them receives three different colors. Hence at most 4 new colors can be used to color the outer edges in the subgraph P_1. By assigning color k_1 and color k_4 respectively to all the colorless edges of the subgraph P_3 and subgraph P_4, the edges of subgraph P_2 can be assigned with at most two new different colors. Therefore in this case the outer edges of graph $G_{m,n}$ can be given at most 6 new colors. Thus, $\psi'_{vi2}(G_{m,n}) = 2n + (m-4) \times \left\lfloor \frac{n}{2} \right\rfloor - 3 + 6 = 2n + (m-4)\left\lfloor \frac{n}{2} \right\rfloor + 3$.

Case-3: Assume when both m and n are odd. As discussed in Case-3 of lemma 2.2, vertex induced 2-edge coloring number of graph $G^*_{m,n}$ in this case is $2n+(m-4)\left\lfloor \frac{n}{2} \right\rfloor + \frac{m-11}{2}$. The outer edges of $G_{m,n}$ can be assigned with at most 6 new colors, refer Case-2. Therefore, $\psi'_{vi2}(G) = 2n + (m-4) \times \left\lfloor \frac{n}{2} \right\rfloor + \frac{(m-11)}{2} + 6 = 2n + (m-4)\left\lfloor \frac{n}{2} \right\rfloor + \frac{(m+1)}{2}$.

Case-4: Assume when m is even and n is odd. As discussed in Case-4 of lemma 2.2, vertex induced 2-edge coloring number of graph $G^*_{m,n}$ in this case is $2n + (m-4)\left\lfloor \frac{n}{2} \right\rfloor + \frac{m-10}{2}$. Thus outer edges of $G_{m,n}$ can be assigned with at most 6 new colors, refer Case-2. Hence $\psi'_{vi2}(G) = 2n + (m-4) \times \left\lfloor \frac{n}{2} \right\rfloor + \frac{m-10}{2} + 6 = 2n + (m-4)\left\lfloor \frac{n}{2} \right\rfloor + \frac{m+2}{2}$. □

Next, we prove the result for the Torus grid graph $T_{m,n}$ which is also the supergraph of the rectangular grid graph $G_{m,n}$.

Theorem 2.4. *The torus grid graph $T_{m,n} \equiv C_m \square C_n$ is obtained by the Cartesian product of the cycle graphs, where $m, n \geq 3$ and $m \leq n$. Then $\psi'_{vi2}(T_{m,n}) = \psi'_{vin2}(T_{m,n}) = m + n$.*

Proof. Let $T_{m,n} \equiv C_m \square C_n$ be the torus grid graph such that $m, n \geq 3$ and $m \leq n$. The torus grid graph $T_{m,n}$ is a $4-$ regular graph of order mn and size $2mn$. Consider the graph $T_{m,n}$ that consists of $m-$copies of cycle C_n as well as $n-$ copies of cycle C_m which can be assigned with m and n distinct colors respectively. Thus the minimum number of colors required to color the edges of the torus grid graph $T_{m,n}$ is $m + n$, i.e.,

$$\psi'_{vi2}(T_{m,n}) \geq m + n \tag{2.1}$$

Let $V(T_{m,n}) = \{v^i_1, v^i_2, \ldots, v^i_n\}$ be the vertex set of graph $T_{m,n}$, where $1 \leq i \leq m$. Consider the subgraphs H_k, where $1 \leq k \leq m$, with the vertex set $V(H_k) = \{v^k_1, v^k_2, \ldots, v^k_n\}$ and the edge set $E(H_k) = \{v^k_i v^k_{i+1}, v^k_j v^{k+1}_j, v^k_1 v^k_n\}$ for $1 \leq i \leq n-1; 1 \leq j \leq n$ of the graph $T_{m,n}$. Since there are four edges incident at each vertex of subgraph H_1 which can be assigned with at most two different colors. So the edges of the subgraph H_1 can be given at most $n+1$ distinct colors (where the edges $v^1_j v^2_j$, $1 \leq j \leq n$ are assigned with n distinct colors and the other uncolored edges of subgraph H_1 are assigned with $(n+1)^{th}$ color). Now consider the edges of the subgraph $H_1 \cup H_2$. These edges can be given at most one more color. Suppose one of the edge, say edge $v^2_2 v^3_2$, is given $(n+3)^{th}$ color then the vertex induced 2-edge coloring condition fails at the vertex v^2_2. Thus at most $n+2$ colors can be assigned to the edges of the subgraph $H_1 \cup H_2$. Continuing in this manner, we can see that the edges of the graph $T_{m,n} \equiv \bigcup_{k=1}^{n} H_k$ can be colored with at most $n + m$ colors, that is,

$$\psi'_{vi2}(T_{m,n}) \leq m + n \tag{2.2}$$

Thus from equation (2.1) and (2.2) the result follows. This implies, $\psi'_{vi2}(T_{m,n}) = m + n$. □

Corollary 2.5. *Let G be the Cartesian product of the following graphs:*

(i) $C_m \square P_n$ where $m \geq 3$ and $n \geq 2$

(ii) $K_m \square P_n$ where $m, n \geq 2$

(iii) $K_m \square K_n$ *where* $m, n \geq 3$

(iv) $K_m \square C_n$ *where* $m, n \geq 3$

then $\psi'_{vi2}(G) = \psi'_{vin2}(G) = m + n.$

Theorem 2.6. *If* G *is the Cartesian product of the path* P_m *with the star graph* $K_{1,n}$*, then* $\psi'_{vi2}(G) = \psi'_{vin2}(G) = m + n + 1.$

Proof. Let graph $G \equiv P_m \square K_{1,n}$ where $m, n \geq 2$ and $n \geq m$. The graph G contains $m-$copies of the star graph $K_{1,n}$ such that each vertex in the first copy of the star graph has an edge joining the corresponding vertex in the second copy and so on. Since there are $n+1$ edges joining any two consecutive copies of the star graph $K_{1,n}$ and these distinct edges can be given $n+1$ distinct colors. Since the vertex induced 2-edge coloring number of the star graph $K_{1,n}$ is 2 (see [8]) thus the all remaining uncolored edges in each copy of the star graph can be given at most one new color. As there are $m-$copies of the star graph so m more distinct colors are required to color the uncolored edges of the graph G. Thus $\psi'_{vi2}(G) = m + n + 1$. □

Theorem 2.7. *For a hypercube* Q_n*,* $\psi'_{vi2}(Q_n) = \psi'_{vin2}(Q_n) = 2^{n-1} + 2.$

Proof. The graph $Q_n \equiv Q_{n-1} \square K_2$. Since the number of edges joining two copies of graph Q_{n-1} is equal to the order of Q_{n-1}. So these 2^{n-1} edges can be assigned with 2^{n-1} distinct colors. The remaining uncolored edges in the two copies of graph Q_{n-1} can be colored using at most 2 colors. Thus the maximum vertex induced 2-edge coloring number is obtained by adding two more colors to the previous color set. This implies, $\psi'_{vi2}(Q_n) = 2^{n-1} + 2$. □

Theorem 2.8. *Let* G *be the strong products* $P_n \boxtimes P_m$ *with* $m, n \geq 2$ *and* $m \leq n$*. Then*
(i) $\psi'_{vi2}(G) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{n}{2} \rfloor + 1$

$$(ii)\quad \psi'_{vin2}(G) = \begin{cases} n + m - 1 + \lfloor \frac{m-2}{2} \rfloor (n-2) + \lfloor \frac{n-2}{2} \rfloor, & \text{when } m \text{ is odd} \\ n + m - 1 + \lfloor \frac{m-2}{2} \rfloor (n-2), & \text{when } m \text{ is even} \end{cases}$$

Proof. Let $G \equiv P_n \boxtimes P_m$ such that $m, n \geq 3$ and $m \leq n$. Let $\{v_1^j, v_2^j, \ldots, v_n^j\}$ be the vertex set of graph G, where $1 \leq j \leq m$.

(i) Let $G_1^1 \equiv \langle v_1^1 \cup v_3^1 \rangle$ be the induced subgraph generated by v_1^1 and v_3^1. In any vertex induced 2-edge coloring, the maximum number of colors required to color the edges of the subgraph G_1^1 is 3. Consider the induced subgraph $G_n^1 \equiv \left\langle \bigcup_{i=1}^{\lfloor \frac{2n-1}{2} \rfloor} v_i^1 \right\rangle$ generated by vertices $v_1^1, v_3^1, \ldots$ and v_{n-1}^1. In order to assign vertex induced 2-edge coloring to the edges of the subgraph G_n^1 at most $\lfloor \frac{n}{2} \rfloor + 1$ colors are required. Similarly, each time $\lfloor \frac{n}{2} \rfloor + 1$ colors are required to color the edges of the induced subgraph $G_n^j \equiv \left\langle \bigcup_{i=1}^{\lfloor \frac{2n-1}{2} \rfloor} v_i^j \right\rangle$, where $1 \leq j \leq m$. Thus $\lfloor \frac{m}{2} \rfloor \times \lfloor \frac{n}{2} \rfloor + 1$ colors are required to assign vertex induced 2-edge coloring to the edges of graph $G \equiv \bigcup_{j=1}^{m} G_n^j$. Therefore, $\psi'_{vi2}(G) = \lfloor \frac{m}{2} \rfloor \lfloor \frac{n}{2} \rfloor + 1.$

(ii) The number of outer edges of the graph $G \equiv P_m \boxtimes P_n$ is equal to $2(m-1) + 2(n-1)$, which always a positive even integer. So these edges are alternatively colored in such a way that the first edge is colored by the color c_1 and the second outer edge of the graph G is given the color c_2 and the third edge is colored by the color c_1 and so on. Thus the maximum number of colors required to color the outer edges of the graph G is $(n + m - 1)$. The edges incident to the vertices with degree three and degree five can be assigned with the color c_1 else the vertex incident 2-edge coloring condition fails. The vertex incident 2-edge coloring of the remaining edges of graph G is discussed in the cases mentioned below.
Case-1: Assume that m is an odd integer. Then clearly $m - 1$ is an even integer. Let $H_2 \equiv$

$\left\langle \bigcup_{i=1}^{n-1} v_i^2 \right\rangle$, $H_4 \equiv \left\langle \bigcup_{i=1}^{n-1} v_i^4 \right\rangle, \ldots, H_{m-1} \equiv \left\langle \bigcup_{i=1}^{n-1} v_i^{m-1} \right\rangle$ be the induced subgraphs of the graph G. It can be noted that $(n-2)$ new colors are required to color the uncolored edges of the subgraph H_2. Similarly, at each time $(n-2)$ new colors are required to color the uncolored edges of the subgraph H_j, where $2 \leq j < n-1$. Thus the edges of subgraph $H_2 \cup H_4 \cup \ldots \cup H_{m-3}$ requires at most $\lfloor \frac{m-2}{2} \rfloor \times (n-2)$ colors. Since m is an odd integer thus the edges of subgraph H_{m-1} can be assigned with $\lfloor \frac{n-2}{2} \rfloor$ distinct colors in such a way that each alternate horizontal edge of the subgraph H_{m-1} gets a new color. Therefore, in this case $\lfloor \frac{m-2}{2} \rfloor (n-2) + \lfloor \frac{n-2}{2} \rfloor$ more colors are required to get the maximum vertex incident 2-edge coloring number of the edges incident to the vertex with degree eight. Hence $\psi'_{vin2}(G) = n + m - 1 + \lfloor \frac{m-2}{2} \rfloor (n-2) + \lfloor \frac{n-2}{2} \rfloor$.

Case-2: Assume that m is an even integer. Let $H_2 \equiv \left\langle \bigcup_{i=1}^{n-1} v_i^2 \right\rangle$, $H_4 \equiv \left\langle \bigcup_{i=1}^{n-1} v_i^4 \right\rangle$, $\ldots, H_{m-2} \equiv \left\langle \bigcup_{i=1}^{n-1} v_i^{m-2} \right\rangle$ be the induced subgraphs of the graph G. As mentioned in case-1, each time $(n-2)$ new colors are required to color the edges of subgraph H_j where $2 \leq j \leq n-2$. Thus in this case, $\lfloor \frac{m-2}{2} \rfloor \times (n-2)$ colors are used to assign the vertex incident 2-edge coloring to the edges which are incident to the vertex with degree eight. Therefore $\psi'_{vin2}(G) = n + m - 1 + \lfloor \frac{m-2}{2} \rfloor (n-2)$. □

Corollary 2.9. *Let G be the strong product of $P_n \boxtimes K_2$, where $n \geq 2$. Then*
(i) $\psi'_{vi2}(G) = \lfloor \frac{n}{2} \rfloor + 1$
(ii) $\psi'_{vin2}(G) = n + 1$.

3 Results on Rooted Product of Some Graphs

In this section, we find the vertex induced 2-edge coloring number and vertex incident 2-edge coloring number of the rooted product of some graphs.

Theorem 3.1. *Let G_n be a simple connected graph of order $n \geq 2$. Then for the rooted product graph $G_n \odot C_m$, where any vertex of the cycle graph C_m is a root, $\psi'_{vi2}(G_n \odot C_m) = \psi'_{vin2}(G_n \odot C_m) = mn - n + 1$ for all $n \geq 2$ and $m \geq 4$.*

Proof. The vertex induced 2-edge coloring number of the cycle C_m is m (see [8]). Since the i^{th} copy of the subgraph C_m of the graph $G_n \odot C_m$ is identified as the i^{th} vertex of the subgraph G_n thus all the edges of the subgraph C_m can be given $(m-1)n$ distinct colors. The remaining uncolored edges of the graph $G_n \odot C_m$ can be assigned with at most one more color. Suppose, if we start by giving vertex induced 2-edge coloring to the edges of the subgraph G_n of the graph $G_n \odot C_m$, then there exists at least one vertex $v \in V(G_n)$ such that the two colorless edges of the cycle subgraph C_m incident to the vertex $v \in V(G_n)$ receives one of the color given to the edges of the subgraph G_n incident to vertex v. So the vertex induced 2-edge coloring number obtained in this manner will be always less than or equal to the above-mentioned coloring pattern since, $|E(G_n)| < n|E(C_m)| = nm$. Hence the only way to obtain the maximum vertex induced 2-edge coloring number for the graph $G_n \odot C_m$ is by giving distinct colors to the edges of the cycle subgraph C_m except the edges incident to each vertex rooted at G_n and then by assigning one new color to all the remaining edges of the graph $G_n \odot C_m$. Therefore, $\psi'_{vi2}(G_n \odot C_m) = (m-1)n + 1 = mn - n + 1$. □

Corollary 3.2. *Let G_n be a simple connected graph of order n and C_3 be the cycle graph of order 3. If G is the rooted product of the graph G_n with C_3, where any vertex of the graph C_3 is a root, then $\psi'_{vi2}(G) = n + 1$ and $\psi'_{vin2}(G) = 2n + 1$.*

Theorem 3.3. *Let G_n be a simple connected graph of order $n \geq 2$ and let P_m be the path of order $m \geq 2$ with vertices denoted by $v_1, v_2, \ldots, v_m$. For the rooted product graph $G \equiv G_n \odot P_m$ where any vertex of the Path P_m is a root,*

$$\psi'_{vi2}(G) = \psi'_{vin2}(G) = \begin{cases} (m-1)n + 1, & \text{when the pendant vertex of } P_m \text{ is a root} \\ (m-2)n + 1, & \text{otherwise.} \end{cases}$$

Proof. Let $G \equiv G_n \odot P_m$ where $n, m \geq 2$. Path P_m is a graph on m vertices. The vertex induced 2-edge coloring number of Path graph P_m is $m-1$ (see [8]). As mentioned in proposition 3.1 as $|E(G_n)| < n|E(P_m)|$, we start the coloring procedure of the graph G by giving vertex induced 2-edge coloring numbers to the edges of the path subgraph P_m. Thus the number of colors required to color the edges of graph G are discussed in the cases mentioned below.
Case-1: Assume that the pendant vertex of P_m is a root ie. either v_1 or v_m. Since there are n copies of P_m thus the edges in these n copies of the path subgraph P_m of the graph G can be assigned $(m-1)n$ distinct colors. The remaining colorless edges of the graph G can be given at most one new color else the vertex induced 2-edge coloring condition fails. Therefore, $\psi'_{vi2}(G) = n(m-1) + 1$.
Case-2: Assume when the vertex with degree 2 of P_m is a root. Thus, in this case the edges in the n copies of the subgraph P_m of the graph G can be given $(m-2)n$ distinct colors. Suppose if assign $(m-1)n$ distinct colors to all the edges of the path subgraph P_m then the vertex induced 2-edge coloring condition fails if we assign any color to the subgraph G_n in G. Thus to maximize the number of colors we need to assign $(m-2)n$ colors to the edges of the path subgraph P_m and the remaining uncolored edges of graph G can be assigned with most one different color. Therefore, the vertex induced 2-edge coloring number of the graph G is $(m-2)n + 1$. □

Next, we consider the rooted product of the Ladder graph with the star graph and determine its vertex induced 2-edge coloring number and vertex incident 2-coloring number. We consider the vertex with a maximum degree in the star graph as the apex vertex.

Theorem 3.4. *Let graph G be the rooted product of the graph $(P_n \square P_2) \odot K_{1,m}$, where $n, m \geq 2$. Then*

$$\psi'_{vi2}(G) = \psi'_{vin2}(G) = \begin{cases} 4n+1, & \text{when pendant vertex of } K_{1,m} \text{ is a root} \\ 2n+1, & \text{when apex vertex of } K_{1,m} \text{ is a root} \end{cases}$$

Proof. The Cartesian product of $L_n \equiv P_n \square P_2$ is also known as the ladder graph. Let $G \equiv L_n \odot K_{1,m}$ where $n, m \geq 2$. The vertex induced 2-edge coloring number of the star graph $K_{1,m}$ is 2 (see [8]). Since $|E(L_n)| < n|E(K_{1,m})|$ so we begin the coloring procedure by assigning the colors to the edges of the subgraph $K_{1,m}$ of graph G (refer proposition 3.1).
Case-1: Assume the case when any one of the pendant vertex of the subgraph $K_{1,n}$ is a root in G. Since there are $2n$ copies of the star graph so the edges of the $2n$-copies of the star graph $K_{1,m}$ can be assigned with $2 \times 2n = 4n$ colors. The remaining uncolored edges of the graph $P_n \square P_2$ can be given at most one new color. Thus $\psi'_{vi2}(G) = 4n + 1$.
Case-2: Assume when the apex vertex of $K_{1,m}$ is a root. In this case, $2n$-copies of the star graph can be given $2n$ distinct colors and all the remaining uncolored edges of the graph G are given a new color. Thus the maximum vertex induced 2-edge coloring number of G is $2n + 1$. □

Theorem 3.5. *Let G_n be a simple connected graph of order $n \geq 2$ and K_m be the complete graph on $m \geq 4$ vertices. If $G \equiv G_n \odot K_m$, where any vertex of the subgraph K_m is a root, then*

$$\psi'_{vin2}(G) = \begin{cases} \left(\left\lfloor \frac{m}{2} \right\rfloor + 1\right) n + 1, & \text{when } m \text{ is odd} \\ \left\lfloor \frac{m}{2} \right\rfloor n + 1, & \text{when } m \text{ is even} \end{cases}$$

Proof. Let $G \equiv G_n \odot K_m$, where any vertex of the subgraph K_m is a root and $n \geq 2; m \geq 4$. The vertex incident 2-edge coloring number of the complete graph K_m is $\left\lfloor \frac{m}{2} \right\rfloor + 1$ (see [8]). We start the coloring procedure by assigning colors to the edges of the subgraph K_m (refer proposition 3.1). The number of colors required to color the edges of the graph G is discussed in the two cases mentioned below.
Case-1: Assume the case when m is odd. Since there are n copies of the complete subgraph K_m so these edges of K_m can be assigned with $\left(\left\lfloor \frac{m}{2} \right\rfloor + 1\right) n$ distinct colors. The remaining uncolored edges of the graph $G_n \odot K_m$ can be given at most one new color. Thus $\psi'_{vin2}(G) = \left(\left\lfloor \frac{m}{2} \right\rfloor + 1\right) n + 1$.
Case-2: Assume the case when m is even. In this case since m is an even integer so the edges of the $n-$copies of complete subgraph can be only given $\left\lfloor \frac{m}{2} \right\rfloor n$ distinct colors. The remaining uncolored edges of the graph G can be given at most one new color. Thus the maximum vertex incident 2-edge coloring number of G is $\left\lfloor \frac{m}{2} \right\rfloor n + 1$. □

Theorem 3.6. *Consider G_n as a simple connected graph of order n and K_m be the complete graph on m vertices. Let G be the rooted product $G_n \bigodot K_m$, where K_m is a root and for $n \geq 2; m \geq 4$. Then $\psi'_{vi2}(G) = n + 1$.*

Proof. The vertex induced 2-edge coloring number of complete graph K_m with m vertices is 2 (see [8]). The graph $G \equiv G_n \bigodot K_m$ has n-copies of complete graph K_m. If all the edges in one copy of complete subgraph K_m of graph G is assigned with one color then n-copies of complete subgraph K_m of graph G can be assigned with n distinct colors. The remaining uncolored edges of subgraph G_n can be given at most one more new color else the vertex induced 2-edge coloring condition fails. Thus the result holds. □

4 Future Scope

In section 2, we gave the exact value of vertex induced 2-edge coloring number of the strong product of path graphs only. It may be interesting to get this value for other strong product graphs as well.

5 Conclusion

In this paper, we have found the exact values of vertex induced 2-edge coloring number and vertex incident 2-edge coloring number of some graph products such as Cartesian product, strong product, and rooted product graphs. Section 3 gives the vertex induced (incident) 2-edge coloring number of some rooted product graphs.

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