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A SECOND-ORDER FRACTIONAL STEP METHOD FOR TWO-DIMENSIONAL DELAY PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS WITH A SMALL PARAMETER

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Abstract We present a higher-order robust numerical method for 2D singularly perturbed delay parabolic differential equation of convection-diffusion type. For the family of such type of differential equations, till now a first-order (reduced by a logarithmic term) numerical scheme is proposed. To develop a more efficient numerical scheme, first, we time-discretize the problem in such a way so that it yields a pair of 1D problems. Then we propose a numerical scheme which consists of upwind scheme along with a post processing technique. We prove analytically that the proposed algorithm provides almost second-order convergent solution. To corroborate the theoretical result, we provide numerical examples.

1 Introduction

Parabolic partial differential equations (PDEs) arise in many areas of applied mathematics. These equations model a large class of physical phenomena involving diffusion processes, where a gradient of temperature, pressure, or concentration cause a transport of matter or energy. In addition reaction-diffusion is simplest example of parabolic PDEs use to model chemical mass transport in porous media, thermal oxidation of silicon, and the motion of a plate through a viscous fluid (refer [1]). If small parameter ε ($0 < \varepsilon \ll 1$) multiplying the highest-order derivative of these PDEs, generally they are known as singularly perturbed PDEs (SPPDEs). Solutions of SPPDEs exhibit layers as ε tends to zero. These in general appears in modeling of semiconductor devices, financial modeling and fluid dynamics. More details of these applications are given in [2].

Here, we deal with a class of 2D SPPDE of convection-diffusion type with delay in time variable. We define the domain $\mathfrak{G} = \mathfrak{D} \times \Omega_t$, $\mathfrak{D} = I_x \times I_y = (0,1)^2$, $\Omega_t = [0,T]$.

$$\begin{cases} z_t + \mathfrak{L}_\varepsilon z(x,y,t) = -c(x,y)z(x,y,t-\tau) + g(x,y,t), & (x,y,t) \in \mathfrak{G}, \\ z(x,y,t) = \phi_b(x,y,t), & (x,y,t) \in \Gamma_b = \overline{\mathfrak{D}} \times [-\tau, 0], \\ z(x,y,t) = 0, & (x,y,t) \in \partial\mathfrak{D} \times \Omega_t, \end{cases} \tag{1.1}$$

where

$$\mathfrak{L}_\varepsilon z = -\varepsilon \Delta z + \mathbf{a}(x,y).\nabla z + b(x,y)z,$$

$0 < \varepsilon \ll 1$ and $\tau > 0$. We assume the terminal time satisfies the condition $T = k\tau$, for some positive integer k. The coefficients $\mathbf{a} = (a_1, a_2)$, b, and c are sufficiently smooth and bounded functions that satisfy

$$a_i(x,y) \geq 2\alpha_i > 0,\ i = 1,\ 2,\ b(x,y) \geq 0, \quad \text{on } \overline{\mathfrak{D}}.$$

The solution of (1.1) exhibits regular boundary layers near the boundaries $x = 1$ and $y = 1$ and corner layer in the neighborhood of the corner $(1,1)$. The solution differs very rapidly within the layer regions and for such behaviour, this will require a uniform mesh with mesh size $O(1/\varepsilon)$ for the classical numerical methods, which is computationally not feasible. To overcome this drawback, in the literature, several finite difference methods (refer [3, 4] and references given therein) and finite element methods on piecewise uniform meshes (refer [5]) are used.

In recent days several numerical methods are used to deal with one-dimensional time delay parabolic problems (refer [6, 7, 8, 9]). A numerical scheme is developed for one dimensional convection-diffusion delay parabolic problems in [10] by the upwind scheme on Shishkin mesh and obtained first-order rate of convergence. To increase the rate of convergence, Das and Natesan in [6] solved the same problem by a combination of implicit Euler method for time direction on uniform mesh and hybrid scheme for spatial direction on Shishkin mesh. Mohapatra and Natesan [11] used the Richardson extrapolation technique as a post processing technique to obtain higher-order accurate solution of a delay singularly perturbed two point BVPs while Shishkin et. al. [12] applied this idea to solve the parabolic reaction-diffusion equation. The authors in [13] used extrapolation technique for singularly perturbed problems of on adaptive mesh.

It is apparent that most of the works are done for 1D cases whereas a natural phenomena can be modelled more accurately with two spatial variables. For the model problem (1.1), few works are available in the literature. In [14] and [15], the discretized equations provide a penta-diagonal matrix and the used methods are of first-order (up to a logarithmic term) convergent and first-order convergent, respectively. To reduce the computational cost, in [16], fractional step method is used, which reduces the discretized equations into two tridiagonal matrices. But the scheme used there is of first-order convergence. For the first time in this paper, we are proposing a method to obtain a second-order accurate solution for a 2D SPPDE with delay (1.1). For that purpose first, we convert the model problem into a pair of 1D problems with the help of a fractional step method. Then, we use the upwind scheme and the Richardson extrapolation technique to obtain the optimal order of convergence.

The article is structured as follows. Some standard finite difference schemes which will be used in the later part of the article are constructed in Section 2. In Section 3, the Richardson extrapolation is implemented and the main result is presented. Some numerical simulations accompanied with graphical explanations are presented in Section 4. The concluding remarks are given in Section 5.

Notations. C and the subscripted C denote positive constants independent of the perturbation parameter ε, and the mesh parameters M, N. Standard L_∞ norm is used which is denoted by $\|.\|_\infty$ and for a continuous function f on $\mathfrak{G}$ it is defined as $||f||_\infty = \sup_{(x,y,t)\in\mathfrak{G}} |f(x,y,t)|$.

2 Discretization and finite difference schemes

Here, we focus on the discretization of the time domain and implementation of the fractional step method on the model problem (1.1) by time semidiscretization.

2.1 Discretization of time domain

Let the positive integer M be the number of mesh interval in the time-direction and Δt be the size of uniform time step. The discretized time domain is given as $\Omega_t^M = \{t_n = n\Delta t,\ \ n = 0,\ldots,M,\ \ t_M = T,\ \ \Delta t = T/M\}$. The temporal mesh size ΔT satisfies the constant $p\Delta T = \tau$, where p is positive integer and $t_n = n\Delta t$, such that $n \geq -p$.

2.2 Time semidiscretization

Now, we do the time semidiscretization which will be helpful to analyse the fully discrete scheme. The fractional step scheme is used for the time semidiscretization process of the IBVP (1.1), which yields

$$z^{-m} = \phi_b(x,y,-t_m), \quad m = 0,\ldots,p,$$

$$\begin{cases} (I + \Delta t\mathfrak{L}_{x,\varepsilon})z^{n+1/2} = z^n - \Delta t c_1(x,y)z^{n+1-p} + \Delta t g_1(x,y,t_{n+1}), \quad y \in (0,1), \\ z^{n+1/2}(0,y) = z^{n+1/2}(1,y) = 0, \end{cases} \tag{2.1}$$

$$\begin{cases} (I+\Delta t\mathfrak{L}_{y,\varepsilon})z^{n+1} = z^{n+1/2} - \Delta t c_2(x,y)z^{n+1-p} + \Delta t g_2(x,y,t_{n+1}), \quad x\in(0,1), \\ z^{n+1}(x,0) = z^{n+1}(x,1) = 0, \end{cases} \tag{2.2}$$

where

$$\mathfrak{L}_{x,\varepsilon} \equiv -\varepsilon\frac{\partial^2}{\partial x^2} + a_1(x,y)\frac{\partial}{\partial x} + b_1(x,y),$$
$$\mathfrak{L}_{y,\varepsilon} \equiv -\varepsilon\frac{\partial^2}{\partial y^2} + a_2(x,y)\frac{\partial}{\partial y} + b_2(x,y),$$

$b(x,y) = b_1(x,y) + b_2(x,y)$ and $g(x,y,t) = g_1(x,y,t) + g_2(x,y,t)$, with

$$g_1(x,0,t) = g_1(x,1,t) = g_2(0,y,t) = g_2(1,y,t) = 0. \tag{2.3}$$

We denote $z^n(x,y)$ as the solution $z(x,y,t)$ of (1.1) at n-th time level, by using the above scheme.

The stability of the scheme (2.1)-(2.2) is ensured as the operators $(I+\Delta t\mathfrak{L}_{x,\varepsilon})$ and $(I+\Delta t\mathfrak{L}_{y,\varepsilon})$ satisfy the following comparison principle.

Lemma 2.1. *(***Comparison Principle***) Consider a mesh function $\Psi(x,y)$ such that, $\Psi(x,y)\geq 0$, $\forall\,(x,y)\in\partial\mathfrak{D}$. Then $(I+\Delta t\mathfrak{L}_{i,\varepsilon})\Psi(x,y)\geq 0$, $i=x,\ y$ on $\mathfrak{D}$ implies that $\Psi(x,y)\geq 0\,\forall\,(x,y)\in\overline{\mathfrak{D}}$.*

Now, we introduce the local error to study the consistency of the method. We define the local error as $e^n = |z(t_n,x,y) - \widetilde{z}^n(x,y)|$, where $\widetilde{z}^n$ is the approximation solution of $z(t_n,x,y)$ at one time step of (2.1) and (2.2). So, $\widetilde{z}^n$ is solution of the problem given as follows:

$$z^n = z(t_n), \quad (x,y)\in\overline{\mathfrak{D}},$$

$$\begin{cases} (I+\Delta t\mathfrak{L}_{x,\varepsilon})\widetilde{z}^{n+1/2} = z^n - \Delta t c_1(x,y)\widetilde{z}^{n+1-p} + \Delta t g_1(x,y,t_{n+1}), \quad y\in(0,1), \\ \widetilde{z}^{n+1/2}(0,y) = \widetilde{z}^{n+1/2}(1,y) = 0, \end{cases} \tag{2.4}$$

$$\begin{cases} (I+\Delta t\mathfrak{L}_{y,\varepsilon})\widetilde{z}^{n+1} = \widetilde{z}^{n+1/2} - \Delta t c_2(x,y)\widetilde{z}^{n+1-p} + \Delta t g_2(x,y,t_{n+1}), \quad x\in(0,1), \\ \widetilde{z}^{n+1}(x,0) = \widetilde{z}^{n+1}(x,1) = 0. \end{cases} \tag{2.5}$$

Lemma 2.2. *The solution $\widetilde{z}^{n+1/2}$ of (2.4) satisfies the following bound*

$$\left|\frac{\partial^i\widetilde{z}^{n+1/2}}{\partial x^i}\right| \leq C\left(1+\varepsilon^{-i}\exp\left(-\alpha_x(1-x)/\varepsilon\right)\right), \quad 0\leq i\leq 4.$$

Proof. Refer [16] for the proof. □

To analyse the convergence of the semidiscrete scheme, we decompose $\widetilde{z}^{n+1/2} = \widetilde{r}^{n+1/2} + \widetilde{s}^{n+1/2}$, where $\widetilde{r}^{n+1/2}$ is required for the outer region and $\widetilde{s}^{n+1/2}$ is required for the inner region.

Now, with the help of the bounds of the derivatives for $\widetilde{z}^{n+1/2}$, we get the bounds as follows:

$$\left\|\frac{\partial^i\widetilde{r}^{n+1/2}}{\partial x^i}\right\|_\infty \leq C\left(1+\varepsilon^{3-i}\right), \tag{2.6}$$

and

$$\left|\frac{\partial^i\widetilde{w}^{n+1/2}}{\partial x^i}\right| \leq C\varepsilon^{-i}\exp(-\alpha_x(1-x)/\varepsilon), \quad 0\leq i\leq 4, \tag{2.7}$$

by following the same idea used in [6].

2.3 Discretization of space domain

We use the tensor product of 1D Shihskin meshes to define the spatial rectangular mesh $\overline{\mathfrak{D}}^N$, *i.e.*, $\overline{\mathfrak{D}}^N = \overline{I}_x^N \times \overline{I}_y^N$. The transition points, which are used to separate the coarse and fine meshes are defined as

$$\rho_l = \min\left\{\frac{1}{2},\, \rho_{l,0}\varepsilon \ln N\right\},\; l = x,\, y,$$

where $\rho_{l,0} \geq 2/\alpha_l$. Throughout the analysis done in the paper, we assume that $\rho_l = \rho_{l,0}\varepsilon \ln N$, because otherwise it will be uniform mesh which is not the case of interest.

Shishkin mesh

Here, we give the Shishkin mesh based on the idea from [3]. We divide the interval $[0, 1]$ into two sub-intervals $[0, 1-\rho_x]$ and $(1-\rho_x, 1]$ where each sub-interval will contain $N/2$ mesh-intervals with uniform step sizes, which we denote by

$$\overline{I}_x^N = \left\{0 = x_0,\, x_1,\, \cdots,\, x_{N/2} = 1-\rho_x,\, \cdots,\, x_N = 1\right\}.$$

In the same way, we construct $\overline{I}_y^N = \left\{0 = y_0, y_1, \cdots, y_{N/2} = 1-\rho_y, \cdots, y_N = 1\right\}$.

Further, we define $h_{x,i}$ and $h_{y,j}$ as the mesh step sizes in the x and y directions, respectively, *i.e.*,

$$h_{x,i} = x_i - x_{i-1},\; i = 1, \cdots, N, \widetilde{h}_{x,i} = h_{x,i} + h_{x,i+1}, i = 1, \cdots, N-1,$$

$$h_{y,j} = y_j - y_{j-1},\; j = 1, \cdots, N, \widetilde{h}_{y,j} = h_{y,j} + h_{y,j+1}, j = 1, \cdots, N-1.$$

Let $h_l = 2\rho_l/N$ and $H_l = 2(1-\rho_l)/N$, $l = x,\, y$, be the mesh widths in the inner layer region and outer layer region respectively. Hence, we have

$$h_l = 2\rho_{l,0}\varepsilon N^{-1}\ln N, \quad N^{-1} \leq H_l \leq 2N^{-1}, \quad l = x,\, y.$$

We denote $\mathfrak{G}^{N,\,M}$ as the discretized domain which is defined by $\mathfrak{G}^{N,\,M} = I_x^N \times I_y^N \times \Omega_t^M$.

2.4 Spatial discretization

We define the forward δ_x^+, the backward δ_x^- differences in the x-direction by

$$\delta_x^+ f_{x_i,y}^n = \frac{f_{x_{i+1},y}^n - f_{x_i,y}^n}{h_{x,i+1}}, \quad \delta_x^- f_{x_i,y}^n = \frac{f_{x_i,y}^n - f_{x_{i-1},y}^n}{h_{x,i}},$$

respectively. The central difference approximation δ_x^2 in the x-direction is given by

$$\delta_x^2 f_{x_i,y}^n = \frac{2(\delta_x^+ f_{x_i,y}^n - \delta_x^- f_{x_i,y}^n)}{\widetilde{h}_{x,i}}.$$

In an analogous manner, we define δ_y^+, δ_y^-, and δ_y^2.

We use the upwind scheme on I_x^N to approximate $\mathfrak{L}_{x,\varepsilon}$ and we denote the approximation as $\mathfrak{L}_{x,\varepsilon}^N$. Hence, one can get the discretized equations as follows: For $y \in I_y^N$,

$$\begin{cases} (I + \Delta t\mathfrak{L}_{x,\varepsilon}^N)\widetilde{Z}_{x_i,y}^{n+1/2} & = (I + \Delta t(-\varepsilon\delta_x^2 + a_1(x_i,y)\delta_x^- + b_1(x_i,y)))\widetilde{Z}_{x_i,y}^{n+1/2} \\ & = z(x_i,y,t_n) - c_1\Delta t\widetilde{Z}(x_i,y,t_{n+1-p}) + \Delta t g_1(x_i,y,t_{n+1}), \\ & i = 1, \cdots, N-1, \\ \widetilde{Z}_{0,y}^{n+1/2} = \widetilde{Z}_{1,y}^{n+1/2} = 0. \end{cases} \tag{2.8}$$

Similarly, for $x \in I_x^N$,

$$\begin{cases} (I+\Delta t\mathfrak{L}_{y,\varepsilon}^N)\widetilde{Z}_{x,y_j}^{n+1} & = (I+\Delta t(-\varepsilon\delta_y^2 + a_2(x,y_j)\delta_y^- + b_2(x,y_j)))\widetilde{Z}_{x,y_j}^{n+1} \\ & = \widetilde{Z}_{x,y_j}^{n+1/2} - c_2\Delta t\widetilde{Z}(x,y_j,t_{n+1-p}) + \Delta t g_2(x,y_j,t_{n+1}), \\ & j = 1,\cdots,N-1, \\ \widetilde{Z}_{x,0}^{n+1} = \widetilde{Z}_{x,1}^{n+1} = 0, \end{cases} \tag{2.9}$$

where $Z(x,y,t_m) = z(x,y,t_m)$, for $m = 0,\cdots,p$. After rearranging the terms in (2.8), we get

$$\begin{cases} (I+\Delta t\mathfrak{L}_{x,\varepsilon}^N)\widetilde{Z}_{x_i,y}^{n+1/2} \equiv v_i^-\widetilde{Z}_{x_{i-1},y}^{n+1/2} + v_i^0\widetilde{Z}_{x_i,y}^{n+1/2} & + v_i^+\widetilde{Z}_{x_{i+1},y}^{n+1/2} = G_{x_i,y}, \\ & i = 1,\cdots,N-1, \\ \widetilde{Z}_{0,y}^{n+1/2} = \widetilde{Z}_{1,y}^{n+1/2} = 0, \end{cases} \tag{2.10}$$

where

$$\begin{cases} v_i^- = \Delta t\left(-\dfrac{2\varepsilon}{\widetilde{h}_{x,i}h_{x,i}} - \dfrac{a_1(x_i,y)}{h_{x,i}}\right), \quad v_i^+ = \Delta t\left(-\dfrac{2\varepsilon}{\widetilde{h}_{x,i}h_{x,i+1}}\right), \\ v_i^0 = 1 + \Delta t b_1(x_i,y) - v_i^- - v_i^+, \\ G_{x_i,y} = z(x_i,y,t_n) - c_1\Delta t\widetilde{Z}(x_i,y,t_{n+1-p}) + \Delta t g_1(x_i,y,t_{n+1}). \end{cases} \tag{2.11}$$

One can observe that, for $i = 1,\cdots,N-1$, $v_i^- < 0$, $v_i^+ < 0$ and $v_i^0 > 0$, moreover the discretized matrix is strictly diagonally dominant. These facts ensure that the discretized matrix is M-matrix. Hence, the scheme is stable, uniformly. In a similar way for the difference operator of (2.9) one can get the above result.

2.5 Error analysis

To do the convergence analysis, we write the fully discretized form as follows:

$$\begin{cases} (I+\Delta t\mathfrak{L}_{x,\varepsilon}^N)Z_{i,j}^{n+1/2} = Z_{i,j}^n & -\Delta t c_1(x_i,y_j)Z_{i,j}^{n+1-p} \\ & +\Delta t g_1(x_i,y_j,t_{n+1}), \; 1 \le i \le N-1, \\ Z_{0,j}^{n+1/2} = Z_{N,j}^{n+1/2} = 0, \quad 0 \le j \le N, \end{cases} \tag{2.12}$$

$$\begin{cases} (I+\Delta t\mathfrak{L}_{y,\varepsilon}^N)Z_{i,j}^{n+1} = Z_{i,j}^{n+1/2} & -\Delta t c_2(x_i,y_j)Z_{i,j}^{n+1-p} \\ & +\Delta t g_2(x_i,y_j,t_{n+1}), \; 1 \le j \le N-1, \\ Z_{i,0}^{n+1} = Z_{i,N}^{n+1} = 0, \quad 0 \le i \le N, \end{cases} \tag{2.13}$$

$$Z_{i,j}^{-m} = \varphi_b(x_i,y_j,-t_m),$$

for $i,\, j = 0,\cdots,N,$ and $m = 0,\cdots,p.$

Theorem 2.3. *At time level $t_n = n\Delta t$, let z and $\{Z^n\}$ be the solution of continuous problem (1.1) and solution of fully discrete problem (2.12)-(2.13), respectively, then*

$$\left\|z(x_i,y_j,t_n) - Z_{i,j}^n\right\|_\infty \le C(\Delta t + N^{-1+\beta}\ln N), \textit{for } (x_i,y_j,t_n) \in \mathfrak{G}^{N,M},$$

for a ε, N independent positive constant C, with $0 < \beta < 1$.

Proof. The proof can be found in [16]. □

For the accuracy improvement, we propose the Richardson extrapolation technique, which is discussed in the next section.

3 Post processing technique for the discrete solution

3.1 Extrapolation for $\widetilde{z}$

By following the technique discussed in [17], we get

$$\widetilde{z}_{extp^t}(t_n) = 2\widetilde{z}(t_n) - \widetilde{z}(t_n), \quad t_n \in \Omega_t^M, \tag{3.1}$$

where $\widetilde{z}$ is semidiscrete solution of (2.4)-(2.5) on Ω_t^{2M}, with $2M$ intervals in the t-direction.

Error bound for $\widetilde{z}_{extp^t}$

Theorem 3.1. *Let $z(t_n)$ be the exact solution of (1.1) and $\widetilde{z}_{extp^t}(t_n)$ be the time extrapolated solution of $z(t_n)$, then for $t_n \in \Omega_t^M$,*

$$|(z - \widetilde{z}_{extp^t})(x, y, t_n)| = O(\Delta t^3).$$

Proof. The proof can be done case-wise depending upon the different time interval.

Case 1. For the first time interval, *i.e.*, for $n = 0, \cdots, p$, the error estimate $|z(t_n) - \widetilde{z}_{extp^t}(t_n)| = O(\Delta t^3)$ can be obtained by following [17], as the initial condition is given for the first time interval.

Case 2. Main difficulty occurs when the initial condition can not be used, *i.e.*, for the case $n \geq p+1$. To derive the nodal error $|z(t_n) - \widetilde{z}_{extp^t}(t_n)|$, for $n = p+1, \ldots, 2p$, first we combine (2.4) and (2.5), which provides

$$\begin{aligned}
&(I + \Delta t\mathfrak{L}_{x,\varepsilon})((I + \Delta t\mathfrak{L}_{y,\varepsilon})\widetilde{z}^{n+1} - \Delta t g_2(x, y, t_{n+1}) + c_2\Delta t\widetilde{z}(x, y, t_{n+1-p})) \\
&= -c_1\Delta t\widetilde{z}(x, y, t_{n+1-p}) + z^n(x, y) + \Delta t g_1(x, y, t_{n+1}),
\end{aligned}$$

then we get

$$\begin{aligned}
(I + \Delta t\mathfrak{L}_{x,\varepsilon} + \Delta t\mathfrak{L}_{y,\varepsilon})\widetilde{z}^{n+1} &= z^n(x, y) + \Delta t g(x, y, t_{n+1}) - (\Delta t)^2\mathfrak{L}_{x,\varepsilon}\mathfrak{L}_{y,\varepsilon}\widetilde{z}^{n+1} \\
&\quad -c_2\Delta t\widetilde{z}(x, y, t_{n+1-p}) - c_1\Delta t\widetilde{z}(x, y, t_{n+1-p}) + \\
&\quad c_2(\Delta t)^2\mathfrak{L}_{x,\varepsilon}\widetilde{z}(x, y, t_{n+1-p}) + (\Delta t)^2\mathfrak{L}_{x,\varepsilon}g_2(x, y, t_{n+1}) \\
&= z^n(x, y) + \Delta t g(x, y, t_{n+1}) - c\Delta t\widetilde{z}(x, y, t_{n+1-p}) \\
&\quad +c_2(\Delta t)^2\mathfrak{L}_{x,\varepsilon}\widetilde{z}(x, y, t_{n+1-p}) \\
&\quad +(\Delta t)^2\mathfrak{L}_x g_2(x, y, t_{n+1}) - (\Delta t)^2\mathfrak{L}_x\mathfrak{L}_y\widetilde{z}^{n+1}.
\end{aligned}$$

Since $t \in (\tau, 2\tau]$, we have

$$\begin{aligned}
(I + \Delta t\mathfrak{L}_{x,\varepsilon} + \Delta t\mathfrak{L}_{y,\varepsilon})\widetilde{z}^{n+1} &= z^n(x, y) + \Delta t g - c\Delta t\widetilde{z}_\tau(x, y, t_{n+1-p}) + \\
&\quad c_2(\Delta t)^2\mathfrak{L}_{x,\varepsilon}\widetilde{z}_\tau(x, y, t_{n+1-p}) + (\Delta t)^2\mathfrak{L}_x g_2(x, y, t_{n+1}) \\
&\quad -(\Delta t)^2\mathfrak{L}_x\mathfrak{L}_y\widetilde{z}^{n+1},
\end{aligned}$$

where $\widetilde{z}_\tau(x, y, t)$ is the solution of (2.4)-(2.5) at $t \in [0, \tau]$.

Again the PDE (1.1) can be written in semidiscrete form as

$$\begin{aligned}
\frac{z(t_{n+1}) - z(t_n)}{\Delta t} + (\mathfrak{L}_{x,\varepsilon} + \mathfrak{L}_{y,\varepsilon})z(t_{n+1}) &= -cz(x, y, t_{n+1-p}) + g(x, y, t_{n+1}) \\
&\quad -\frac{\Delta t}{2}z_{tt}(t_{n+1}) + O(\Delta t^2),
\end{aligned}$$

which implies that

$$\begin{aligned}
(I + \Delta t\mathfrak{L}_{x,\varepsilon} + \Delta t\mathfrak{L}_{y,\varepsilon})z(t_{n+1}) &= z^n - c\Delta t z(x, y, t_{n+1-p}) + \Delta t g(x, y, t_{n+1}) \\
&\quad -\frac{(\Delta t)^2}{2}z_{tt}(t_{n+1}) + O(\Delta t^3).
\end{aligned}$$

Therefore, the local truncation error is

$$
\begin{aligned}
&(I+\Delta t\mathfrak{L}_{x,\varepsilon}+\Delta t\mathfrak{L}_{y,\varepsilon})(\widetilde{z}^{n+1}-z(t_{n+1})) = -c\Delta t(\widetilde{z}_\tau(x,y,t_{n+1-p})\\
&-z_\tau(x,y,t_{n+1-p})) + c_2(\Delta t)^2\mathfrak{L}_{x,\varepsilon}\widetilde{z}_\tau(x,y,t_{n+1-p}) - (\Delta t)^2\mathfrak{L}_{x,\varepsilon}\mathfrak{L}_{y,\varepsilon}\widetilde{z}^{n+1}\\
&+(\Delta t)^2\mathfrak{L}_{x,\varepsilon}g_2(x,y,t_{n+1}) + \frac{(\Delta t)^2}{2}z_{tt}(t_{n+1}) + O(\Delta t^3),
\end{aligned}
\tag{3.2}
$$

where $z_\tau(x,y,t)$ is the solution of (1.1) at $t\in[0,\tau]$.

Since $|\widetilde{z}_\tau(x,y,t_{n+1-p}) - z_\tau(x,y,t_{n+1-p})| \le C(\Delta t)^2$,

$$
\begin{aligned}
&(I+\Delta t\mathfrak{L}_{x,\varepsilon}+\Delta t\mathfrak{L}_{y,\varepsilon})(\widetilde{z}^{n+1}-z(t_{n+1})) = c_2(\Delta t)^2\mathfrak{L}_{x,\varepsilon}\widetilde{z}_\tau(x,y,t_{n+1-p}) -\\
&(\Delta t)^2\mathfrak{L}_{x,\varepsilon}\mathfrak{L}_{y,\varepsilon}\widetilde{z}^{n+1} + (\Delta t)^2\mathfrak{L}_{x,\varepsilon}g_2(x,y,t_{n+1}) + \frac{(\Delta t)^2}{2}z_{tt}(t_{n+1}) + O(\Delta t^3).
\end{aligned}
\tag{3.3}
$$

Next, by assuming $\chi = c_2\mathfrak{L}_{x,\varepsilon}\widetilde{z}_\tau(x,y,t_{n+1-p}) - \mathfrak{L}_{x,\varepsilon}\mathfrak{L}_{y,\varepsilon}\widetilde{z}(x,y,t_{n+1}) + \mathfrak{L}_{x,\varepsilon}(g_2) + (1/2)z_{tt}(t_{n+1})$ and following the process used in [17], we can get $(z-\widetilde{z}_{extp^t})(x,y,t_{n+1}) = O(\Delta t^3)$.

We can find the same outcome for $t\ge 2\tau$ in an analogous way. □

3.2 Extrapolation for $\widetilde{Z}$

Here, we introduce the extrapolation formula for the space discrete solution as follows:

$$
\widetilde{Z}_{extp}(x_i,y_j) = 2\widetilde{\widetilde{Z}}(x_i,y_j) - \widetilde{Z}(x_i,y_j), \quad (x_i,y_j)\in\mathfrak{D}^N, \tag{3.4}
$$

The forumula (3.4) can be derived easily by following the idea used in [17] and [16], where, $\widetilde{Z}^n_{x_i,y_j}$ and $\widetilde{\widetilde{Z}}(\widetilde{x}_i,\widetilde{y}_j)$ are the numerical solutions of the discrete problem (2.8)-(2.9) on the mesh $\mathfrak{D}^N$ and $\mathfrak{D}^{2N}$, respectively.

For the sake of further analysis, we decompose z in right hand side of (2.4) as $r+s$, where r and s are the outer and inner components, respectively. Similarly $\widetilde{R}+\widetilde{S}$ and $\widetilde{\mathcal{R}}+\widetilde{\mathcal{S}}$ are the decompositions of the solution $\widetilde{Z}^{n+1/2}_{x_i,y}$ of (2.8) in the domains $\mathfrak{G}^{N,M}$ and $\mathfrak{G}^{N,2M}$, respectively. Further, we decompose $\widetilde{z}$ as $\widetilde{z} = \widetilde{\psi}_r + \widetilde{\psi}_s$. The outer components $\widetilde{r}$ and r are further decomposed into $\widetilde{r} = \widetilde{r}_0 + \varepsilon\widetilde{r}_1 + \varepsilon^2\widetilde{r}_2$ and $r = r_0 + \varepsilon r_1 + \varepsilon^2 r_2$, which satisfy the following equation:

$$
\begin{cases}
-\varepsilon\Delta t\,\widetilde{r}''(x) + a_1(x,y)\Delta t\,\widetilde{r}'(x) + (1+\Delta t b_1(x,y))\widetilde{r}(x)\\
\qquad = r(x,y,t_n) - c_1(x,y)\Delta t\widetilde{r}(x,y,t_{n+1-p}) + \Delta t g_1(x,y,t_{n+1}), \quad y\in(0,1),\\
\widetilde{r}(0)=0, \quad \widetilde{r}(1) = \displaystyle\sum_{i=0}^{3}\varepsilon^i\,\widetilde{r}_i(1,t),\\
\widetilde{r}(x,y,t_{n+1-p}) = \widetilde{z}_\tau(x,y,t_{n+1-p}).
\end{cases}
\tag{3.5}
$$

The inner components $\widetilde{s}$ and s satisfy the equation given below:

$$
\begin{cases}
-\varepsilon\Delta t\,\widetilde{s}''(x) + a_1(x,y)\Delta t\,\widetilde{s}'(x) + (1+\Delta t b_1(x,y))\widetilde{s}(x) = s(x,y,t_n) -\\
c_1(x,y)\Delta t\widetilde{s}(x,y,t_{n+1-p}), \quad y\in(0,1),\\
\widetilde{s}(0)=0, \quad \widetilde{s}(1) = \widetilde{z}^{n+1/2}|_{x=1} - \widetilde{r}(1),\\
\widetilde{s}(x,y,t_{n+1-p}) = 0.
\end{cases}
\tag{3.6}
$$

The discrete solutions $\widetilde{R}$ and $\widetilde{S}$ satisfy the following discrete equations

$$\begin{cases} (I+\Delta t\mathfrak{L}^N_{x,\varepsilon})\widetilde{R} & = (I+\Delta t(-\varepsilon\delta_x^2 + a_1(x_i,y)\delta_x^- + b_1(x_i,y)))\widetilde{R} \\ & = r(x_i,y,t_n) - \Delta t c_1(x_i,y)\widetilde{R}(x_i,y,t_{n+1-p}) \\ & \quad +\Delta t g_1(x_i,y,t_{n+1}), \quad i=1,\cdots,N-1, \\ \widetilde{R}_0 = \widetilde{r}(0), & \widetilde{R}_1 = \widetilde{r}(1), \\ \widetilde{R}(x_i,y,t_{n+1-p}) & = \widetilde{Z}_1(x_i,y,t_{n+1-p})), \end{cases}$$

$$\begin{cases} (I+\Delta t\mathfrak{L}^N_{x,\varepsilon})\widetilde{S} & = (I+\Delta t(-\varepsilon\delta_x^2 + a_1(x_i,y)\delta_x^- + b_1(x_i,y)))\widetilde{S} \\ & = s(x_i,y,t_n) - \Delta t c_1(x_i,y)\widetilde{S}(x_i,y,t_{n+1-p}), \\ & \quad i=1,\cdots,N-1, \\ \widetilde{S}_0 = \widetilde{s}(0), & \widetilde{S}_1 = \widetilde{s}(1), \\ \widetilde{S}(x_i,y,t_{n+1-p}) & = 0, \end{cases} \tag{3.7}$$

respectively.

We write the error in the following form:

$$(\widetilde{Z}-\widetilde{z})(x_i,y,t_{n+1/2}) = (\widetilde{R}-\widetilde{r})(x_i,y,t_{n+1/2}) + (\widetilde{S}-\widetilde{s})(x_i,y,t_{n+1/2}).$$

Extrapolation technique for the outer and inner components

Lemma 3.2. *The local truncation error after extrapolation in the time direction associated to the smooth component satisfies*

$$(\widetilde{R}_{extp^t} - \widetilde{r}_{extp^t})(x_i,y,t_{n+1/2}) = h_{x,i}\eta(x_i,y) + O(N^{-2}),$$

where $\widetilde{R}^{n+1/2}_{extp^t}$ *and* $\widetilde{r}^{n+1/2}_{extp^t}$ *are the extrapolated solutions in the time direction for* $\widetilde{R}^{n+1/2}$ *and* $\widetilde{r}^{n+1/2}$*, respectively.*

Proof. By using Taylor's expansion, (2.6) and $\varepsilon < N^{-1} < H_x$, we get,

$$\begin{aligned} &(I+\Delta t\mathfrak{L}^N_{x,\varepsilon})(\widetilde{R}-\widetilde{r})(x_i,y,t_{n+1/2}) \\ &= r(x_i,y,t_n) - \Delta t c_1(x_i,y)\widetilde{R}(x_i,y,t_{n+1-p}) + \Delta t g_1(x_i,y,t_n) - (I+\Delta t\mathfrak{L}^N_{x,\varepsilon})\widetilde{r}(x_i,y,t_{n+1/2}) \\ &= r(x_i,y,t_n) - \Delta t c_1(x_i,y)\widetilde{R}(x_i,y,t_{n+1-p}) + (I+\Delta t\mathfrak{L}_{x,\varepsilon})\widetilde{r}^{n+1/2} - r(x_i,y,t_n) \\ &\quad +\Delta t c_1(x_i,y)\widetilde{r}(x_i,y,t_{n+1-p}) - (I+\Delta t\mathfrak{L}^N_{x,\varepsilon})\widetilde{r}(x_i,y,t_{n+1/2}) \\ &= -c_1\Delta t(\widetilde{R}(x_i,y,t_{n+1-p}) - \widetilde{r}(x_i,y,t_{n+1-p})) + ((I+\Delta t\mathfrak{L}_{x,\varepsilon}) - (I+\Delta t\mathfrak{L}^N_{x,\varepsilon}))\widetilde{r}(x_i,y,t_{n+1/2}). \end{aligned}$$

To prove the required result, we discuss case-wise depending on the time intervals.

Case 1. For $t \in (0,\tau]$, we can use the given initial condition, and the required bound can be obtained by following the idea of [17].

Case 2. For $t \in (\tau, 2\tau]$

$$\begin{aligned} &(I+\Delta t\mathfrak{L}^N_{x,\varepsilon})(\widetilde{R}-\widetilde{r})(x_i,y,t_{n+1/2}) \\ &= -c_1\Delta t N^{-1}\ln N + \frac{\Delta t}{2}\frac{h_{x,i}}{2}a_1(x_i,y)\frac{\partial^2\widetilde{r}}{\partial x^2}(x_i,y,t_{n+1/2}) + O(H_x^2) \\ &= -c_1\Delta t N^{-1}\ln N + h_{x_i,y}(I+\Delta t\mathfrak{L}_{x,\varepsilon})\eta(x_i,y) + O(H_x^2), \end{aligned}$$

where η denotes the smooth component of E and E is defined as the solution of

$$\begin{cases} (I+\Delta t\mathfrak{L}_{x,\varepsilon})E = \dfrac{\Delta t}{2}a_1(x,y)\dfrac{\partial^2\widetilde{r}}{\partial x^2}(x,y,t_{n+1}), \quad y\in(0,1), \\ E(0) = E(1) = 0. \end{cases} \tag{3.8}$$

With the application of comparison principle together with an appropriate barrier function, we get $(\widetilde{R}-\widetilde{r})(x_i,y,t_{n+1/2}) = -c_1\Delta t N^{-1}\ln N + h_{x_i,y}\eta(x_i,y) + O(H_x^2)$.

Similarly, in the finer mesh

$$(\widetilde{\mathcal{R}}-\widetilde{\psi}_r)(x_i,y,t_{n+1/2}) = -c_1\frac{\Delta t}{2}N^{-1}\ln N + h_{x_i,y}\eta(x_i,y) + O(H_x^2).$$

Therefore, we get

$$(\widetilde{R}_{extp^t}-\widetilde{r}_{extp^t})(x_i,y,t_{n+1/2}) = h_{x,i}\eta(x_i,y) + O(N^{-2}).$$

Case 3. Analogously, the required bound can be obtained for $t_n \geq 2\tau$. □

Now, by using the Lemma 3.2 and following the process done in [17], we can have the following two lemmas.

Lemma 3.3. *The error bound for smooth component $\widetilde{R}(x_i,y,t_{n+1/2})$ after extrapolation is given as*

$$\left|\left(\widetilde{R}_{extp}-\widetilde{r}_{extp^t}\right)(x_i,y,t_{n+1/2})\right| \leq CN^{-2},$$

where $\varepsilon \leq N^{-1}$.

Lemma 3.4. *The error bound for singular component $\widetilde{S}(x_i,\, y,\, t_{n+1/2})$ after extrapolation is given as*

$$\left(\widetilde{S}_{extp}-\widetilde{s}_{extp^t}\right)(x_i,\, y,\, t_{n+1/2}) \leq CN^{-2}, \quad \textit{for} \quad 1\leq i\leq N/2,$$

where $\varepsilon \leq N^{-1}$.

Next, we will find the effect of the Richardson extrapolation on $x_i \in (1-\rho_x, 1]$. We have,

$$\begin{aligned}
&\left(I+\Delta t\mathfrak{L}^N_{x,\varepsilon}\right)\left(\widetilde{S}-\widetilde{s}\right)(x_i,\, y,\, t_{n+1/2})\\
&= \left(I+\Delta t\mathfrak{L}^N_{x,\varepsilon}\right)\widetilde{S}(x_i,\, y,\, t_{n+1/2}) - \left(I+\Delta t\mathfrak{L}^N_{x,\varepsilon}\right)\widetilde{s}(x_i,\, y,\, t_{n+1/2})\\
&= s(x_i,y,t_n) - \Delta t c_1(x_i,y)\widetilde{S}(x_i,y,t_{n+1-p}) - \left(I+\Delta t\mathfrak{L}^N_{x,\varepsilon}\right)\widetilde{s}, \text{ by using (3.7)}\\
&= \left((I+\Delta t\mathfrak{L}_{x,\varepsilon}) - \left(I+\Delta t\mathfrak{L}^N_{x,\varepsilon}\right)\right)\widetilde{s}\\
&+c_1(x_i,y)\Delta t\left(\widetilde{S}(x_i,y,t_{n+1-p}) - \widetilde{s}(x_i,y,t_{n+1-p})\right).
\end{aligned} \tag{3.9}$$

For $t\in(0,\tau]$, by using the given initial conditions of (3.6) and (3.7), we can have

$$\left(\widetilde{S}(x_i,y,t_{n+1-p}) - \widetilde{s}(x_i,y,t_{n+1-p})\right) = 0.$$

Now by putting that in (3.9), we get

$$\left(I+\Delta t\mathfrak{L}^N_{x,\varepsilon}\right)\left(\widetilde{S}-\widetilde{s}\right) = \left((I+\Delta t\mathfrak{L}_{x,\varepsilon}) - \left(I+\Delta t\mathfrak{L}^N_{x,\varepsilon}\right)\right)\widetilde{s}.$$

Hence, by using the analogues idea of [17], one can have

$$(\widetilde{S}-\widetilde{s})(x_i,y,t_{n+1/2}) = (N^{-1}\ln N)F(x_i,y) + O(N^{-2}\ln^2 N).$$

Next, for $t\in(\tau,\, 2\tau]$, we know $\left(\widetilde{S}(x_i,y,t_{n+1-p}) - \widetilde{s}(x_i,y,t_{n+1-p})\right)$ will lie on $(0,\, \tau]$.

By using the a result obtained in [16], we can get

$$\left(\widetilde{S}(x_i,y,t_{n+1-p}) - \widetilde{s}(x_i,y,t_{n+1-p})\right) \leq CN^{-1}\ln N.$$

Hence,

$$\begin{aligned}
&\left(I+\Delta t\mathfrak{L}^N_{x,\varepsilon}\right)\left(\widetilde{S}-\widetilde{s}\right)(x_i,y,t_{n+1/2}) = \left((I+\Delta t\mathfrak{L}_{x,\varepsilon}) - \left(I+\Delta t\mathfrak{L}^N_{x,\varepsilon}\right)\right)\widetilde{s}(x_i,y,t_{n+\frac{1}{2}})\\
&+Cc_1(x_i,y)\Delta t N^{-1}\ln N.
\end{aligned}$$

With the help of Taylor's expansion, we get

$$\left((I+\Delta t\mathfrak{L}_{x,\varepsilon})-\left(I+\Delta t\mathfrak{L}^N_{x,\varepsilon}\right)\right)\widetilde{s}(x_i,y,t_{n+1/2})=-\frac{\varepsilon\Delta t h_x^2}{4!}\left[\frac{\partial^4\widetilde{s}}{\partial x^4}(\xi_1)+\frac{\partial^4\widetilde{s}}{\partial x^4}(\xi_2)\right]$$
$$+\frac{h_x\Delta t}{2}a_1(x_i,y)\frac{\partial^2\widetilde{s}}{\partial x^2}(x_i,y)-\frac{h_x^2\Delta t}{3!}a_1(x_i,y)\frac{\partial^3\widetilde{s}}{\partial x^3}(\xi_3,y).$$

where $\xi_1\in(x_i,x_{i+1})$ and $\xi_2,\ \xi_3\in(x_{i-1},x_i)$.

Hence, it can be deduced that

$$\begin{aligned}\left(I+\Delta t\mathfrak{L}^N_{x,\varepsilon}\right)\left(\widetilde{S}-\widetilde{s}\right)(x_i,y,t_{n+1/2})=\ &\frac{2\varepsilon\Delta t}{\alpha_x}\left(N^{-1}\ln N\right)a_1(x_i,y)\frac{\partial^2\widetilde{s}}{\partial x^2}(x_i,y)\\&+O(\varepsilon^{-1}\exp(-\alpha_x(1-x_{i+1})/\varepsilon)N^{-2}\ln^2 N)\\&+Cc_1(x_i,y)\Delta t N^{-1}\ln N,\end{aligned}$$

by using (2.7).

To deal with the right side of the above equation, we assume

$$\begin{aligned}&(I+\Delta t\mathfrak{L}_{x,\varepsilon})\,F=\frac{2\varepsilon}{\alpha_x}\Delta t a_1(x,y)\frac{\partial^2\widetilde{s}}{\partial x^2}(x,y)+Cc_1(x_i,y)\Delta t,\quad y\in(0,1),\\&F(1-\rho_x)=F(1)=0,\end{aligned}$$

when $x\in(1-\rho_x,1)$.

Approaching in an analogues way as done in [18], we can obtain

$$(\widetilde{S}-\widetilde{s})(x_i,y,t_{n+1/2})=(N^{-1}\ln N)F(x_i,y)+O(N^{-2}\ln^2 N).$$

By proceeding in a similar way, for the entire time domain $(0,\,T]$, we can have

$$(\widetilde{S}-\widetilde{s})(x_i,y,t_{n+1/2})=(N^{-1}\ln N)F(x_i,y)+O(N^{-2}\ln^2 N).$$

Likewise,we get

$$(\widetilde{\mathcal{S}}-\widetilde{\psi}_s)(x_i,y,t_{n+1/2})=(N^{-1}\ln N)F(x_i,y)+O(N^{-2}\ln^2 N),$$

in the finer mesh of temporal direction.

Now, by some simple calculations one can get

$$\left(\widetilde{S}_{extp^t}-\widetilde{s}_{extp^t}\right)(x_i,y,t_{n+1/2})=(N^{-1}\ln N)F(x_i,y)+O(N^{-2}\ln^2 N).$$

As we know that

$$\left|(\widetilde{S}_{extp}-\widetilde{s}_{extp^t})(x_i,y,t_{n+1/2})\right|=\left|(2\widetilde{\mathcal{S}}_{extp^t}-\widetilde{S}_{extp^t}-\widetilde{s}_{extp^t})(x_i,y,t_{n+1/2})\right|,$$

we can deduce that $\left|\widetilde{S}_{extp}(x_i,y,t_{n+1/2})-\widetilde{s}_{extp^t}(x_i,y,t_{n+1/2})\right|\le CN^{-2}\ln^2 N$.

By considering the above bounds, we are ready to state the lemma given below.

Lemma 3.5. *For $N/2+1\le i\le N-1$, the error related to the inner component $\widetilde{S}$ after the post processing technique satisfies*

$$\left|(\widetilde{S}_{extp}-\widetilde{s}_{extp^t})(x_i,y,t_{n+1/2})\right|\le C\left(N^{-2}\ln^2 N\right).$$

By merging the results of Lemmas 3.3, 3.4 and 3.5, we get

$$\left|\left(\widetilde{z}^{\,n+1/2}_{extp^t}-\widetilde{Z}^{n+1/2}_{extp}\right)(x_i,y)\right|\le CN^{-2}\ln^2 N. \tag{3.10}$$

Semidiscrete problems (2.4) and (2.5) are of almost the same form except for the first term of both equations in right hand side; *i.e.*, the RHS of (2.5) involves $\widetilde{z}^{n+1/2}$ contrary to z^n as in

(2.4). But, by using $\left|(\widetilde{z}^{n+1/2} - \widetilde{Z}^{n+1/2})(x_i, y)\right| \leq CN^{-1} \ln N$, [16] and deriving in an analogues way as before, for $1 \leq j \leq N-1$ we get the error bound as follows:

$$\left|\left(\widetilde{z}^{n+1}_{extp^t} - \widetilde{Z}^{n+1}_{extp}\right)(x, y_j)\right| \leq CN^{-2} \ln^2 N. \tag{3.11}$$

We can write

$$\begin{aligned}&\|(z - Z_{extp})(x_i, y_j, t_{n+1})\|_\infty \\ &\leq \|(z - \widetilde{z}_{extp^t})(x_i, y_j, t_{n+1})\|_\infty + \left\|\left(\widetilde{z}_{extp^t} - \widetilde{Z}_{extp}\right)(x_i, y_j, t_{n+1})\right\|_\infty \\ &+ \left\|\left(\widetilde{Z}_{extp} - Z_{extp}\right)(x_i, y_j, t_{n+1})\right\|_\infty.\end{aligned}$$

Next, by using Theorem 3.1 and equation (3.11), we obtain that

$$\|(z - Z_{extp})(x_i, y_j, t_{n+1})\|_\infty \leq C\left(\Delta t^3 + N^{-2} \ln^2 N\right) + \left\|\left(\widetilde{Z}_{extp} - Z_{extp}\right)(x_i, y_j, t_{n+1})\right\|_\infty,$$

Hence, we have

$$\begin{aligned}&\|(z - Z_{extp})(x_i, y_j, t_{n+1})\|_\infty \leq C\Delta t\left(\Delta t^2 + N^{-2+\beta} \ln^2 N\right) \\ &+ \left\|\left(\widetilde{Z}_{extp} - Z_{extp}\right)(x_i, y_j, t_{n+1})\right\|_\infty,\end{aligned}$$

where $0 < \beta < 1$ and $N^{-\beta} \leq C\Delta t$.

Since the fully discrete scheme is stable, we can have

$$\left\|\left(\widetilde{Z}_{extp} - Z_{extp}\right)(x_i, y_j, t_{n+1})\right\|_\infty \leq \left\|z(x_i, y_j, t_{n-1}) - {Z_{extp}}^{n-1}_{x_i, y_j}\right\|_\infty.$$

Therefore, we are ready to state the main convergence theorem for the proposed method applied on (1.1).

Theorem 3.6. *(Error after extrapolation) If z and Z_{extp} denote the solution of (1.1) and the extrapolated solution of (2.12)-(2.13) at time level $t_n = n\Delta t$, respectively. Then, we have the error bound as follows:*

$$\|z(x_i, y_j, t_n) - Z_{extp}(x_i, y_j, t_n)\|_\infty \leq C\left(N^{-2+\beta} \ln^2 N + \Delta t^2\right), \textit{for } (x_i, y_j, t_n) \in \mathfrak{G}^{N,M},$$

where $0 < \beta < 1$.

4 Numerical Results

This section is dedicated to confirm the effectiveness of the proposed scheme numerically. We apply the scheme on two test problems and tabulated the results, where we start with $M = 40$ and $N = 32$. Further we doubled the values of M and N. Under the condition of compatibility

$$g(x, y, t) = 0, \text{for } x \in \{0, 1\},\ y \in \{0, 1\}, \text{ and } t \in [0, T],$$

to satisfy the property given in (2.3), we write $g(x, y, t)$ as

$$g_2(x, y, t) = y\left(g(x, 1, t) - g(x, 0, t)\right) + g(x, 0, t), \quad g_1(x, y, t) = g(x, y, t) - g_2(x, y, t).$$

Example 4.1. Consider the following test problem:

$$\begin{cases} z_t - \varepsilon\Delta z + (1+x)z_x + (2-y)z_y + (x^2+y^2+1)z & = z(x, y, t-1) + g(x, y, t), \\ & (x, y, t) \in \mathfrak{D} \times (0, 2], \\ z(x, y, t) = \varphi_b(x, y, t), & (x, y, t) \in \overline{\mathfrak{D}} \times [-1, 0], \\ z(x, y, t) = 0, & (x, y, t) \in \partial\mathfrak{D} \times [0, 2]. \end{cases} \tag{4.1}$$

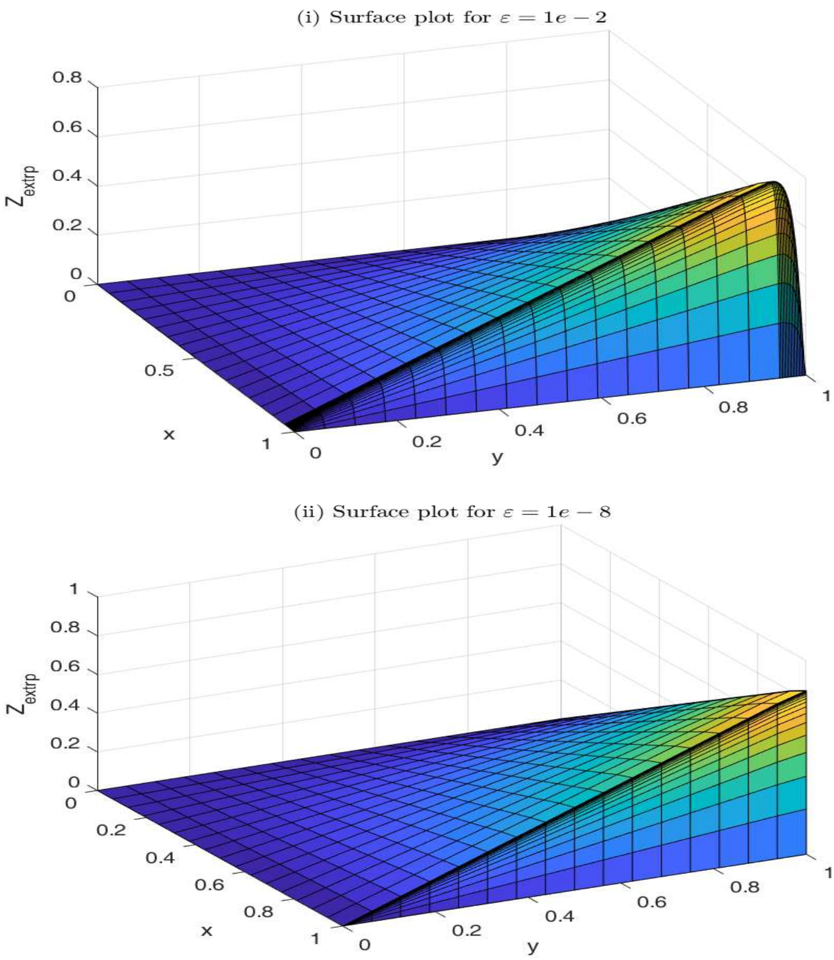

Figure 1. *Extrapolated numerical solutions at $N = 32$ and $t = 2$ for Example 4.2.*

We choose the source function $g(x,y,t)$ and initial data $\varphi_b(x,y,t)$ to fit with the exact solution

$$z(x,y,t) = e^{-t}xy(\gamma_1(x) - 1)(\gamma_2(y) - 1),$$

where

$$\gamma_1(x) = e^{\frac{-(3-2x-x^2)}{2\varepsilon}} \quad \text{and} \quad \gamma_2(y) = e^{\frac{-(3-4y+y^2)}{2\varepsilon}}.$$

Example 4.2. Consider the following test problem:

$$\begin{cases} z_t - \varepsilon\Delta z + z_x + z_y = z(x,y,t-1) + g(x,y,t), & (x,y,t) \in \mathfrak{D} \times (0,2] \\ z(x,y,t) = \varphi_b(x,y,t), & (x,y,t) \in \overline{\mathfrak{D}} \times [-1,0], \\ z(x,y,t) = 0, & (x,y,t) \in \partial\mathfrak{D} \times [0,2]. \end{cases} \tag{4.2}$$

We choose the source function $g(x,y,t)$ and initial data $\varphi_b(x,y,t)$ to fit with the exact solution

$$z(x,y,t) = (1 - \exp(-t))\,(m_1 + m_2 x + \exp(-(1-x)/\varepsilon))\,(m_1 + m_2 y + \exp(-(1-y)/\varepsilon))\,,$$

where $m_1 = -\exp(-1/\varepsilon), m_2 = -1 - m_1$. To visualize the numerical solution, we have given surface plots in Figure 1, for different values of ε, *i.e.* 10^{-2} and 10^{-8}. One can observe the sharp layer, when ε is closed to zero.

For each ε, to calculate the maximum pointwise error, we use the formula

$$e_{\varepsilon}^{N,\Delta t} = \max_{(x_i,y_j,t_n)\in\mathfrak{G}^{N,M}} |z(x_i,y_j,t_n) - Z(x_i,y_j,t_n)|, \quad \text{(before extrapolation)},$$

and

$$e_{\varepsilon,\,extp}^{N,\Delta t} = \max_{(x_i,y_j,t_n)\in\mathfrak{G}^{N,M}} |z(x_i,y_j,t_n) - Z_{extp}(x_i,y_j,t_n)|, \quad \text{(after extrapolation)},$$

where $z(x_i,y_j,t_n)$ denotes the exact solution, $Z(x_i,y_j,t_n)$ and $Z_{extp}(x_i,y_j,t_n)$ denote the numerical solutions obtained before and after extrapolation.

For each ε, we calculate the corresponding order of convergence as

$$p_{\varepsilon}^{N,\Delta t} = \log_2\left(\frac{e_{\varepsilon}^{N,\Delta t}}{e_{\varepsilon}^{2N,\Delta t/2}}\right), \quad \text{(before extrapolation)},$$

and

$$p_{\varepsilon,\,extp}^{N,\Delta t} = \log_2\left(\frac{e_{\varepsilon,\,extp}^{N,\Delta t}}{e_{\varepsilon,\,extp}^{2N,\Delta t/2}}\right), \quad \text{(after extrapolation)}.$$

Parameter-uniform maximum nodal error is calculated by

$$e^{N,\Delta t} = \max_{\varepsilon} e_{\varepsilon}^{N,\Delta t}, \quad \text{(before extrapolation)},$$

and

$$e_{extp}^{N,\Delta t} = \max_{\varepsilon} e_{\varepsilon,\,extp}^{N,\Delta t}, \quad \text{(after extrapolation)},$$

and associated order of convergence is calculated as

$$p^{N,\Delta t} = \log_2\left(\frac{e^{N,\Delta t}}{e^{2N,\Delta t/2}}\right), \quad \text{(before extrapolation)},$$

and

$$p_{extp}^{N,\Delta t} = \log_2\left(\frac{e_{extp}^{N,\Delta t}}{e_{extp}^{2N,\Delta t/2}}\right), \quad \text{(after extrapolation)}.$$

In the Table 1 and Table 2, we provide the maximum pointwise errors and the corresponding order of convergence. Numerical results given in the Table 1 and Table 2, reflect the fact that the solution converges parameter-uniformly. The effectiveness of the scheme can also be seen from the tables, *i.e.*, after applying the proposed technique, maximum point wise errors and the corresponding order of convergence are improved.

To show the numerical order of convergence (before and after extrapolation) for the Example 4.1 and Example 4.2, we provide the loglog plot for maximum pointwise errors in the Figure 2 and Figure 3 respectively. For the problem considered in this paper, the numerical rate of convergence obtained is maximum till now.

5 Conclusion

In this paper, we propose a parameter-uniform numerical method for a class of 2D singularly perturbed parabolic time delay problem having convection and diffusion terms of the form (1.1). First, to transform the problem into a set of 1D problems, we used the fractional-step method. Then, we applied the upwind finite difference scheme along with the Richardson extrapolation technique. We proved theoretically that the proposed method is almost second-order convergence. To support the theoretical finding, we provided numerical results. It is evident from both the ways that the method is quite effective to increase the order of convergence for our model problem. Few deficiencies also can be observed in the proposed strategy, such as, to capture the layer region numerically, we have used Shishkin mesh. To construct the Shishkin mesh, we need apriori information regarding the layer location, layer width etc. Sometime it may not be possible to have those information apriori. This issue can be taken care off by using adaptive

Table 1. *Before and after extrapolations, maximum pointwise errors and order of convergence for Example 4.1.*

ε	Extra-polation	Number of spatial mesh intervals N					
		16	32	64	128	256	512
10^{-2}	Before	1.0654e-01	8.4968e-02	5.8123e-02	3.6785e-02	2.2100e-02	1.2794e-02
		0.3265	0.5478	0.6600	0.7351	0.7886	
	After	3.5710e-02	1.7511e-02	8.6349e-03	3.9237e-03	1.5548e-03	5.4760e-04
		1.0281	1.0200	1.1380	1.3355	1.5055	
10^{-4}	Before	1.2005e-01	9.6667e-02	6.5608e-02	4.1396e-02	2.4861e-02	1.4364e-02
		0.3126	0.5591	0.6644	0.7356	0.7914	
	After	3.8248e-02	1.6651e-02	7.0367e-03	2.8532e-03	1.2644e-03	7.0958e-04
		1.1998	1.2427	1.3023	1.1741	0.8334	
10^{-6}	Before	1.2025e-01	9.6993e-02	6.6018e-02	4.1650e-02	2.4964e-02	1.4413e-02
		0.3101	0.5550	0.6645	0.7385	0.7925	
	After	3.8260e-02	1.6725e-02	7.0318e-03	2.6252e-03	9.1127e-04	3.0106e-04
		1.1939	1.2500	1.4215	1.5264	1.5978	
10^{-8}	Before	1.2025e-01	9.6997e-02	6.6023e-02	4.1656e-02	2.4970e-02	1.4420e-02
		0.3100	0.5550	0.6644	0.7383	0.7921	
	After	3.8261e-02	1.6726e-02	7.0319e-03	2.6254e-03	9.1141e-04	3.0336e-04
		1.1938	1.2501	1.4214	1.5264	1.5871	
$e^{N,\Delta t}$	Before	**1.2025e-01**	**9.6997e-02**	**6.6023e-02**	**4.1656e-02**	**2.4970e-02**	**1.4420e-02**
$p^{N,\Delta t}$		**0.3100**	**0.5550**	**0.6644**	**0.7383**	**0.7921**	
$e^{N,\Delta t}_{extp}$	After	**3.8261e-02**	**1.7511e-02**	**8.6349e-03**	**3.9237e-03**	**1.5548e-03**	**7.0958e-04**
$p^{N,\Delta t}_{extp}$		**1.1276**	**1.0200**	**1.1380**	**1.3355**	**1.1316**	

Table 2. *Before and after extrapolations, maximum pointwise errors and order of convergence for Example 4.2.*

ε	Extra-polation	Number of spatial mesh intervals N					
		16	32	64	128	256	512
10^{-2}	Before	1.0798e-01	6.6775e-02	4.0097e-02	2.3696e-02	1.3720e-02	7.8000e-03
		0.6934	0.7358	0.7589	0.7883	0.8147	
	After	1.9807e-02	6.3272e-03	2.1082e-03	7.0689e-04	2.3252e-04	7.4426e-05
		1.6463	1.5856	1.5765	1.6041	1.6435	
10^{-4}	Before	1.1904e-01	7.2681e-02	4.3029e-02	2.5086e-02	1.4396e-02	8.1308e-03
		0.7118	0.7563	0.7784	0.8012	0.8242	
	After	2.9079e-02	1.0106e-02	3.5975e-03	1.2599e-03	4.2641e-04	1.3867e-04
		1.5247	1.4901	1.5137	1.5630	1.6205	
10^{-6}	Before	1.1916e-01	7.2751e-02	4.3068e-02	2.5107e-02	1.4409e-02	8.1384e-03
		0.7118	0.7563	0.7784	0.8012	0.8242	
	After	2.9227e-02	1.0186e-02	3.6471e-03	1.2906e-03	4.4634e-04	1.5170e-04
		1.5208	1.4817	1.4987	1.5319	1.5569	
10^{-8}	Before	1.1916e-01	7.2752e-02	4.3068e-02	2.5108e-02	1.4409e-02	8.1385e-03
		0.7118	0.7564	0.7785	0.8012	0.8241	
	After	2.9228e-02	1.0186e-02	3.6476e-03	1.2909e-03	4.4655e-04	1.5241e-04
		1.5207	1.4816	1.4985	1.5315	1.5509	
$e^{N,\Delta t}$	Before	**1.1916e-01**	**7.2752e-02**	**4.3068e-02**	**2.5108e-02**	**1.4409e-02**	**8.1385e-03**
$p^{N,\Delta t}$		**0.7118**	**0.7564**	**0.7785**	**0.8012**	**0.8241**	
$e^{N,\Delta t}_{extp}$	After	**2.9228e-02**	**1.0186e-02**	**3.6476e-03**	**1.2909e-03**	**4.4655e-04**	**1.5241e-04**
$p^{N,\Delta t}_{extp}$		**1.5207**	**1.4816**	**1.4985**	**1.5315**	**1.5509**	

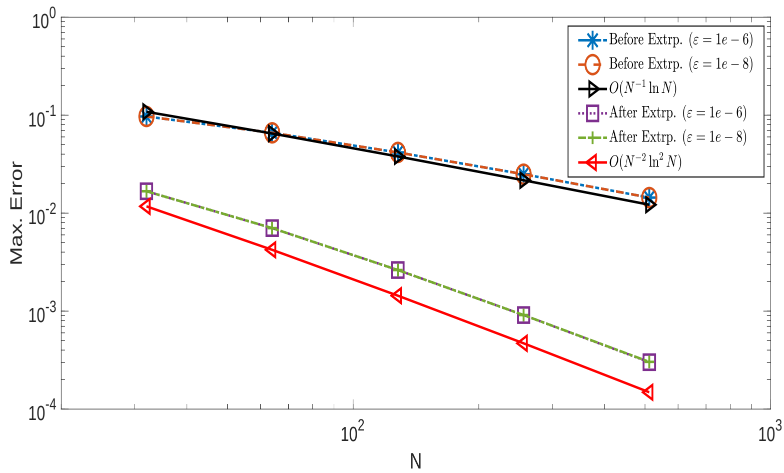

Figure 2. *Comparison of maximum pointwise errors through loglog plot for Example 4.1.*

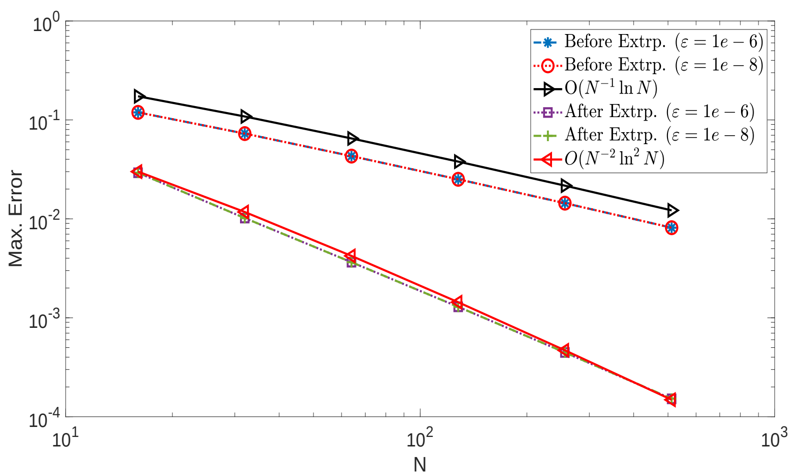

Figure 3. *Comparison of maximum pointwise errors through loglog plot for Example 4.2.*

grid equidistribution technique, for which, no such prior information is required. Another observation which can be treated as a limitation of the proposed strategy is the order of convergence is not exactly second-order, more precisely, it is reduced by logarithmic factor. Since, we are using Shishkin mesh, it is not possible to remove that term, as because, that logarithmic term appears from the transition parameter of the Shishkin mesh. As a practical interest, for our future work, we shall try to use the proposed method to solve a 2D degenerate SPPDE with delay term where the degenerate term causes boundary turning point.

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