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INCLUSION RESULTS ON SUBCLASSES OF HARMONIC UNIVALENT FUNCTIONS ASSOCIATED WITH PASCAL DISTRIBUTION SERIES

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Abstract In this existing paper, it is aimed to set up bonds among diverse subclasses of harmonic univalent functions by implementing specific convolution operator including Pascal Distribution Series. To be more accurate, we research this kind of relations with Goodman-Rønning type harmonic univalent functions in the open unit disc.

1 Introduction

Let $\mathcal{A}$ denote the class of functions h of the form

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k,$$

that are analytic in the open unit disk $\mathcal{U} = \{z \in \mathbb{C} : |z| < 1\}$ and satisfy the normalization condition $h(0) = h'(0) - 1 = 0$. Let $\mathcal{H}$ be the family of all harmonic functions of the form

$$f = h + \overline{g}, \tag{1.1}$$

where

$$h(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad g(z) = \sum_{k=1}^{\infty} b_k z^k, \quad (z \in \mathcal{U}) \tag{1.2}$$

are in the class $\mathcal{A}$ and then $f(z)$ is given by,

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k + \overline{\sum_{k=1}^{\infty} b_k z^k}, \quad (z \in \mathcal{U}). \tag{1.3}$$

Denote by $S_{\mathcal{H}}$ the subclass of $\mathcal{H}$ that are univalent and sense-preserving in $\mathcal{U}$. One shows easily that the sense-preserving property implies that $|b_1| < 1$. Note that $\frac{f-\overline{b_1 f}}{1-|b_1|^2} \in S_{\mathcal{H}}$ whenever $f \in S_{\mathcal{H}}$. We also let the subclass $S^0_{\mathcal{H}}$ of $S_{\mathcal{H}}$,

$$S^0_{\mathcal{H}} = \{f = h + \overline{g} \in S_{\mathcal{H}} : g'(0) = b_1 = 0\}.$$

The classes $S^0_{\mathcal{H}}$ and $S_{\mathcal{H}}$ were first studied in [5]. Also, we let $S^{*,0}_{\mathcal{H}}$, $C^0_{\mathcal{H}}$ and $K^0_{\mathcal{H}}$, denote the subclasses of $S^0_{\mathcal{H}}$ of harmonic functions which are, respectively, starlike, close-to-convex and convex in $\mathcal{U}$. For definitions and properties of these classes, one may refer to ([1], [5] or [11]).

Motivated by the earlier works on the subject of harmonic functions, in this paper we consider a subclass of harmonic univalent functions $f \in \mathcal{H}$ given by (1.3) introduced and studied by Porwal et al.[16] and made an attempt has been made to study inclusion relations making use of Pascal Distribution Series.

2 The Class $\mathcal{G}_{\mathcal{H}}(\lambda, \alpha, \gamma)$

For $0 \leq \lambda \leq 1,\ 0 \leq \gamma < 1$ and $\alpha \in \mathbb{R}$, we let, $\mathcal{G}_{\mathcal{H}}(\lambda, \alpha, \gamma)$ a new subclass of $\mathcal{H}$, consist of all functions of the form (1.3) satisfying the condition

$$\Re\left((1+e^{i\alpha})\frac{zf'(z)}{(1-\lambda)z' + \lambda f(z)} - e^{i\alpha}\right) > \gamma \tag{2.1}$$

where

$$z' = \frac{\partial}{\partial\theta}\left(z = re^{i\theta}\right),\ f'(z) = \frac{\partial}{\partial\theta} f\left(re^{i\theta}\right).$$

Equivalently, we have,

$$\Re\left((1+e^{i\alpha})\frac{z(h(z))' - \overline{z(g(z))'}}{(1-\lambda)z + \lambda[h(z) + \overline{g(z)}]} - e^{i\alpha}\right) > \gamma, (z \in \mathcal{U}). \tag{2.2}$$

Example 2.1. For $\lambda = 0,\ 0 \leq \gamma < 1$, we define a new class $\mathcal{G}_{\mathcal{H}}(0, \alpha, \gamma) \equiv \mathcal{N}_{\mathcal{H}}(\alpha, \gamma)$ satisfying the analytic criteria

$$\Re\left((1+e^{i\alpha})\frac{zf'(z)}{z'} - e^{i\alpha}\right) > \gamma,\ \alpha \in \mathbb{R}. \tag{2.3}$$

Equivalently,

$$\Re\left((1+e^{i\alpha})\frac{z(h(z))' - \overline{z(g(z))'}}{z} - e^{i\alpha}\right) > \gamma,\ (z \in \mathcal{U}). \tag{2.4}$$

Example 2.2. ([17])For $\lambda = 1,\ 0 \leq \gamma < 1$, we define a new class $\mathcal{G}_{\mathcal{H}}(1, \alpha, \gamma) \equiv \mathcal{R}_{\mathcal{H}}(\alpha, \gamma)$ satisfying the analytic criteria

$$\Re\left((1+e^{i\alpha})\frac{zf'(z)}{f(z)} - e^{i\alpha}\right) > \gamma,\ \alpha \in \mathbb{R}. \tag{2.5}$$

Equivalently,

$$\Re\left((1+e^{i\alpha})\frac{z(h(z))' - \overline{z(g(z))'}}{h(z) + \overline{g(z)}} - e^{i\alpha}\right) > \gamma,\ (z \in \mathcal{U}). \tag{2.6}$$

Also let,

(i) $\mathcal{GV}_{\mathcal{H}}(\lambda, \alpha, \gamma) = \mathcal{G}_{\mathcal{H}}(\lambda, \alpha, \gamma) \cap \mathcal{T}_{\mathcal{H}}$

(ii) $\mathcal{RV}_{\mathcal{H}}(\alpha, \gamma) = \mathcal{R}_{\mathcal{H}}(\alpha, \gamma) \cap \mathcal{T}_{\mathcal{H}}$

(iii) $\mathcal{NV}_{\mathcal{H}}(\alpha, \gamma) = \mathcal{N}_{\mathcal{H}}(\alpha, \gamma) \cap \mathcal{T}_{\mathcal{H}}$

where $\mathcal{T}_{\mathcal{H}}$ the subclass of $\mathcal{H}$ consisting of harmonic functions $f = h + \overline{g}$ of the form

$$f(z) = z - \sum_{k=2}^{\infty} |a_k| z^k + \sum_{k=1}^{\infty} |b_k| \overline{z^k}. \tag{2.7}$$

We recall, a sufficient coefficient condition for harmonic functions in $\mathcal{G}_{\mathcal{H}}(\lambda, \alpha, \gamma)$.

Theorem 2.3. *([16]) Let $f = h + \overline{g}$ be given by (1.3). If*

$$\sum_{k=2}^{\infty} \frac{2k - \lambda(1+\gamma)}{1-\gamma}|a_k| + \sum_{k=1}^{\infty} \frac{2k + \lambda(1+\gamma)}{1-\gamma}|b_k| \leq 1, \tag{2.8}$$

where $a_1 = 1$ and $0 \leq \gamma < 1$, then $f \in \mathcal{G}_{\mathcal{H}}(\lambda, \alpha, \gamma)$.

Theorem 2.4. *([16]) For $a_1 = 1$ and $0 \leq \gamma < 1$, $f = h + \overline{g} \in \mathcal{GV}_{\mathcal{H}}(\lambda, \alpha, \gamma)$ if and only if*

$$\sum_{k=2}^{\infty} \frac{2k - \lambda(1+\gamma)}{1-\gamma}|a_k| + \sum_{k=1}^{\infty} \frac{2k + \lambda(1+\gamma)}{1-\gamma}|b_k| \leq 1. \tag{2.9}$$

Theorem 2.5. *For* $a_1 = 1$ *and* $0 \leq \gamma < 1$, $f \in \mathcal{NV_H}(\alpha, \gamma)$ *if and only if*

$$\sum_{k=2}^{\infty} \frac{2k}{1-\gamma}|a_k| + \sum_{k=1}^{\infty} \frac{2k}{1-\gamma}|b_k| \leq 1. \tag{2.10}$$

Theorem 2.6. *([17]) For* $a_1 = 1$ *and* $0 \leq \gamma < 1$, $f = h + \overline{g} \in \mathcal{RV_H}(\alpha, \gamma)$ *if and only if*

$$\sum_{k=2}^{\infty} \frac{2k-1-\gamma}{1-\gamma}|a_k| + \sum_{k=1}^{\infty} \frac{2k+1+\gamma}{1-\gamma}|b_k| \leq 1. \tag{2.11}$$

Further, we recall the following Remarks:

Remark 2.7. In [16], it is also shown that $f \in \mathcal{GV_H}(\lambda, \alpha, \gamma)$ if and only if

$$|a_k| \leq \frac{1-\gamma}{2k - \lambda(1+\gamma)}, \quad k \geq 2,$$

$$|b_k| \leq \frac{1-\gamma}{2k + \lambda(1+\gamma)}, \quad k \geq 1.$$

Accepting $\lambda = 0$, it is expressed as follows.

Remark 2.8. $f \in \mathcal{NV_H}(\alpha, \gamma)$ if and only if $|a_k| \leq \frac{1-\gamma}{2k}$ and $|b_k| \leq \frac{1-\gamma}{2k}, \ \ k \geq 2$.

By taking $\lambda = 1$ we state the following.

Remark 2.9. ([17]) $f \in \mathcal{RV_H}(\alpha, \gamma)$ if and only if $|a_k| \leq \frac{1-\gamma}{2k-1-\gamma}$ and $|b_k| \leq \frac{1-\gamma}{2k+1+\gamma}, \ \ k \geq 2$.

3 Applications to Pascal Distribution Series

The Pascal distribution series is a current subject of study in Geometric Function Theory (e.g., see, [6], [7], [8], [9], [10]). Taking into account the consequences on relations between various subclasses of analytic and harmonic univalent functions by using hypergeometric functions (see [2], [3], [4], [12], [13], [14], [15] and [18]), we establish a number of relations between the classes $\mathcal{G}^0_{\mathcal{H}}(\lambda, \alpha, \gamma)$, $K^0_{\mathcal{H}}$, $S^{*,0}_{\mathcal{H}}$, and $\mathcal{N}^0_{\mathcal{H}}(\alpha, \gamma)$ by applying the convolution operator $P^{r,s}_{p,q}$ associated with Pascal distribution series are built.

Let us consider a non-negative discrete random variable $\mathcal{X}$ with a Pascal probability generating function

$$P(\mathcal{X} = k) = \binom{k+r-1}{r-1} p^k (1-p)^r, \quad k \in \{0, 1, 2, 3, ...\}$$

where p, r are named as the parameters.
Currently, a power series whose coefficients are probabilities of the Pascal distribution are introduced

$$P^r_p(z) = z + \sum_{k=2}^{\infty} \binom{k+r-2}{r-1} p^{k-1} (1-p)^r z^k, \quad (r \geq 1, \ 0 \leq p \leq 1, \ z \in \mathcal{U}). \tag{3.1}$$

Pay attention to, when the ratio test is utilized, it is deduced that the radius of convergence of the above power series is infinity. In the conclusion, the formulas used are as follows

$$\sum_{k=0}^{\infty} \binom{k+r-1}{r-1} p^k = \frac{1}{(1-p)^r}, \quad \sum_{k=0}^{\infty} \binom{k+r-2}{r-2} p^k = \frac{1}{(1-p)^{r-1}},$$

$$\sum_{k=0}^{\infty} \binom{k+r}{r} p^k = \frac{1}{(1-p)^{r+1}}, \quad \sum_{k=0}^{\infty} \binom{k+r+1}{r+1} p^k = \frac{1}{(1-p)^{r+2}}, \ |p| < 1.$$

Further, throughout this paper unless otherwise stated we let $r \geq 1$ and $0 \leq p < 1$.

Now, for $r, s \geq 1$ and $0 \leq p, q < 1$, the operator is being introduced

$$P_{p,q}^{r,s}(f)(z) = P_p^r(z) * h(z) + \overline{P_q^s(z) * g(z)} = H(z) + \overline{G(z)}$$

where

$$\begin{aligned} H(z) &= z + \sum_{k=2}^{\infty} \binom{k+r-2}{r-1} p^{k-1} (1-p)^r a_k z^k \\ G(z) &= b_1 z + \sum_{k=2}^{\infty} \binom{k+s-2}{s-1} q^{k-1} (1-q)^s b_k z^k \end{aligned} \tag{3.2}$$

and "$*$" represents the convolution (or Hadamard product) of power series.

To be able to build relations between harmonic convex functions and Goodman-Rønning-type harmonic univalent functions, the following conclusions are needed :

Lemma 3.1. *([9]) Let $r, s \geq 1$ and $0 \leq p, q < 1$. Also, let $f = h + \overline{g} \in \mathcal{H}$ is given by (1.3). If the inequalities*

$$\sum_{k=2}^{\infty} |a_k| + \sum_{k=1}^{\infty} |b_k| \leq 1, \quad (|b_1| < 1) \tag{3.3}$$

and

$$(1-p)^r + (1-q)^s \geq 1 + |b_1| + \frac{rp}{1-p} + \frac{sq}{1-q} \tag{3.4}$$

are hold, then $P_{p,q}^{r,s}(f) \in \mathcal{SH}^*$.

Lemma 3.2. *([5], [11]) If $f = h + \overline{g} \in K_{\mathcal{H}}^0$ where h and g are given by (1.2) with $b_1 = 0$, then*

$$|a_k| \leq \frac{k+1}{2}, \quad |b_k| \leq \frac{k-1}{2}, \quad k \geq 2.$$

Theorem 3.3. *Let $0 \leq \alpha < 1$, $r, s \geq 1$ and $0 \leq p, q < 1$. If the inequality*

$$\begin{aligned} &\frac{r(r+1)p^2}{(1-p)^2} + \frac{\left(4 - \frac{\lambda}{2}(1+\gamma)\right) rp}{1-p} + 2\left(1 - \frac{\lambda}{2}(1+\gamma)\right)\left[1 - (1-p)^r\right] \\ &+ \frac{s(s+1)q^2}{(1-q)^2} + \frac{\left(2 + \frac{\lambda}{2}(1+\gamma)\right) sq}{1-q} \leq 1 - \gamma \end{aligned} \tag{3.5}$$

is satisfied then $P_{p,q}^{r,s}\left(K_{\mathcal{H}}^0\right) \subset \mathcal{G}_{\mathcal{H}}^0(\lambda, \alpha, \gamma)$.

Proof. Let $f = h + \overline{g} \in K_{\mathcal{H}}^0$ where h and g are of the form (1.2) with $b_1 = 0$. We need to show that $P_{p,q}^{r,s}(f) = H + \overline{G} \in \mathcal{G}_{\mathcal{H}}^0(\lambda, \alpha, \gamma)$, where H and G defined by (3.2) with $B_1 = 0$ are analytic functions in $\mathcal{U}$. In view of Theorem 2.3, we need to prove that

$$\begin{aligned} \Phi_1 = &\textstyle\sum_{k=2}^{\infty} [2k - \lambda(1+\gamma)] \left|\binom{k+r-2}{r-1} (1-p)^r p^{k-1} a_k\right| \\ &+ \textstyle\sum_{k=2}^{\infty} [2k + \lambda(1+\gamma)] \left|\binom{k+s-2}{s-1} q^{k-1} (1-q)^s b_k\right| \leq 1 - \gamma. \end{aligned}$$

Considering Lemma 3.2, we get

$$\begin{aligned} \Phi_1 \leq \frac{1}{2} \Bigg\{ &\sum_{k=2}^{\infty} [2k - \lambda(1+\gamma)] (k+1) \binom{k+r-2}{r-1} (1-p)^r p^{k-1} \\ &+ \sum_{k=2}^{\infty} [2k + \lambda(1+\gamma)] (k-1) \binom{k+s-2}{s-1} (1-q)^s q^{k-1} \Bigg\} \end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=2}^{\infty}\left(k-\frac{\lambda}{2}(1+\gamma)\right)(k+1)\binom{k+r-2}{r-1}(1-p)^r p^{k-1} \\
&\quad +\sum_{k=2}^{\infty}\left(k+\frac{\lambda}{2}(1+\gamma)\right)(k-1)\binom{k+s-2}{s-1}(1-q)^s q^{k-1} \\
&= \sum_{k=2}^{\infty}\left[(k-1)(k-2)+\left(4-\frac{\lambda}{2}(1+\gamma)\right)(k-1)\right]\binom{k+r-2}{r-1}(1-p)^r p^{k-1} \\
&\quad +\sum_{k=2}^{\infty}2\left(1-\frac{\lambda}{2}(1+\gamma)\right)\binom{k+r-2}{r-1}(1-p)^r p^{k-1} \\
&\quad +\sum_{k=2}^{\infty}\left[(k-1)(k-2)+\left(2+\frac{\lambda}{2}(1+\gamma)\right)(k-1)\right]\binom{k+s-2}{s-1}(1-q)^s q^{k-1} \\
&= r(r+1)p^2(1-p)^r\sum_{k=3}^{\infty}\binom{k+r-2}{r+1}p^{k-3} \\
&\quad +\left(4-\frac{\lambda}{2}(1+\gamma)\right)rp(1-p)^r\sum_{k=2}^{\infty}\binom{k+r-2}{r}p^{k-2} \\
&\quad +2\left(1-\frac{\lambda}{2}(1+\gamma)\right)(1-p)^r\sum_{k=2}^{\infty}\binom{k+r-2}{r-1}p^{k-2} \\
&\quad +s(s+1)q^2(1-q)^s\sum_{k=3}^{\infty}\binom{k+s-2}{s+1}q^{k-3} \\
&\quad +\left(2+\frac{\lambda}{2}(1+\gamma)\right)sq(1-q)^s\sum_{k=2}^{\infty}\binom{k+s-2}{s}q^{k-2}
\end{aligned}
$$

$$
\begin{aligned}
&= r(r+1)p^2(1-p)^r\sum_{k=0}^{\infty}\binom{k+r+1}{r+1}p^k+\left(4-\frac{\lambda}{2}(1+\gamma)\right)rp(1-p)^r\sum_{k=0}^{\infty}\binom{k+r}{r}p^k \\
&\quad +2\left(1-\frac{\lambda}{2}(1+\gamma)\right)(1-p)^r\sum_{k=0}^{\infty}\binom{k+r-1}{r-1}p^k-2(1-\alpha)(1-p)^r \\
&\quad +s(s+1)q^2(1-q)^s\sum_{k=0}^{\infty}\binom{k+s+1}{s+1}q^k+\left(2+\frac{\lambda}{2}(1+\gamma)\right)sq(1-q)^s\sum_{k=0}^{\infty}\binom{k+s}{s}q^k \\
&= \frac{r(r+1)p^2}{(1-p)^2}+\frac{\left(4-\frac{\lambda}{2}(1+\gamma)\right)rp}{1-p}+2\left(1-\frac{\lambda}{2}(1+\gamma)\right)\left[1-(1-p)^r\right] \\
&\quad +\frac{s(s+1)q^2}{(1-q)^2}+\frac{\left(2+\frac{\lambda}{2}(1+\gamma)\right)sq}{1-q}.
\end{aligned}
$$

The last expression is bounded above by $(1-\gamma)$ by the given condition. Thus the proof of Theorem 3.3 is complete. □

Analogous to Theorem 3.3, we next find conditions of the class $S_{\mathcal{H}}^{*,0}$, with $\mathcal{G}_{\mathcal{H}}^{0}(\lambda,\alpha,\gamma)$. However, we first need the following result, which may be found in [10], [11].

Lemma 3.4. *If* $f=h+\overline{g}\in S_{\mathcal{H}}^{*,0}$ *where* h *and* g *are given by* (1.2) *with* $b_1=0$*, then*

$$
|a_k|\le\frac{(2k+1)(k+1)}{6},\quad |b_k|\le\frac{(2k-1)(k-1)}{6}.
$$

Theorem 3.5. *Suppose* $0 \le \alpha < 1$, $r, s \ge 1$ *and* $0 \le p, q < 1$. *If the inequality*

$$\frac{2r\,(r+1)\,(r+2)\,p^3}{(1-p)^3} + \frac{(15-\lambda(1+\gamma))\,r\,(r+1)\,p^2}{(1-p)^2}$$

$$+\frac{\left(24-\frac{9\lambda}{2}(1+\gamma)\right)rp}{1-p} + 6\left(1-\tfrac{\lambda}{2}(1+\gamma)\right)\left[1-(1-p)^r\right]$$

$$+\frac{2s\,(s+1)\,(s+2)\,q^3}{(1-q)^3} + \frac{(9+\lambda(1+\gamma))\,s\,(s+1)\,q^2}{(1-q)^2}$$

$$+\frac{\left(6+\frac{3\lambda}{2}(1+\gamma)\right)sq}{1-q} \le 3\,(1-\gamma)$$

is hold then then $P^{r,s}_{p,q}(S^{*,0}_{\mathcal{H}}) \subset \mathcal{G}^0_{\mathcal{H}}(\lambda,\alpha,\gamma)$.

Proof. Let $f = h + \overline{g} \in S^{*,0}_{\mathcal{H}}$ where h and g are given by (1.2) with $b_1 = 0$. We need to show that $P^{r,s}_{p,q}(f) = H + \overline{G} \in \mathcal{G}^0_{\mathcal{H}}(\lambda,\alpha,\gamma)$, where H and G defined by (3.2) with $B_1 = 0$ are analytic functions in $\mathcal{U}$. In view of Theorem 2.3, it is enough to show that

$$\Phi_2 = \textstyle\sum_{k=2}^{\infty}\left[2k-\lambda(1+\gamma)\right]\binom{k+r-2}{r-1}(1-p)^r\,p^{k-1}\,|a_k|$$

$$+\textstyle\sum_{k=2}^{\infty}\left[2k+\lambda(1+\gamma)\right]\binom{k+s-2}{s-1}(1-q)^s\,q^{k-1}\,|b_k| \le 1-\gamma.$$

In view of Lemma 3.4, we have

$$\begin{aligned}
\Phi_2 \le\ & \frac{1}{3}\Bigg\{\sum_{k=2}^{\infty}\left(k-\frac{\lambda}{2}(1+\gamma)\right)(2k+1)\,(k+1)\binom{k+r-2}{r-1}(1-p)^r\,p^{k-1}\\
&+\sum_{k=2}^{\infty}\left(k+\frac{\lambda}{2}(1+\gamma)\right)(2k-1)\,(k-1)\binom{k+s-2}{s-1}(1-q)^s\,q^{k-1}\Bigg\}
\end{aligned}$$

$$\begin{aligned}
=\ & \frac{1}{3}\Bigg\{2\sum_{k=2}^{\infty}\binom{k+r-2}{r-1}(k-1)\,(k-2)\,(k-3)\,(1-p)^r\,p^{k-1}\\
&+(15-\lambda(1+\gamma))\sum_{k=2}^{\infty}\binom{k+r-2}{r-1}(k-1)\,(k-2)\,(1-p)^r\,p^{k-1}\\
&+\left(24-\frac{9\lambda}{2}(1+\gamma)\right)\sum_{k=2}^{\infty}\binom{k+r-2}{r-1}(k-1)\,(1-p)^r\,p^{k-1}\\
&+6\left(1-\frac{\lambda}{2}(1+\gamma)\right)\sum_{k=2}^{\infty}\binom{k+r-2}{r-1}(1-p)^r\,p^{k-1}\\
&+2\sum_{k=2}^{\infty}\binom{k+s-2}{s-1}(k-1)\,(k-2)\,(k-3)\,(1-q)^s\,q^{k-1}\\
&+(9+\lambda(1+\gamma))\sum_{k=2}^{\infty}\binom{k+s-2}{s-1}(k-1)\,(k-2)\,(1-q)^s\,q^{k-1}\\
&+\left(6+\frac{3\lambda}{2}(1+\gamma)\right)\sum_{k=2}^{\infty}\binom{k+s-2}{s-1}(k-1)\,(1-q)^s\,q^{k-1}\Bigg\}
\end{aligned}$$

$$
\begin{aligned}
= \quad & \frac{1}{3}\Bigg\{2r\,(r+1)\,(r+2)\,p^3\,(1-p)^r\sum_{k=4}^{\infty}\binom{k+r-2}{r+2}p^{k-4} \\
& +(15-\lambda(1+\gamma))\,r\,(r+1)\,p^2\,(1-p)^r\sum_{k=3}^{\infty}\binom{k+r-2}{r+1}p^{k-3} \\
& +\left(24-\frac{9\lambda}{2}(1+\gamma)\right)rp\,(1-p)^r\sum_{k=2}^{\infty}\binom{k+r-2}{r}p^{k-2} \\
& +6\left(1-\frac{\lambda}{2}(1+\gamma)\right)\sum_{k=2}^{\infty}\binom{k+r-2}{r-1}(1-p)^r\,p^{k-1} \\
& +2s\,(s+1)\,(s+2)\,q^3\,(1-q)^s\sum_{k=4}^{\infty}\binom{k+s-2}{s+2}q^{k-4} \\
& +(9+\lambda(1+\gamma))\,s\,(s+1)\,q^2\,(1-q)^s\sum_{k=3}^{\infty}\binom{k+s-2}{s+1}q^{k-3} \\
& +\left(6+\frac{3\lambda}{2}(1+\gamma)\right)sq\,(1-q)^s\sum_{k=2}^{\infty}\binom{k+s-2}{s}q^{k-2}\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{1}{3}\Bigg\{2r\,(r+1)\,(r+2)\,p^3\,(1-p)^r\sum_{k=0}^{\infty}\binom{k+r+2}{r+2}p^{k} \\
& +(15-\lambda(1+\gamma))\,r\,(r+1)\,p^2\,(1-p)^r\sum_{k=0}^{\infty}\binom{k+r+1}{r+1}p^{k} \\
& +\left(24-\frac{9\lambda}{2}(1+\gamma)\right)rp\,(1-p)^r\sum_{k=0}^{\infty}\binom{k+r}{r}p^{k} \\
& +6\left(1-\frac{\lambda}{2}(1+\gamma)\right)(1-p)^r\sum_{k=0}^{\infty}\binom{k+r-1}{r-1}p^{k}-6\left(1-\frac{\lambda}{2}(1+\gamma)\right)(1-p)^r \\
& +2s\,(s+1)\,(s+2)\,q^3\,(1-q)^s\sum_{k=0}^{\infty}\binom{k+s+2}{s+2}q^{k} \\
& +(9+\lambda(1+\gamma))\,s\,(s+1)\,q^2\,(1-q)^s\sum_{k=0}^{\infty}\binom{k+s+1}{s+1}q^{k} \\
& +\left(6+\frac{3\lambda}{2}(1+\gamma)\right)sq\,(1-q)^s\sum_{k=0}^{\infty}\binom{k+s}{s}q^{k}\Bigg\}
\end{aligned}
$$

$$
\begin{aligned}
= \quad & \frac{1}{3}\Bigg\{\frac{2r\,(r+1)\,(r+2)\,p^3}{(1-p)^3}+\frac{(15-\lambda(1+\gamma))\,r\,(r+1)\,p^2}{(1-p)^2} \\
& +\frac{\left(24-\frac{9\lambda}{2}(1+\gamma)\right)rp}{1-p}+6\left(1-\frac{\lambda}{2}(1+\gamma)\right)[1-(1-p)^r \\
& +\frac{2s\,(s+1)\,(s+2)\,q^3}{(1-q)^3}+\frac{(9+\lambda(1+\gamma))\,s\,(s+1)\,q^2}{(1-q)^2}+\frac{\left(6+\frac{3\lambda}{2}(1+\gamma)\right)sq}{1-q}\Bigg\} \\
\leq \quad & 1-\gamma
\end{aligned}
$$

by the given condition. Now $\Phi_2 \leq 1-\gamma$ follows from the given condition. □

In the next theorem, we establish connections between $\mathcal{GV}_{\mathcal{H}}(\lambda,\alpha,\gamma)$ and $\mathcal{GV}_{\mathcal{H}}(\lambda,\alpha,\gamma)$.

Theorem 3.6. *Let* $0 \leq \gamma < 1$, $r, s \geq 1$ *and* $0 \leq p, q < 1$. *If the inequality*

$$(1-p)^r + (1-q)^s \geq 1 + \frac{(2+\lambda(1+\gamma))\,|b_1|}{1-\gamma} \tag{3.6}$$

is satisfied, then $P^{r,s}_{p,q}\left(\mathcal{GV}_{\mathcal{H}}(\lambda,\alpha,\gamma)\right) \subset \mathcal{GV}_{\mathcal{H}}(\lambda,\alpha,\gamma)$.

Proof. Suppose $f = h + \overline{g} \in \mathcal{GV}_{\mathcal{H}}(\lambda,\alpha,\gamma)$ where h and g are provided by (2.7). We need to prove that the function

$$\begin{aligned} P^{r,s}_{p,q}(f)(z) &= z - \sum_{k=2}^{\infty}\binom{k+r-2}{r-1}(1-p)^r p^{k-1}|a_k|\, z^k \\ &\quad + |b_1|\,\overline{z} + \sum_{k=2}^{\infty}\binom{k+s-2}{s-1}(1-q)^s q^{k-1}|b_k|\,\overline{z}^k \end{aligned}$$

is in $\mathcal{GV}_{\mathcal{H}}(\lambda,\alpha,\gamma)$ if $\Phi_3 \leq 1-\alpha$, where

$$\begin{aligned} \Phi_3 &= \sum_{k=2}^{\infty}(2k-\lambda(1+\gamma))\binom{k+r-2}{r-1}(1-p)^r p^{k-1}|a_k| \\ &\quad + (2+\lambda(1+\gamma))\,|b_1| + \sum_{k=2}^{\infty}(2k+\lambda(1+\gamma))\binom{k+s-2}{s-1}(1-q)^s q^{k-1}|b_k|. \end{aligned}$$

By Remark 2.7, we have

$$\begin{aligned} \Phi_3 &\leq (1-\gamma)\left\{\sum_{k=2}^{\infty}\binom{k+r-2}{r-1}(1-p)^r p^{k-1}\right. \\ &\quad \left. + \sum_{k=2}^{\infty}\binom{k+s-2}{s-1}(1-q)^s q^{k-1}\right\} + (2+\lambda(1+\gamma))\,|b_1| \\ &= (1-\gamma)\left\{(1-p)^r\sum_{k=0}^{\infty}\binom{k+r-1}{r-1}p^k - (1-p)^r\right. \\ &\quad \left. + (1-q)^s\sum_{k=0}^{\infty}\binom{k+s-1}{s-1}q^k - (1-q)^s\right\} + (2+\lambda(1+\gamma))\,|b_1| \\ &= (1-\gamma)\left\{2-(1-p)^r-(1-q)^s\right\} + (2+\lambda(1+\gamma))\,|b_1| \\ &\leq 1-\gamma \end{aligned}$$

by the given condition and thus the proof of the theorem is complete. □

Remark 3.7. By suitably specializing the parameter λ, one can deduce the results for the subclasses $\mathcal{NV}_{\mathcal{H}}(\alpha,\gamma)$ and $f \in \mathcal{RV}_{\mathcal{H}}(\alpha,\gamma)$ which are defined, respectively, in Example 2.1 and 2.2 and associated with the Pascal distribution series. The details involved may be given as a practice for the reader willing.

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