Left annihilator of identity with pair of generalized derivations in prime and semiprime rings

Asma Ali, 1 Md Hamidur Rahaman and Farhat Ali

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Abstract Let $\pi_1$ and $\pi_2$ be two generalized derivations of a ring $R$ with associated derivations $\delta_1$ and $\delta_2$ respectively. Let $m, n \geq 1$ are fixed positive integers and $k$ be a nonzero ideal of $R$. 
In the present paper we discuss the left annihilator of the following two sets: \{ $\pi_1(a) \circ_m \pi_2(b) - a \circ_m b | a, b \in k$\} and \{ $[\pi_1(a), b]_m + [a, \delta_1(b)]_n - [a, b] | a, b \in k$\} and give a characterization of $\pi_1$ and $\pi_2$. Moreover, we examine the case when $R$ is a semiprime ring. Finally, we provide examples to show that various restrictions imposed in the hypotheses of our theorems are not superfluous.

1 Introduction

Throughout the paper $R$ is always an associative ring with centre $Z(R)$, $C$ the extended centroid of $R$, $U$ its Utumi quotient ring and $Q$ the Martindale ring of quotients of $R$. Let $x, y \in R$, $[x, y]$ and $x \circ y$ stand for commutator $xy - yx$ and anti-commutator $xy +yx$ respectively. Also, we set $x_0 = x$, $x_1 = y = x \circ y = xy +yx$ and $x_0 = (x_0-1)y + y(x_0-1)y$ for $m \geq 2$ and $[x, y]_0 = x$, $[x, y]_1 = xy -yx$ and $[x, y]_m = [x,y]_m - y[x,y]_m$ for all $x, y \in R$ in non-commuting indeterminates $x$ and $y$. Recall that a ring $R$ is prime if $xRy = \{0\}$ gives that either $x = 0$ or $y = 0$ for all $x, y \in R$ and is semiprime if $xRx = \{0\}$ gives that $x = 0$ for all $x \in R$. Our target is to establish a relation between structure of ring and the nature of favorable mapping defined on it. A map $\delta : R \rightarrow R$ is called a derivation of $R$ if $\delta$ is additive and $\delta(ab) = \delta(a)b + a\delta(b)$ for any $a, b \in R$. If $\delta$ can be expressed as $\delta(a) = [b, a]$ for some element $b \in R$, then $\delta$ is called an inner derivation. We use generally the notation $I_b(a)$ to denote inner derivation. By a generalized inner derivation on $R$, we mean a self mapping $\pi$ on $R$ if $\pi$ is additive and $\pi(a) = ba + ac$ for some fixed $b, c \in R$. For suchlike mapping $\pi$, we can see that $\pi(ab) = a[c, b] + \pi(a)b = I_c(b) + \pi(a)b$, where $I_c$ denotes the inner derivation. This observation gives the following definition: a map $\pi : R \rightarrow R$ is said to be a generalized derivation on $R$ if $\pi(zw) = \pi(z)w + z\delta(w)$ for all $w, z \in R$, where $\delta$ is a derivation on $R$.

Ashraf et al. [1] investigates the commutativity of a prime ring $R$ admitting a derivation $\delta$ satisfying $\delta(a) \circ \delta(b) = a \circ b$ for all $a, b \in L$ where $L$ is a nonzero ideal of $R$. Further, Huang [11] proved that if $L$ is a square closed Lie ideal of a prime ring $R$ with characteristic different from 2 and generalized derivation $\pi$ with associated derivation $\delta$ satisfying $\pi(a) \circ \delta(b) = a \circ b$ for all $a, b \in L$, then either $R$ is commutative or $\delta = 0$.

Motivated by the above mentioned results, we prove the following:

Theorem 1.1. Let $m$ be the fixed positive integer and $k$ be a nonzero ideal of a prime ring $R$ with characteristic different from 2. If $R$ admits generalized derivations $\pi_1$ and $\pi_2$ with associated derivations $\delta_1$ and $\delta_2$ respectively and $0 \neq a \in R$ such that $a(\pi_1(x) \circ_m \pi_2(y) - x \circ_m y) = 0$ for all $x, y \in k$, then either $R$ is commutative or there exist $\alpha$ and $\beta \in C$, extended centroid of $R$ such that $\pi_1(x) = \alpha x$ and $\pi_2(x) = \beta x$ for all $x \in R$ with $\alpha(\alpha^m \beta^m - 1) = 0$.

Huang [10] proved that if $k$ is a nonzero ideal of a prime ring $R$ with characteristic different from 2 admitting a nonzero derivation $\delta$ satisfying $[\delta(x), \delta(y)]_m = [x, y]_m$ for any $x, y \in k$, for some positive integers $m, n$, then $R$ is commutative. In this line of investigation, Dhara et al. [2] proved the following: Let $K$ be a nonzero ideal of a 2-torsion free semiprime ring
admitting a generalized derivation \( \pi \) with associated derivation \( \delta \) such that \( \delta(\mathbb{K}) \neq \{0\} \). If \( [\delta(y), \pi(x)] = \pm [y, x] \) holds for all \( x, y \in \mathbb{K} \), then \( \mathbb{R} \) contains a nonzero central ideal.

Tendentious by the above results, we prove

**Theorem 1.2.** Let \( m, n \) be fixed positive integers, \( \mathbb{K} \) be a nonzero ideal of a prime ring \( \mathbb{R} \) with characteristic different from 2 and \( 0 \neq a \in \mathbb{R} \). If \( \pi \) is a generalized derivation of \( \mathbb{R} \) with associated derivation \( \delta \) satisfying \( a([\pi(x), y]_{m} + [x, \delta(y)]_{n} - [x, y]) = 0 \) for all \( x, y \in \mathbb{K} \), then either \( \mathbb{R} \) is commutative or there exist \( b \in \mathbb{U} \) such that \( \pi(x) = bx \) for all \( x \in \mathbb{R} \).

**Theorem 1.3.** Let \( m, n \) be fixed positive integers and \( \mathbb{R} \) is a semiprime ring with characteristic different from 2 and \( 0 \neq a \in \mathbb{R} \). If \( \pi \) is a generalized derivation of \( \mathbb{R} \) with associated derivation \( \delta \) satisfying \( a([\pi(x), y]_{m} + [x, \delta(y)]_{n} - [x, y]) = 0 \) for all \( x, y \in \mathbb{K} \), then \( \mathbb{R} \) contains a nonzero central ideal.

### 2 Main Results

We will use frequently the following important result due to Kharchenko [15]: Let \( 0 \neq \delta \) be a derivation of a prime ring \( \mathbb{R} \) and \( \{0\} \neq \mathbb{K} \) be an ideal of \( \mathbb{R} \). Let \( g(p_{1}, ..., p_{n}, \delta(p_{1}), ..., \delta(p_{n})) \) be a differential identity in \( \mathbb{K} \), i.e.,

\[
g(w_{1}, ..., w_{n}, \delta(w_{1}), ..., \delta(w_{n})) = 0 \quad \text{for all} \quad w_{1}, w_{2}, ..., w_{n} \in \mathbb{K}.
\]

Then we have exactly one of the following:

(i) \( \delta \) is an inner in \( \mathbb{Q} \), Martindale ring of quotient of \( \mathbb{R} \)

(ii) \( \delta \) is \( \mathbb{Q} \)-outer and the following GPI is satisfied by \( \mathbb{K} \)

\[
g(w_{1}, ..., w_{n}, y_{1}, ..., y_{n}) = 0.
\]

**Remark 2.1.** Let \( \mathbb{K} \) be an ideal of \( \mathbb{R} \). Then

(i) \( \mathbb{U}, \mathbb{R} \) and \( \mathbb{K} \) satisfy the same differential identities. [14, Theorem 2]

(ii) \( \mathbb{U}, \mathbb{R} \) and \( \mathbb{K} \) satisfy the same GPI with coefficients in \( \mathbb{U} \). [4, Theorem 2]

**Remark 2.2.** Let \( \pi \) be a generalized derivation defined on a dense right ideal of a semiprime ring \( \mathbb{R} \). Then \( \pi \) can be uniquely extended to \( \mathbb{U} \) which takes the form \( \pi(x) = ax + \delta(x) \), where \( \delta \) is a derivation on \( \mathbb{U} \) and for some \( a \in \mathbb{U} \). Moreover, \( a \) and \( \delta \) are uniquely determined by the generalized derivation \( \pi \). [13, Theorem 4]

**Remark 2.3.** Let \( F \) be a field, \( \mathbb{R} \) a dense ring of \( F \)-linear transformations (over a vector space \( \mathbb{V} \) of \( char(\mathbb{R}) \neq 2 \) with \( dim_{\mathbb{C}}\mathbb{V} \geq 2 \), \( p, c \in \mathbb{R} \), and \( 0 \neq c \notin \mathbb{Z}(\mathbb{R}) \). Assume \( pv = 0 \) for any \( v \in \mathbb{V} \) such that \( \{v, cv\} \) is linear \( F \)-independent. Then \( p = 0 \). [16, Lemma 2.1]

**Proof of Theorem 1.1** By hypothesis

\[
a(\pi_{1}(x) \circ_{m} \pi_{2}(y) - x \circ_{m} y) = 0 \quad \text{for all} \quad x, y \in \mathbb{K}.
\]

By Remark 2.2, \( \pi_{1}(x) = bx + \delta_{1}(x) \) and \( \pi_{2}(x) = cx + \delta_{2}(x) \) for some \( b, a \in \mathbb{U} \) and derivations \( \delta_{1}, \delta_{2} \) on \( \mathbb{U} \). Hence

\[
a((bx + \delta_{1}(x)) \circ_{m} (cy + \delta_{2}(y)) - x \circ_{m} y) = 0 \quad \text{for all} \quad x, y \in \mathbb{K}.
\]

By Remark 2.1, we have

\[
a((bx + \delta_{1}(x)) \circ_{m} (cy + \delta_{2}(y)) - x \circ_{m} y) = 0 \quad \text{for all} \quad x, y \in \mathbb{U}
\]

that is

\[
a(bz \circ_{m} cy + \delta_{1}(x) \circ_{m} cy + cx \circ_{m} \delta_{2}(y) + \delta_{1}(x) \circ_{m} \delta_{2}(y) - x \circ_{m} y) = 0
\]

for all \( x, y \in \mathbb{U} \).

Here the proof is divided into the following cases:
Case 1 If both $\delta_1$ and $\delta_2$ are inner derivations, then $\delta_1(x) = [q, x]$ and $\delta_2(x) = [p, x]$ for any $x \in U$ and for some $q$ and $p \in U$ respectively. So, we have

$$F(x, y) = a(bx \circ_m cy + [q, x] \circ_m cy + cx \circ_m [p, y] + [q, x] \circ_m [p, y] - x \circ_m y) = 0 \text{ for any } y, x \in U.$$  \hspace{1cm} (2.5)

If $C$ is infinite, then $U \otimes_C \overline{E}$ satisfies (2.5), where $\overline{E}$ stands for algebraic closure of $C$. By [12], $U$ and $U \otimes_C E$ are centrally closed and prime. Therefore, we may replace $R$ by $U \otimes_C E$ or $U$ according to $C$ is infinite or finite. Thus we may assume that $R$ is centrally closed over $C$ which is either algebraically closed and $F(x, y) = 0$ for any $x, y \in R$ or finite. By the use of Martindale’s theorem [12], $R$ is primitive ring with $D$ as associative division ring as well as $R$ has nonzero socle, $\text{soc}(R)$. By [9], $R$ and dense ring of linear transformations for some vector space $V$ over $C$ are isomorphic i.e $R \cong M_k(D)$, where $k = \text{dim}_D V$. Assume that $\text{dim}_D V \geq 2$, otherwise we are done. Also assume that there exists $v \in V$ such that $qv$ and $v$ are linearly $D$-independent.

If $pv$ is not a member of the span of $\{v, qv\}$, then $\{qv, pv, qv\}$ is linearly independent. By the density of ring $R$, there exist $x, y \in R$ such that

$$xqv = -v, xv = 0, ypv = v, yv = 0, xpv = 0, yqv = v.$$  \hspace{1cm} (2.6)

Multiplying equation (2.5) by $v$ from right and using conditions in equation (2.6), we get

$$a(-1)^{m-1}2^{m-1}v = 0.$$  

Since $R$ has characteristic different from 2, we have $av = 0$. If $a \in \mathbb{Z}(R)$, then $v = 0$, a contradiction. If $a \notin \mathbb{Z}(R)$, then by Remark 2.3, we have $a = 0$, again a contradiction.

If $pv$ is a member of the span of $\{v, qv\}$, then $v = \alpha v + qv\beta$ for some $\alpha, \beta \in D$. Again by the density of ring $R$, there exist $x, y \in R$ such that

$$xv = 0, yqv = v, xqv = -v, yv = 0.$$  \hspace{1cm} (2.7)

Again multiplying equation (2.5) by $v$ from right and using conditions in equations (2.7), we get

$$a(-1)^{m-1}2^{m-1}v\beta = 0.$$  

Again using that $R$ has characteristic different from 2, we have $av = 0$. Using the same arguments as used, we get $a = 0$, a contradiction.

Therefore, $\{v, qv\}$ is linearly dependent over $D$ and hence $q \in \mathbb{Z}(R)$ i.e $\delta_1 = 0$. Similarly, we can show that $\delta_2 = 0$. From (2.4), we have the following

$$a(bx \circ_m cy - x \circ_m y) = 0 \text{ for all } x, y \in U.$$  \hspace{1cm} (2.8)

Let for any $u \in V$, $\{u, bu\}$ is linearly independent. Since $\text{dim}_D V \geq 2$, we can choose $t \in V$ such that $\{u, bu, t\}$ is also linearly independent. By density of $R$, there exist $x, y \in R$ such that

$$xu = 0, xbu = 0, xt = u, yu = t, ybu = 0, yt = 0.$$  \hspace{1cm} (2.9)

Now multiplying (2.8) by $u$ from right, we get $auv = 0$. Using the arguments that have been used above, we get contradiction. Therefore, $\{u, bu, t\}$ is linearly dependent i.e $b \in C$. Similary, we can show that $c \in C$. Using these in (2.8), we get

$$a(b^m c^m - 1)x \circ_m y = 0.$$  \hspace{1cm} (2.10)

In particular, for $x = y$, we have $a(b^m c^m - 1)_{c^m+1} = 0$. Using primenes of $R$, we get $a(b^m c^m - 1_{c^m+1} = 0$.

Case 2 Let $\delta_1$ and $\delta_2$ are not both inner derivations of $U$. Then $\delta_1$ and $\delta_2$ are $C$-linearly dependent modulo $D_{\text{int}}$ i.e $\delta_2(y) = [p, y] + \beta \delta_1(y)$ for some $p \in U$ and $\beta \in C$. If either $\beta = 0$ or $\delta_2$ is inner, then $\delta_1$ is also inner which is a contradiction. So, $\beta \neq 0$ and $\delta_2$ is not inner. Then by (2.4), we have

$$a(bx \circ_m cy + \delta_1(x) \circ_m cy + cx \circ_m ([p, y] + \beta \delta_1(y)) + \delta_1(x) \circ_m ([p, y] + \beta \delta_1(y)) - x \circ_m y) = 0 \text{ for any } x, y \in U.$$
Use of Kharchenko’s Theorem [15] gives that
\[
a(b x \circ_m c y + x_1 \circ_m c y + cx \circ_m ([p, y] + \beta y_1) + x_1 \circ_m ([p, y] + \beta y_1) - x \circ_m y) = 0
\]
for all \(x_1, y_1, x, y \in K\). Taking \(y = 0 = x\), we obtain
\[
a(x_1 \circ_m y_1) = 0
\]  \hspace{1cm} (2.11)
for all \(x_1, y_1 \in I\). By [4, Theorem 2], \(Q\) as well as \(R\) satisfy the polynomial identity \(a(x_1 \circ_m y_1) = 0\). By [3, Lemma 1], we have \(R \subseteq M_n(F)\), the ring of \(n \times n\) matrices over some field \(F\), where \(n \geq 1\). Also, \(M_n(F)\) and \(R\) satisfy the same polynomial identity, i.e. \(a(x_1 \circ_m y_1) = 0\), for any \(x_1, y_1 \in M_n(F)\). To denote matrix unit with 1 in \((i, j)^{th}\)-entry and zero elsewhere, we use the notation \(e_{ij}\). Taking \(y_1 = e_{11}, a = e_{11} x_1 = e_{12}\), we see that \(e_{11} (x_1 \circ_m y_1) = e_{12} \neq 0\), a contradiction.

The case \(\delta_1(x) = [q, x] + \gamma \delta_2(x)\) for some \(\gamma \in C\) and \(q \in U\) is analogous.

**Case 3** Now assume \(\delta_1\) and \(\delta_2\) are Outer. Now by Kharchenko’s Theorem [15], we have
\[
a(b x \circ_m c y + x_1 \circ_m c y + cx \circ_m y_1 + x_1 \circ_m y_1 - x \circ_m y) = 0
\]
for any \(x_1, y_1, x, y \in K\). For \(y = x = 0\), we have
\[
a(x_1 \circ_m y_1) = 0
\]  \hspace{1cm} (2.12)
which is same as (2.11). Therefore, by the similar arguments as above this leads that \(R\) is commutative. This overpast the proof of theorem.

If we take \(\pi_1 = \pi_2 = \pi\), we have the following corollary:

**Corollary 2.4.** Let \(m\) be the fixed positive integer and \(K\) be a nonzero ideal of a prime ring \(R\) with characteristic different from 2. If \(R\) admits a generalized derivation \(\pi\) with associated derivation \(\delta\) and \(0 \neq a \in R\) such that \(a(\pi(x) \circ_m \pi(y) - x \circ_m y) = 0\) for all \(x, y \in K\), then either \(R\) is commutative or there exists \(\alpha \in C\), extended centroid of \(R\) such that \(\pi(x) = \alpha x\) for all \(x \in R\) with \(a(\alpha^m - 1) = 0\).

**Proof of Theorem 1.2** By hypothesis
\[
a([\pi(x), y]_m + [x, \delta(y)]_n - [x, y]) = 0 \text{ for any } y, x \in K.
\]  \hspace{1cm} (2.13)
By Remark 2.1, we have
\[
a([\pi(x), y]_m + [x, \delta(y)]_n - [x, y]) = 0 \text{ for any } y, x \in U.
\]  \hspace{1cm} (2.14)
By Remark 2.2, \(\pi(x) = bx + \delta(x)\) for some \(b \in U\) and derivation \(\delta\) on \(U\). Then we have
\[
a([bx + \delta(x), y]_m + [x, \delta(y)]_n - [x, y]) = 0 \text{ for any } y, x \in U.
\]  \hspace{1cm} (2.15)
That is
\[
a([bx, y]_m + [\delta(x), y]_m + [x, \delta(y)]_n - [x, y]) = 0 \text{ for any } y, x \in U.
\]  \hspace{1cm} (2.16)
The proof is divided into the following cases on the basis of Kharchenko’s theorem [15, Theorem 2]:

**Case 1** Let \(\delta\) be an inner derivation i.e \(\delta(x) = [q, x]\) for any \(x \in U\) and for some \(q \in U\). Then
\[
F(x, y) = a([bx, y]_m + [\delta(x), y]_m + [x, \delta(y)]_n - [x, y]) = 0 \text{ for any } y, x \in U.
\]  \hspace{1cm} (2.17)
If \(C\) is infinite, then \(U \otimes_C E\) satisfies (2.5), where \(E\) stands for algebraic closure of \(C\). By [12], \(U\) and \(U \otimes_C E\) are centrally closed and prime. Therefore, we may replace \(R\) by \(U \otimes_C E\) or \(U\) according to \(C\) is infinite or finite. Thus we may assume that \(R\) is centrally closed over \(C\) which is
either algebraically closed and $F(x, y) = 0$ for any $y, x \in R$ or finite. By the use of Martindale’s theorem [12], $R$ is primitive ring with $D$ as associative division ring as well as $R$ has nonzero socle, $\text{soc}(R)$. By [9], $R$ and dense ring of linear transformations for some vector space $V$ over $C$ are isomorphic i.e $R \cong M_k(D)$, where $k = \text{dim}_D V$. Assume that $\text{dim}_D V \geq 2$, otherwise we are done. Also assume that there exists $v \in V$ such that $qv$ and $v$ are linearly $D$-independent.

Since $\text{dim}_D V \geq 2$, we can find an element $w \in V$ such that $\{w, qv, v\}$ is linearly independent over $D$. By the density of the ring $R$, we can find $x, y \in R$ such that

$$xv = 0, yv = v, xqv = w, xv = 0, yv = 0, yqv = v.$$

Multiplying equation (2.17) from right by $v$ and using conditions in equation (2.18), we get $av = 0$. By the same argument that we have used in preecedent, we have $\{qv, v\}$ is linearly dependent and hence $q \in Z(R)$ i.e $d = 0$.

Case 2 Let $d$ be an outer derivation. Then

$$a([bx, y]_{m} + [x_1, y]_{m} + [x, y]_{n} - [x, y]) = 0 \text{ for any } y, x, x_1, y_1, s \in K.$$  \ ((2.19)

In particular, choosing $y = 0$, we get $a([x_1, y]_{n}) = 0$ for any $x_1, y \in K$. By [4, Theorem 2], $Q$ as well as $R$ satisfy the polynomial identity $a([x, y]_{n}) = 0$. By [3, Lemma 1], we have $R \subseteq M_n(F)$, the ring of $n \times n$ matrices over some field $F$, where $n \geq 1$. Also, $M_n(F)$ and $R$ satisfy the same polynomial identity, i.e $a([x, y]_{n}) = 0$, for any $x, y \in M_n(F)$. To denote matrix unit with 1 in $(i, j)^{th}$-entry and zero elsewhere, we use the notation $e_{ij}$. Taking $y_1 = e_{11}, a = e_{11}x_1 = e_{12}$, we see that $c_{11}([x, y]_{m}) = c_{12} \neq 0$, a contradiction.

**Proof of Theorem 1.3** We know that any derivation defined on $R$, a semiprime ring can be uniquely extended to a derivation on $U$, left Utumi ring of quotients of $R$ and hence every derivation of $R$ can be defined on $U$ [14, Lemma 2]. Also, $U$ and $R$ satisfy the same generalized polynomial identity (GPI) and differential identities (see [4] and [14]). By [13, Theorem 4], $\pi$ can be expressed as $\pi(x) = \delta(x) + bx$ for some $b \in U$ and a derivation $\delta$ defined on $U$. We have

$$a([bx, y]_{m} + [\delta(x), y]_{m} + [x, y]_{n} - [x, y]) = 0 \text{ for any } y, x \in U.$$  \ ((2.20)

Let $M(C) = \{A \mid A \text{ is maximal ideal of } C\}$ and let $P \in M(C)$. Then $PU$ is prime ideal of $U$ which is invariant under all derivation of $U$ by the theory of orthogonal completions of semiprime ring ([14, p.31-32]). Also, $\bigcap\{P \mid P \in M(C)\} = \{0\}$. Setting $\bar{U} = U/PU$. Now any derivation $\delta$ of $R$ canonically induces a derivation $\bar{\delta}$ on $\bar{U}$ defined by $\bar{\delta}(x) = \bar{\delta}(x)$ for any $x \in U$. Then

$$\bar{\delta}([bx, y]_{m} + [\delta(x), y]_{m} + [x, y]_{n} - [x, y]) = 0$$

for all $x, y \in \bar{U}$. It is clear that $\bar{U}$ is a prime ring. So by the use of Theorem 1.2, we have, either $[U, U] \subseteq PU$ or $\delta(U) \subseteq PU$ for any $P \in M(C)$. This gives that $\delta(U)[U, U] \subseteq PU$ for any $P \in M(C)$. Thus, $\bigcap\{U \mid P \in M(C)\} = \{0\}$, we have $\delta(U)[U, U] = \{0\}$. In particular, we have $\bar{\delta}(R)[R, R] = \{0\}$. Further, this can be written as $[\bar{\delta}(R), R][\bar{\delta}(R), R] = 0$. Since $R$ is a semiprime ring, we obtain that $[\bar{\delta}(R), R] = 0$. Then by [17, Theorem 3], $R$ contains a nonzero central ideal.

The following examples demonstrate that $R$ to be prime can not be omitted in the hypothesis of Theorem 1.1 and Theorem 1.2.

**Example 2.5.** For any ring $R_1$ which has characteristic different from two, let $R = \{ \begin{pmatrix} z & w \\ 0 & 0 \end{pmatrix} \mid z, w \in R_1 \}$ and $K = \{ \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix} \mid w \in R_1 \}$. Then $K$ is a nonideal ideal of $R$. Define maps $\pi_1, \pi_2, \delta_2, \delta_1 : R \rightarrow R$ by $\pi_1 \left( \begin{pmatrix} z & w \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} z & 2w \\ 0 & 0 \end{pmatrix}, \pi_2 \left( \begin{pmatrix} z & w \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} z & 0 \\ 0 & 0 \end{pmatrix}, \delta_1 \left( \begin{pmatrix} z & w \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & -w \\ 0 & 0 \end{pmatrix}$ and $\delta_2 \left( \begin{pmatrix} z & w \\ 0 & 0 \end{pmatrix} \right) = \begin{pmatrix} 0 & w \\ 0 & 0 \end{pmatrix}$ Then $\pi_1$
and $\pi_2$ are generalized derivations on $\mathbb{R}$ associated with derivations $\delta_1$ and $\delta_2$ respectively satisfying $a(\pi_1(x) \circ_m \pi_2(y) - x \circ_m y) = 0$ for all $x, y \in K$. However neither $\mathbb{R}$ is commutative nor $\pi_1(x) = \alpha x$ and $\pi_2(x) = \beta x$ for all $x \in \mathbb{R}$ as $\delta_1$ and $\delta_2$ are nonzero. Hence Theorem 1.1 is not true for arbitrary rings.

**Example 2.6.** Let $\mathbb{R} = \left\{ \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \right\}$ and $K = \left\{ \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} \right\}$, where $\mathbb{R}_1$ is a ring which has characteristic different from two. Then $K$ is a nonzero ideal of $\mathbb{R}$. Define maps $\pi, \delta : \mathbb{R} \to \mathbb{R}$ by $\pi \left( \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \right) = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}$ and $\delta \left( \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} \right) = \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$. Then $\pi$ is a generalized derivation associated with the derivation $\delta$ satisfying $a(\pi(x), y)_{m} + [x, \delta(y)]_{m} - [x, y] = 0$ for all $x, y \in K$. However neither $\mathbb{R}$ is commutative nor $\pi(x) = bx$ for all $x \in \mathbb{R}$ as $\delta$ is nonzero. Hence Theorem 1.2 does not hold for arbitrary rings.

### References


### Author information

Asma Ali, 1 Md Hamidur Rahaman and Farhat Ali, Department of Mathematics, Aligarh Muslim University, Aligarh-202002, 1 Department of Mathematics, A.P.C Roy Govt. College, Siliguri-734010, India. E-mail: asma.ali2@rediffmail.com, rahamanhamidmath@gmail.com, 04farhatama@gmail.com

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