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Left annihilator of identity with pair of generalized derivations in prime and semiprime rings

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Abstract Let π_1 and π_2 be two generalized derivations of a ring $\mathbb{R}$ with associated derivations δ_1 and δ_2 respectively. Let $m, n \geq 1$ are fixed positive integers and $\mathbb{K}$ be a nonzero ideal of $\mathbb{R}$. In the present paper we discuss the left annihilator of the following two sets: $\{\pi_1(a) \circ_m \pi_2(b) - a \circ_m b | \ \ a, b \in \mathbb{K}\}$ and $\{[\pi_1(a), b]_m + [a, \delta_1(b)]_n - [a, b]| \ \ a, b \in \mathbb{K}\}$ and give a characterization of π_1 and π_2. Moreover, we examine the case when $\mathbb{R}$ is a semiprime ring. Finally, we provide examples to show that various restrictions imposed in the hypotheses of our theorems are not superfluous.

1 Introduction

Throughout the paper $\mathbb{R}$ is always an associative ring with centre $\mathbb{Z}(\mathbb{R})$, $\mathbb{C}$ the extended centroid of $\mathbb{R}$, $\mathbb{U}$ its Utumi quotient ring and $\mathbb{Q}$ is the Martindale ring of quotients of $\mathbb{R}$. Let $x, y \in \mathbb{R}$, $[x, y]$ and $x \circ y$ stand for commutator $xy - yx$ and anti-commutator $xy + yx$ respectively. Also, we set $x \circ_0 y = x$, $x \circ_1 y = x \circ y = xy + yx$ and $x \circ_m y = (x \circ_{m-1} y)y + y(x \circ_{m-1} y)$ for $m \geqslant 2$ and $[x, y]_0 = x, [x, y]_1 = xy - yx$ and $[x, y]_m = [x, y]_{m-1}y - y[x, y]_{m-1}, m \geq 2$ in non-commuting indeterminates x and y. Recall that a ring $\mathbb{R}$ is prime if $x\mathbb{R}y = \{0\}$ gives that either $x = 0$ or $y = 0$ for all $x, y \in \mathbb{R}$ and is semiprime if $x\mathbb{R}x = \{0\}$ gives that $x = 0$ for all $x \in \mathbb{R}$. Our target is to establish a relation between structure of ring and the nature of favorable mapping defined on it. A map $\delta : \mathbb{R} \to \mathbb{R}$ is called a derivation of $\mathbb{R}$ if δ is additive and $\delta(ab) = \delta(a)b + a\delta(b)$ for any $b, a \in \mathbb{R}$. If δ can be expressed as $\delta(a) = [b, a]$ for some element $b \in \mathbb{R}$, then δ is called an inner derivation. We use generally the notation $I_b(a)$ to denote inner derivation. By a generalized inner derivation on $\mathbb{R}$, we mean a self mapping π on $\mathbb{R}$ if π is additive and $\pi(a) = ba + ac$ for some fixed $b, c \in \mathbb{R}$. For suchlike mapping π, we can see that $\pi(ab) = a[c, b]+\pi(a)b = aI_c(b)+\pi(a)b$, where I_c denotes the inner derivation. This observation gives the following definition: a map $\pi : \mathbb{R} \to \mathbb{R}$ is said to be a generalized derivation on $\mathbb{R}$ if $\pi(zw) = \pi(z)w + z\delta(w)$ for all $w, z \in \mathbb{R}$, where δ is a derivation on $\mathbb{R}$.

Ashraf et al. [1] investigates the commutativity of a prime ring $\mathbb{R}$ admitting a derivation δ satisfying $\delta(a) \circ \delta(b) = a \circ b$ for all $a, b \in \mathbb{I}$, where $\mathbb{I}$ is a nonzero ideal of $\mathbb{R}$. Further, Huang [11] proved that if $\mathbb{L}$ is a square closed Lie ideal of a prime ring $\mathbb{R}$ with characteristic different from 2 and generalized derivation π with associated derivation δ satisfying $\pi(a) \circ \delta(b) = a \circ b$ for all $a, b \in \mathbb{L}$, then either R is commutative or $\delta = 0$.

Motivated by the above mentioned results, we prove the following:

Theorem 1.1. Let m be the fixed positive integer and $\mathbb{K}$ be a nonzero ideal of a prime ring $\mathbb{R}$ with characteristic different from 2. If $\mathbb{R}$ admits generalized derivations π_1 and π_2 with associated derivations δ_1 and δ_2 respectively and $0 \neq a \in \mathbb{R}$ such that $a(\pi_1(x) \circ_m \pi_2(y) - x \circ_m y) = 0$ for all $x, y \in \mathbb{K}$, then either $\mathbb{R}$ is commutative or there exist α and $\beta \in \mathbb{C}$, extended centroid of $\mathbb{R}$ such that $\pi_1(x) = \alpha x$ and $\pi_2(x) = \beta x$ for all $x \in \mathbb{R}$ with $a(\alpha^m\beta^m - 1) = 0$.

Huang [10] proved that if $\mathbb{K}$ is a nonzero ideal of a prime ring $\mathbb{R}$ with characteristic different from 2 admitting a nonzero derivation δ satisfying $[\delta(x), \delta(y)]_m = [x, y]^n$ for any $y, x \in \mathbb{K}$, for some positive integers m, n, then R is commutative. In this line of investigation, Dhara et al. [2] proved the following: Let K be a nonzero ideal of a 2-torsion free semiprime ring

R admitting a generalized derivation π with associated derivation δ such that $\delta(\mathbb{K}) \neq \{0\}$. If $[\delta(y), \pi(x)] = \pm[y, x]$ holds for all $x, y \in \mathbb{K}$, then $\mathbb{R}$ contains a nonzero central ideal.

Tendentious by the above results, we prove

Theorem 1.2. Let m, n be fixed positive integers, $\mathbb{K}$ be a nonzero ideal of a prime ring $\mathbb{R}$ with characteristic different from 2 and $0 \neq a \in \mathbb{R}$. If π is a generalized derivation of $\mathbb{R}$ with associated derivation δ satisfying $a([\pi(x), y]_m + [x, \delta(y)]_n - [x, y]) = 0$ for all $x, y \in \mathbb{K}$, then either $\mathbb{R}$ is commutative or there exist $b \in \mathbb{U}$ such that $\pi(x) = bx$ for all $x \in \mathbb{R}$.

Theorem 1.3. Let m, n be fixed positive integers and $\mathbb{R}$ is a semiprime ring with characteristic different from 2 and $0 \neq a \in \mathbb{R}$. If π is a generalized derivation of $\mathbb{R}$ with associated derivation δ satisfying $a([\pi(x), y]_m + [x, \delta(y)]_n - [x, y]) = 0$ for all $x, y \in \mathbb{R}$, then $\mathbb{R}$ contains a nonzero central ideal.

2 Main Results

We will use frequently the following important result due to Kharchenko [15]:
Let $0 \neq \delta$ be a derivation of a prime ring $\mathbb{R}$ and $\{0\} \neq \mathbb{K}$ be an ideal of $\mathbb{R}$. Let $g(p_1, ..., p_n, \delta(p_1), ..., \delta(p_n))$ be a differential identity in $\mathbb{K}$ i.e,

$$g(w_1, ..., w_n, \delta(w_1), ..., \delta(w_n)) = 0 \text{ for all } w_1, w_2, ..., w_n \in \mathbb{K}.$$

Then we have exactly one of the following
(i) δ is an inner in $\mathbb{Q}$, Martindale ring of quotient of $\mathbb{R}$
(ii) δ is $\mathbb{Q}$-outer and the following GPI is satisfied by $\mathbb{K}$

$$g(w_1, ..., w_n, y_1, ..., y_n) = 0.$$

Remark 2.1. Let $\mathbb{K}$ be an ideal of $\mathbb{R}$. Then
(i) $\mathbb{U}$, $\mathbb{R}$ and $\mathbb{K}$ satisfy the same differential identities. [14, Theorem 2]
(ii) $\mathbb{U}$, $\mathbb{R}$ and $\mathbb{K}$ satisfy the same GPI with coefficients in U.[4, Theorem 2]

Remark 2.2. Let π be a generalized derivation defined on a dense right ideal of a semiprime ring $\mathbb{R}$. Then π can be uniquely extended to $\mathbb{U}$ which takes the form $\pi(x) = ax + \delta(x)$, where δ is a derivation on $\mathbb{U}$ and for some $a \in \mathbb{U}$. Moreover, a and δ are uniquely determined by the generalized derivation π. [13, Theorem 4]

Remark 2.3. Let $\mathbb{F}$ be a field, $\mathbb{R}$ a dense ring of $\mathbb{F}$-linear transformations (over a vector space $\mathbb{V}$) of $char(\mathbb{R}) \neq 2$ with $dim_{\mathbb{C}}\mathbb{V} \geq 2$, $p, c \in \mathbb{R}$, and $0 \neq c \notin \mathbb{Z}(\mathbb{R})$. Assume $pv = 0$, for any $v \in \mathbb{V}$ such that $\{v, cv\}$ is linear $\mathbb{F}$-independent. Then $p = 0$. [16, Lemma 2.1]

Proof of Theorem 1.1 By hypothesis

$$a(\pi_1(x) \circ_m \pi_2(y) - x \circ_m y) = 0 \text{ for all } x, y \in \mathbb{K}. \tag{2.1}$$

By Remark 2.2, $\pi_1(x) = bx + \delta_1(x)$ and $\pi_2(x) = cx + \delta_2(x)$ for some $b, a \in \mathbb{U}$ and derivations δ_1, δ_2 on $\mathbb{U}$. Hence

$$a((bx + \delta_1(x)) \circ_m (cy + \delta_2(y)) - x \circ_m y) = 0 \text{ for all } x, y \in \mathbb{K}. \tag{2.2}$$

By Remark 2.1, we have

$$a((bx + \delta_1(x)) \circ_m (cy + \delta_2(y)) - x \circ_m y) = 0 \text{ for all } x, y \in \mathbb{U} \tag{2.3}$$

that is

$$a(bx \circ_m cy + \delta_1(x) \circ_m cy + cx \circ_m \delta_2(y) + \delta_1(x) \circ_m \delta_2(y) - x \circ_m y) = 0 \tag{2.4}$$

for all $x, y \in \mathbb{U}$.
Here the proof is divided into the following cases:

Case 1 If both δ_1 and δ_2 are inner derivations, then $\delta_1(x) = [q, x]$ and $\delta_2(x) = [p, x]$ for any $x \in \mathbb{U}$ and for some q and $p \in \mathbb{U}$ respectively. So, we have

$$\mathbb{F}(x, y) = a(bx \circ_m cy + [q, x] \circ_m cy + cx \circ_m [p, y] + [q, x] \circ_m [p, y] - x \circ_m y) = 0 \text{ for any } y, x \in \mathbb{U}. \tag{2.5}$$

If $\mathbb{C}$ is infinite, then $\mathbb{U} \bigotimes_{\mathbb{C}} \bar{\mathbb{E}}$ satisfies (2.5), where $\bar{\mathbb{E}}$ stands for algebraic closure of $\mathbb{C}$. By [12], $\mathbb{U}$ and $\mathbb{U} \otimes_{\mathbb{C}} \mathbb{E}$ are centrally closed and prime. Therefore, we may replace $\mathbb{R}$ by $\mathbb{U} \otimes_{\mathbb{C}} \mathbb{E}$ or $\mathbb{U}$ according to $\mathbb{C}$ is infinite or finite. Thus we may assume that $\mathbb{R}$ is centrally closed over $\mathbb{C}$ which is either algebraically closed and $\mathbb{F}(x, y) = 0$ for any $x, y \in \mathbb{R}$ or finite. By the use of Martindale's theorem [12], $\mathbb{R}$ is primitive ring with $\mathbb{D}$ as associative division ring as well as $\mathbb{R}$ has nonzero socle, $soc(\mathbb{R})$. By [9], $\mathbb{R}$ and dense ring of linear transformations for some vector space $\mathbb{V}$ over $\mathbb{C}$ are isomorphic i.e $\mathbb{R} \cong \mathbb{M}_k(\mathbb{D})$, where $k = dim_{\mathbb{D}}\mathbb{V}$. Assume that $dim_{\mathbb{D}}\mathbb{V} \geqslant 2$, otherwise we are done. Also assume that there exists $v \in \mathbb{V}$ such that qv and v are linearly $\mathbb{D}$-independent.

If pv is not a member of the span of $\{v, qv\}$, then $\{v, pv, qv\}$ is linearly independent. By the density of ring $\mathbb{R}$, there exist $x, y \in \mathbb{R}$ such that

$$xqv = -v, xv = 0, ypv = v, yv = 0, xpv = 0, yqv = v. \tag{2.6}$$

Multiplying equation (2.5) by v from right and using conditions in equation (2.6), we get

$$a(-1)^{m-1}2^{m-1}v = 0.$$

Since $\mathbb{R}$ has characteristic different from 2, we have $av = 0$. If $a \in \mathbb{Z}(\mathbb{R})$, then $v = 0$, a contradiction. If $a \notin \mathbb{Z}(\mathbb{R})$, then by Remark 2.3, we have $a = 0$, again a contradiction.

If pv is a member of the span of $\{v, qv\}$, then $p = v\alpha + qv\beta$ for some $\alpha, 0 \neq \beta \in \mathbb{D}$. Again by the density of ring $\mathbb{R}$, there exist $x, y \in \mathbb{R}$ such that

$$xv = 0, yqv = v, xqv = -v, yv = 0. \tag{2.7}$$

Again multiplying equation (2.5) by v from right and using conditions in equations (2.7), we get

$$a(-1)^{m-1}2^{m-1}v\beta = 0.$$

Again using that $\mathbb{R}$ has characteristic different from 2, we have $av = 0$. Using the same arguments as used, we get $a = 0$, a contradiction.

Therefore, $\{v, qv\}$ is linearly dependent over $\mathbb{D}$ and hence $q \in \mathbb{Z}(\mathbb{R})$ i.e $\delta_1 = 0$. Similarly, we can show that $\delta_2 = 0$. From (2.4), we have the following

$$a(bx \circ_m cy - x \circ_m y) = 0 \text{ for all } x, y \in \mathbb{U}. \tag{2.8}$$

Let for any $u \in \mathbb{V}$, $\{u, bu\}$ is linearly independent. Since $dim_{\mathbb{D}}\mathbb{V} \geqslant 2$, we can choose $t \in \mathbb{V}$ such that $\{u, bu, t\}$ is also linearly independent. By density of $\mathbb{R}$, there exist $x, y \in \mathbb{R}$ such that

$$xu = 0, xbu = 0, xt = u, yu = t, ybu = 0, yt = 0. \tag{2.9}$$

Now mulplying (2.8) by u from right, we get $au = 0$. Using the arguments that have been used above, we get contradiction. Therefore, $\{u, bu\}$ is linearly dependent i.e $b \in \mathbb{C}$. Similary, we can show that $c \in \mathbb{C}$. Using these in (2.8), we get

$$a(b^m c^m - 1)x \circ_m y = 0. \tag{2.10}$$

In particular, for $x = y$, we have $a(b^m c^m - 1)x^{m+1} = 0$. Using primeness of $\mathbb{R}$, we get $a(b^m c^m - 1) = 0$

Case 2 Let δ_1 and δ_2 are not both inner derivations of U. Then δ_1 and δ_2 are $\mathbb{C}$-linearly dependent modulo $\mathbb{D}_{int}$ i.e $\delta_2(y) = [p, y] + \beta\delta_1(y)$ for some $p \in \mathbb{U}$ and $\beta \in \mathbb{C}$. If either $\beta = 0$ or δ_2 is inner, then δ_1 is also inner which is a contradiction. So, $\beta \neq 0$ and δ_2 is not inner. Then by (2.4), we have

$$a(bx \circ_m cy + \delta_1(x) \circ_m cy + cx \circ_m ([p, y] + \beta\delta_1(y)) + \delta_1(x) \circ_m ([p, y] + \beta\delta_1(y)) - x \circ_m y) = 0 \text{ for any } x, y \in U.$$

Use of Kharchenko's Theorem [15] gives that

$$a(bx \circ_m cy + x_1 \circ_m cy + cx \circ_m ([p,y] + \beta y_1) + x_1 \circ_m ([p,y] + \beta y_1) - x \circ_m y) = 0$$

for all $x_1, y_1, x, y \in \mathbb{K}$. Taking $y = 0 = x$, we obtain

$$a(x_1 \circ_m y_1) = 0 \tag{2.11}$$

for all $x_1, y_1 \in I$. By [4, Theorem 2], $\mathbb{Q}$ as well as $\mathbb{R}$ satisfy the polynomial identity $a(x_1 \circ_m y_1) = 0$. By [3, Lemma 1], we have $\mathbb{R} \subseteq \mathbb{M}_n(\mathbb{F})$, the ring of $n \times n$ matrices over some field $\mathbb{F}$, where $n \geq 1$. Also, $\mathbb{M}_n(\mathbb{F})$ and $\mathbb{R}$ satisfy the same polynomial identity, i.e, $a(x_1 \circ_m y_1) = 0$, for any $x_1, y_1 \in \mathbb{M}_n(\mathbb{F})$. To denote matrix unit with 1 in $(i,j)^{th}$-entry and zero elsewhere, we use the notation e_{ij}. Taking $y_1 = e_{11}, a = e_{11}x_1 = e_{12}$, we see that $e_{11}(x_1 \circ_m y_1) = e_{12} \neq 0$, a contradiction.

The case $\delta_1(x) = [q,x] + \gamma\delta_2(x)$ for some $\gamma \in \mathbb{C}$ and $q \in \mathbb{U}$ is analogous.

Case 3 Now assume δ_1 and δ_2 are Outer. Now by Kharchenko's Theorem [15], we have

$$a(bx \circ_m cy + x_1 \circ_m cy + cx \circ_m y_1 + x_1 \circ_m y_1 - x \circ_m y) = 0$$

for any $x_1, y_1, x, y \in \mathbb{K}$. For $y = x = 0$, we have

$$a(x_1 \circ_m y_1) = 0 \tag{2.12}$$

which is same as (2.11). Therefore, by the similar arguments as above this leads that R is commutative. This overpast the proof of theorem.

If we take $\pi_1 = \pi_2 = \pi$, we have the following corollary:

Corollary 2.4. Let m be the fixed positive integer and $\mathbb{K}$ be a nonzero ideal of a prime ring $\mathbb{R}$ with characteristic different from 2. If $\mathbb{R}$ admits a generalized derivation π with associated derivation δ and $0 \neq a \in \mathbb{R}$ such that $a(\pi(x) \circ_m \pi(y) - x \circ_m y) = 0$ for all $x, y \in \mathbb{K}$, then either $\mathbb{R}$ is commutative or there exists $\alpha \in \mathbb{C}$, extended centroid of $\mathbb{R}$ such that $\pi(x) = \alpha x$ for all $x \in \mathbb{R}$ with $a(\alpha^{2m} - 1) = 0$.

Proof of Theorem 1.2 By hypothesis

$$a([\pi(x), y]_m + [x, \delta(y)]_n - [x,y]) = 0 \text{ for any } y, x \in \mathbb{K}. \tag{2.13}$$

By Remark 2.1, we have

$$a([\pi(x), y]_m + [x, \delta(y)]_n - [x,y]) = 0 \text{ for any } y, x \in \mathbb{U}. \tag{2.14}$$

By Remark 2.2, $\pi(x) = bx + \delta(x)$ for some $b \in \mathbb{U}$ and derivation δ on $\mathbb{U}$. Then we have

$$a([bx + \delta(x), y]_m + [x, \delta(y)]_n - [x,y]) = 0 \text{ for any } y, x \in \mathbb{U}. \tag{2.15}$$

That is

$$a([bx, y]_m + [\delta(x), y]_m + [x, \delta(y)]_n - [x,y]) = 0 \text{ for any } y, x \in \mathbb{U}. \tag{2.16}$$

The proof is divided into the following cases on the basis of Kharchenko's theorem [15, Theorem 2]:

Case I Let δ be an inner derivation i.e $\delta(x) = [q,x]$ for any $x \in \mathbb{U}$ and for some $q \in \mathbb{U}$. Then

$$\mathbb{F}(x,y) = a([bx, y]_m + [\delta(x), y]_m + [x, \delta(y)]_n - [x,y]) = 0 \text{ for any } y, x \in \mathbb{U}. \tag{2.17}$$

If $\mathbb{C}$ is infinite, then $\mathbb{U} \bigotimes_{\mathbb{C}} \bar{\mathbb{E}}$ satisfies (2.5), where $\bar{\mathbb{E}}$ stands for algebraic closure of $\mathbb{C}$. By [12], $\mathbb{U}$ and $\mathbb{U} \otimes_{\mathbb{C}} \mathbb{E}$ are centrally closed and prime. Therefore, we may replace $\mathbb{R}$ by $\mathbb{U} \otimes_{\mathbb{C}} \mathcal{E}$ or $\mathbb{U}$ according to $\mathbb{C}$ is infinite or finite. Thus we may assume that $\mathbb{R}$ is centrally closed over $\mathbb{C}$ which is either algebraically closed and $\mathbb{F}(x,y)=0$ for any $y,x\in\mathbb{R}$ or finite. By the use of Martindale's theorem [12], $\mathbb{R}$ is primitive ring with $\mathbb{D}$ as associative division ring as well as $\mathbb{R}$ has nonzero socle, $soc(\mathbb{R})$. By [9], $\mathbb{R}$ and dense ring of linear transformations for some vector space $\mathbb{V}$ over $\mathbb{C}$ are isomorphic i.e $\mathbb{R}\cong\mathbb{M}_k(\mathbb{D})$, where $k=dim_{\mathbb{D}}\mathbb{V}$. Assume that $dim_{\mathbb{D}}\mathbb{V}\geqslant 2$, otherwise we are done. Also assume that there exists $v\in\mathbb{V}$ such that qv and v are linearly $\mathbb{D}$-independent.

Since $dim_{\mathbb{D}}\mathbb{V}\geqslant 2$, we can find an element $w\in\mathbb{V}$ such that $\{w,qv,v\}$ is linearly independent over $\mathbb{D}$. By the density of the ring $\mathbb{R}$, we can find $x,y\in\mathbb{R}$ such that

$$xv=0, yw=v, xqv=w, xw=0, yv=0, yqv=v. \tag{2.18}$$

Multiplying equation (2.17) from right by v and using conditions in equation (2.18), we get $av=0$. By the same argument that we have used in preecedant, we have $\{qv,v\}$ is linearly dependent and hence $q\in Z(R)$ i.e $d=0$.

Case 2 Let d be an outer derivation. Then

$$a([bx,y]_m+[x_1,y]_m+[x,y_1]_n-[x,y])=0 \text{ for any } y,x,x_1,y_1,s\in\mathbb{K}. \tag{2.19}$$

In particular, choosing $y=0$, we get $a([x,y_1]_n)=0$ for any $y_1,x\in\mathbb{K}$. By [4, Theorem 2], $\mathbb{Q}$ as well as $\mathbb{R}$ satisfy the polynomial identity $a([x,y_1]_n)=0$. By [3, Lemma 1], we have $\mathbb{R}\subseteq\mathbb{M}_n(\mathbb{F})$, the ring of $n\times n$ matrices over some field $\mathbb{F}$, where $n\geq 1$. Also, $\mathbb{M}_n(\mathbb{F})$ and $\mathbb{R}$ satisfy the same polynomial identity, i.e, $a([x,y_1]_n)=0$, for any $x,y_1\in\mathbb{M}_n(\mathbb{F})$. To denote matrix unit with 1 in $(i,j)^{th}$-entry and zero elsewhere, we use the notation e_{ij}. Taking $y_1=e_{11}, a=e_{11}x_1=e_{12}$, we see that $e_{11}([x,y_1]_m)=e_{12}\neq 0$, a contradiction.

Proof of Theorem 1.3 We know that any derivation defined on $\mathbb{R}$, a semiprime ring can be uniquely extended to a derivation on $\mathbb{U}$, left Utumi ring of quotient of $\mathbb{R}$ and hence every derivation of $\mathbb{R}$ can be defined on $\mathbb{U}$ [14, Lemma 2]. Also, $\mathbb{U}$ and $\mathbb{R}$ satisfy the same generalized polynomial identity (GPI) and differential identities (see [4] and [14]). By [13, Theorem 4], π can be expressed as $\pi(x)=\delta(x)+bx$ for some $b\in\mathbb{U}$ and a derivation δ defined on $\mathbb{U}$. We have

$$a([bx,y]_m+[\delta(x),y]_m+[x,\delta(y)]_n-[x,y])=0 \text{ for any } y,x\in U. \tag{2.20}$$

Let $\mathbb{M}(\mathbb{C})=\{\mathbb{A}\mid\mathbb{A}\text{ is maximal ideal of }\mathbb{C}\}$ and let $\mathbb{P}\in\mathbb{M}(\mathbb{C})$. Then $\mathbb{PU}$ is prime ideal of $\mathbb{U}$ which is invariant under all derivation of $\mathbb{U}$ by the theory of orthogonal completions of semiprime ring ([14, p.31-32]). Also, $\bigcap\{\mathbb{PU}\mid\mathbb{P}\in\mathbb{M}(\mathbb{C})\}=\{0\}$. Setting $\overline{\mathbb{U}}=\mathbb{U}/\mathbb{PU}$. Now any derivation δ of $\mathbb{R}$ canonically induces a derivation $\overline{\delta}$ on $\overline{\mathbb{U}}$ defined by $\overline{\delta}(\overline{x})=\overline{\delta(x)}$ for any $x\in\bar{\mathbb{U}}$. Then

$$\bar{a}([\bar{b}\bar{x},\bar{y}]_m+[\overline{\delta(x)},\bar{y}]_m+[\bar{x},\overline{\delta(y)}]_n-[x,y])=0$$

for all $\overline{x},\overline{y}\in\overline{\mathbb{U}}$. It is clear that $\overline{\mathbb{U}}$ is a prime ring. So by the use of Theorem 1.2, we have, either $[\mathbb{U},\mathbb{U}]\subseteq\mathbb{PU}$ or $\delta(\mathbb{U})\subseteq\mathbb{PU}$ for any $\mathbb{P}\in\mathbb{M}(\mathbb{C})$. This gives that $\delta(\mathbb{U})[\mathbb{U},\mathbb{U}]\subseteq\mathbb{PU}$ for any $\mathbb{P}\in\mathbb{M}(\mathbb{C})$. Since $\bigcap\{\mathbb{U}\mid\mathbb{P}\in\mathbb{M}(\mathbb{C})\}=\{0\}$, we have $\delta(\mathbb{U})[\mathbb{U},\mathbb{U}]=\{0\}$. In particular, we have $\delta(\mathbb{R})[\mathbb{R},\mathbb{R}]=\{0\}$. Further, this can be written as $[\delta(\mathbb{R}),\mathbb{R}]\mathbb{R}[\delta(\mathbb{R}),\mathbb{R}]=0$. Since $\mathbb{R}$ is a semiprime ring, we obtain that $[\delta(\mathbb{R}),\mathbb{R}]=0$. Then by [17, Theorem 3], $\mathbb{R}$ contains a nonzero central ideal.

The following examples demonstrate that $\mathbb{R}$ to be *prime* can not be omitted in the hypothesis of Theorem 1.1 and Theorem 1.2.

Example 2.5. For any ring $\mathbb{R}_1$ which has characteristic different from two, let $R=\left\{\begin{pmatrix} z & w\\ 0 & 0\end{pmatrix}\mid z,w\in\mathbb{R}_1\right\}$ and $\mathbb{K}=\left\{\begin{pmatrix} 0 & w\\ 0 & 0\end{pmatrix}\mid w\in\mathbb{R}_1\right\}$. Then $\mathbb{K}$ is a nonzero ideal of $\mathbb{R}$. Define maps $\pi_1,\pi_2,\delta_2,\delta_1:\mathbb{R}\to\mathbb{R}$ by $\pi_1\left(\begin{pmatrix} z & w\\ 0 & 0\end{pmatrix}\right)=\begin{pmatrix} z & 2w\\ 0 & 0\end{pmatrix}$, $\pi_2\left(\begin{pmatrix} z & w\\ 0 & 0\end{pmatrix}\right)=\begin{pmatrix} z & 0\\ 0 & 0\end{pmatrix}$, $\delta_1\left(\begin{pmatrix} z & w\\ 0 & 0\end{pmatrix}\right)=\begin{pmatrix} 0 & -w\\ 0 & 0\end{pmatrix}$ and $\delta_2\left(\begin{pmatrix} z & w\\ 0 & 0\end{pmatrix}\right)=\begin{pmatrix} 0 & w\\ 0 & 0\end{pmatrix}$ Then π_1

and π_2 are generalized derivations on $\mathbb{R}$ associated with derivations δ_1 and δ_2 respectively satisfying $a(\pi_1(x) \circ_m \pi_2(y) - x \circ_m y) = 0$ for all $x, y \in \mathbb{K}$. However neither $\mathbb{R}$ is commutative nor $\pi_1(x) = \alpha x$ and $\pi_2(x) = \beta x$ for all $x \in \mathbb{R}$ as δ_1 and δ_2 are nonzero. Hence Theorem 1.1 is not true for arbitrary rings.

Example 2.6. Let $\mathbb{R} = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \mid x, y, z \in \mathbb{R}_1 \right\}$ and $\mathbb{K} = \left\{ \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \mid y \in \mathbb{R}_1 \right\}$, where $\mathbb{R}_1$ is a ring which has characteristic different from two. Then $\mathbb{K}$ is a nonzero ideal of $\mathbb{R}$. Define maps $\pi, \delta : \mathbb{R} \to \mathbb{R}$ by $\pi\left(\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\right) = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$ and $\delta\left(\begin{pmatrix} x & y \\ 0 & z \end{pmatrix}\right) = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$. Then π is a generalized derivation associated with the derivation δ satisfying $a([\pi(x), y]_m + [x, \delta(y)]_n - [x, y]) = 0$ for all $x, y \in \mathbb{K}$. However neither $\mathbb{R}$ is commutative nor $\pi(x) = bx$ for all $x \in \mathbb{R}$ as δ is nonzero. Hence Theorem 1.2 does not hold for arbitrary rings.

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