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GENERALIZED SZÁSZ-CHLODOWSKY TYPE OPERATORS INVOLVING d-ORTHOGONAL BRENKE POLYNOMIALS FOR FUNCTIONS WITH ONE AND TWO VARIABLES

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Abstract In this paper is discussed and investigated the global approximation of the generalization of the Szasz-Chlodowsky operators based on d-othogonal Brenke polynomials for functions of one and two variables. Further, the Voronovskaya-type asymptotic formula, local approximation, error estimation in terms of the modulus of continuity of second order for functions in a Lipschitz type space are included.

1 Introduction

Brenke polynomials $P_n(x)$ [5], which are d-orthogonal polynomials sets of are generated by

$$\mathcal{Z}(t^{d+1})\mathcal{V}(xt) = \sum_{n=0}^{\infty} P_n(x)t^n, \tag{1.1}$$

where $\mathcal{Z}$ and $\mathcal{V}$ are two formal power series satisfying:

$$\mathcal{Z}(t) = \sum_{n=0}^{\infty} z_n t^n,\ \mathcal{V}(t) = \sum_{n=0}^{\infty} v_n t^n;\ \ z_0 v_0 = 0,\ n \in N.$$

An explicit formula for $p_n(x)$ is given by:

$$P_n(x) = \sum_{j=0}^{n} z_{n-j} v_j x^j,\ n = 0, 1, ... \tag{1.2}$$

In the special case if $d = 0$ we get standard orthogonal Brenke which are defined in [15].
Now, inspired by [9] let us define Szasz-Chlodowsky type generalization of the Szasz operators with the help of generating function (1.1), as follows:

$$\mathcal{H}_\eta^{(d)}(\tau; x) = \sum_{k=0}^{\infty} \frac{\mathcal{L}_k\left(\frac{\eta x}{q_\eta}\right)}{\mathcal{Z}(1)\mathcal{V}(\frac{\eta x}{q_\eta})} \tau\left(\frac{k}{\eta} q_\eta\right), x \in [0, \infty), \tag{1.3}$$

where $\mathcal{Z}(\cdot)$, $\mathcal{V}(\cdot)$ and $\mathcal{L}_k(\cdot)$ have some properties given by (1.1) and q_η a positve increasing sequence with the properties

$$\lim_{\eta\to\infty} q_\eta = \infty, \lim_{\eta\to\infty} \frac{q_\eta}{\eta} = 0.$$

We shall restrict ourselves to the operators given by (1.3) satisfying:
(i) $\mathcal{Z}(1) \neq 0;\ \frac{z_{n-j}v_j}{\mathcal{Z}(1)} \geq 0,\ n \in \mathbb{N} \cup \{0\};$

(ii) $\mathcal{V}:\mathbb{R}_0^+ \to \mathbb{R}$;
(iii) (1.1) converges for $|t|<R, (R>1)$.
We have the following few special case of the operators $\mathcal{H}_\eta^{(d)}$ as:
(1) For $d=0$ these operators reduce to well know Szasz-Chlodowsky-Brenke type operators discussed by Mursaleen and Ansari in [9].
(2) For $d=0$ and $q_\eta=1$ these operators reduce to Szasz-Brenke operators defined by Varma et al. in [13].
(3) For $d=0$ and $\mathcal{Z}(1)=1,\ \mathcal{V}(u)=e^u$ and $q_\eta=1$ these operators reduced Szasz-Mirakyan operators defined by Szasz in [10].
The generalization of Szasz type operators involving by othogonal polynomials have been studied in [4, 2, 11, 1, 6, 12, 8, 14, 3].
The aim of the this paper is to give some direct results in terms of the modulus of continuity of second order, convergence of derivative operators to derivative functions , Lipschitz class function. We also give the theorem convergence and the degree of convergence is established for functions of two variables. In addition, we consider the simultaneous approximation of these operators.

2 Notations and auxiliary results

Let us denote by $C_E[0,\infty) := \{\tau \in C[0,\infty) : |\tau(t)| \le Me^{Nt}, t\in[0,\infty)$, for some $M,N>0\}$. For a fixed $r\in N$ we denote $C_E^r[0,\infty)$ the set of real-valued r-times continuostly differentiable functions on the $[0,\infty)$ and it is a subspace of $C_E[0,\infty)$.
Using the generating function of the d-Brenke polynomials given by (1.1) and the fundamental properties of $\mathcal{H}_\eta^{(d)}$ operators, we get the following lemmas which are helpful to obtain the main result. In what follows, let $e_i(t)=t^i$, $i\in N^0$ be the test functions.

Lemma 2.1. *If the operators $\mathcal{H}_\eta^{(d)}$, are defined by (1.3), then for all $x\in[0,\infty)$, the following identities hold:*

$$(i)\ \mathcal{H}_\eta^{(d)}(e_0;x)=1;$$

$$(ii)\ \mathcal{H}_\eta^{(d)}(e_1;x)=\frac{\mathcal{V}'\left(\frac{\eta x}{q_\eta}\right)}{\mathcal{V}\left(\frac{\eta x}{q_\eta}\right)}x+\frac{q_\eta(d+1)\mathcal{Z}'(1)}{\eta\mathcal{Z}(1)};$$

$$(iii)\ \mathcal{H}_\eta^{(d)}(e_2;x)=\frac{\mathcal{V}''\left(\frac{\eta x}{q_\eta}\right)}{\mathcal{V}\left(\frac{\eta x}{q_\eta}\right)}x^2+\frac{q_\eta}{\eta}\frac{(\mathcal{Z}(1)+2(d+1)\mathcal{Z}'(1))\mathcal{V}'(\frac{\eta x}{q_\eta})}{\mathcal{Z}(1)\mathcal{V}\left(\frac{\eta x}{q_\eta}\right)}x$$
$$+\frac{q_\eta^2}{\eta^2}\frac{(d+1)^2(\mathcal{Z}''(1)+\mathcal{Z}'(1))}{\eta^2\mathcal{Z}(1)};$$

$$(iv)\ \mathcal{H}_\eta^{(d)}(e_3;x)=\frac{\mathcal{V}'''(\frac{\eta x}{q_\eta})}{\mathcal{V}\left(\frac{\eta x}{q_\eta}\right)}x^3+\frac{3q_\eta}{\eta}\frac{(\mathcal{Z}(1)+\mathcal{Z}'(1)(d+1))\mathcal{V}''(\frac{\eta x}{q_\eta})}{\mathcal{Z}(1)\mathcal{V}\left(\frac{\eta x}{q_\eta}\right)}x^2$$
$$+\frac{(\mathcal{Z}(1)+\mathcal{Z}'(1)(d+1)(6+3d)+3\mathcal{Z}''(1)(\,d+1)^2)\mathcal{V}'(\frac{\eta x}{q_\eta})}{\mathcal{Z}(1)\mathcal{V}\left(\frac{\eta x}{q_\eta}\right)}x$$
$$+\frac{q_\eta^3}{\eta^3}\frac{(\mathcal{Z}'(1)+3\mathcal{Z}''(1)+\mathcal{Z}'''(1))\,(d+1)^3}{\mathcal{Z}(1)};$$

$$(v)\ \mathcal{H}_{\eta}^{(d)}(e_4;x)=\frac{\mathcal{V}^{(4)}(\frac{\eta x}{q_\eta})}{\mathcal{V}\left(\frac{\eta x}{q_\eta}\right)}x^4+\frac{2q_\eta}{\eta}\frac{(3\mathcal{Z}(1)+2\mathcal{Z}'(1)(d+1))\mathcal{V}'''\left(\frac{\eta x}{q_\eta}\right)}{\mathcal{Z}(1)\mathcal{V}\left(\frac{\eta x}{q_\eta}\right)}x^3$$
$$+\frac{q_\eta^2}{\eta^2}\frac{(7\mathcal{Z}(1)+\mathcal{Z}'(1)6(d+1)(3+d)+6\mathcal{Z}''(1)(d+1)^2)\mathcal{V}''(\frac{\eta x}{q_\eta})}{\mathcal{Z}(1)\mathcal{V}\left(\frac{\eta x}{q_\eta}\right)}x^2$$
$$+\frac{q_\eta^3}{\eta^3}\frac{\left(\mathcal{Z}(1)+2(d+1)(2d^2+7d+7)\mathcal{Z}'(1)+\mathcal{Z}''(1)6(d+1)^2(3+2d)+4(d+1)^3\mathcal{Z}'''(1)\right)\mathcal{V}'\left(\frac{\eta x}{q_\eta}\right)x}{\mathcal{Z}(1)\mathcal{V}\left(\frac{\eta x}{q_\eta}\right)}$$
$$+\frac{q_\eta^4}{\eta^4}\frac{(d+1)^4\left(\mathcal{Z}'(1)+2\mathcal{Z}''(1)+6\mathcal{Z}'''(1)+\mathcal{Z}^{(4)}(1)\right)}{Z(1)}.$$

Proof. By using generating function given by (1.1) we have

$$\sum_{k=0}^{\infty}\mathcal{L}_k\left(\frac{\eta x}{q_\eta}\right)=\mathcal{Z}(1)\mathcal{V}\left(\frac{\eta x}{q_\eta}\right);$$

$$\sum_{k=0}^{\infty}k\mathcal{L}_k\left(\frac{\eta x}{q_\eta}\right)=\mathcal{Z}'(1)(d+1)\mathcal{V}\left(\frac{\eta x}{q_\eta}\right)+\mathcal{Z}(1)\frac{\eta x}{q_\eta}\mathcal{V}'\left(\frac{\eta x}{q_\eta}\right);$$

$$\sum_{k=0}^{\infty}k^2\mathcal{L}_k\left(\frac{\eta x}{q_\eta}\right)=\frac{\eta^2x^2}{q_\eta^2}\mathcal{Z}(1)\mathcal{V}''\left(\frac{\eta x}{q_\eta}\right)+\frac{\eta x}{q_\eta}(\mathcal{Z}(1)+2\mathcal{Z}'(1)(d+1))\mathcal{V}'\left(\frac{\eta x}{q_\eta}\right)$$
$$+\mathcal{V}\left(\frac{\eta x}{q_\eta}\right)((d+1)^2\mathcal{Z}'(1)+\mathcal{Z}''(1)(d+1)^2);$$

$$\sum_{k=0}^{\infty}k^3\mathcal{L}_k\left(\frac{\eta x}{q_\eta}\right)=\frac{\eta^3x^3}{q_\eta^3}\mathcal{Z}(1)\mathcal{V}'''\left(\frac{\eta x}{q_\eta}\right)+\frac{3\eta^2x^2}{q_\eta^2}(\mathcal{Z}(1)+\mathcal{Z}''(1)(d+1))\mathcal{V}''\left(\frac{\eta x}{q_\eta}\right)$$
$$+\frac{\eta x}{q_\eta}\mathcal{V}'\left(\frac{\eta x}{q_\eta}\right)(\mathcal{Z}(1)+\mathcal{Z}'(1)(d+1)(6+3d)+3\mathcal{Z}''(1)(d+1)^2)$$
$$+\mathcal{V}\left(\frac{\eta x}{q_\eta}\right)(\mathcal{Z}'(1)(d+1)^3+3\mathcal{Z}''(1)(d+1)^3+\mathcal{Z}'''(1)(d+1)^3);$$

$$\sum_{k=0}^{\infty}k^4\mathcal{L}_k\left(\frac{\eta x}{q_\eta}\right)=\frac{\eta^4x^4}{q_\eta^4}\mathcal{Z}(1)\mathcal{V}^{(IV)}(\frac{\eta x}{q_\eta})+\frac{2\eta^3x^3}{q_\eta^3}(3\mathcal{Z}(1)+2(d+1)\mathcal{Z}'(1))\mathcal{V}'''\left(\frac{\eta x}{q_\eta}\right)$$
$$+\frac{\eta^2x^2}{q_\eta^2}\mathcal{V}''(\frac{\eta x}{q_\eta})(7\mathcal{Z}(1)+\mathcal{Z}'(1)6(d+1)(3+d)+6\mathcal{Z}''(1)(d+1)^2)$$
$$+\frac{\eta x}{q_\eta}\mathcal{V}'\left(\frac{\eta x}{q_\eta}\right)(\mathcal{Z}(1)+\mathcal{Z}'(1)2(d+1)(2d^2+7d+7)+6\mathcal{Z}''(1)(d+1)^2(3+2d)$$
$$+4\mathcal{Z}'''(1)(d+1)^3)+\mathcal{V}\left(\frac{\eta x}{q_\eta}\right)(\mathcal{Z}'(1)+7\mathcal{Z}''(1)+6\mathcal{Z}'''(1)+\mathcal{Z}^{(4)}(1))(d+1)^4.$$

Using above equalities, we obtain Lemma 2.1.

Lemma 2.2. *Let $\Theta_i(s)=(s-x)^i$, for $i=1,2,...$; and operators $\mathcal{H}$ be defined by (1.3); Then for all $x\in[0,\infty)$, the following identites hold:*

$$\mathcal{H}_{\eta}^{(d)}(\Theta_1(s);x)=\frac{\mathcal{V}'\left(\frac{\eta x}{q_\eta}\right)-\mathcal{V}\left(\frac{\eta x}{q_\eta}\right)}{\mathcal{V}\left(\frac{\eta x}{q_\eta}\right)}x+\frac{q_\eta(d+1)\mathcal{Z}'(1)}{\eta\mathcal{Z}(1)};$$

$$\mathcal{H}_{\eta}^{(d)}(\Theta_2(s);x)=\frac{\mathcal{V}''(\frac{\eta x}{q_\eta})-2\mathcal{V}'(\frac{\eta x}{q_\eta})+\mathcal{V}(\frac{\eta x}{q_\eta})}{\mathcal{V}(\frac{\eta x}{q_\eta})}x^2+\frac{q_\eta}{\eta\mathcal{Z}(1)\mathcal{V}(\frac{\eta x}{q_\eta})}\Big\{\mathcal{Z}(1)\mathcal{V}'(\frac{\eta x}{q_\eta})$$
$$+2(d+1)\mathcal{Z}'(1)(\mathcal{V}'(\frac{\eta x}{q_\eta})-\mathcal{V}(\frac{\eta x}{q_\eta}))x+\frac{q_\eta^2}{\eta^2}\frac{(d+1)^2(\mathcal{Z}'(1)+\mathcal{Z}''(1))}{\mathcal{Z}(1)}\Big\}$$
$$=A_1(\eta)x^2+A_2(\eta)x+A_3(\eta);$$

$$\mathcal{H}_{\eta}^{(d)}(\Theta_4(s);x)=\frac{\mathcal{V}^{(IV)}(\frac{\eta x}{q_\eta})-4\mathcal{V}'''(\frac{\eta x}{q_\eta})+6\mathcal{V}''(\frac{\eta x}{q_\eta})-4\mathcal{V}'(\frac{\eta x}{q_\eta})+\mathcal{V}(\frac{\eta x}{q_\eta})}{\mathcal{V}(\frac{\eta x}{q_\eta})}x^4$$
$$+\frac{2q_\eta}{\eta\mathcal{Z}(1)\mathcal{V}(\frac{\eta x}{q_\eta})}\Big\{3\mathcal{Z}(1)(\mathcal{V}'''(\frac{\eta x}{q_\eta})-2\mathcal{V}''(\frac{\eta x}{q_\eta})+2\mathcal{V}'\left(\frac{\eta x}{q_\eta}\right))$$
$$+2\mathcal{Z}'(1)(d+1)(\mathcal{V}'''(\frac{\eta x}{q_\eta})-3\mathcal{V}''(\frac{\eta x}{q_\eta}))+3\mathcal{V}'(\frac{\eta x}{q_\eta})-\mathcal{V}(\frac{\eta x}{q_\eta}))\Big\}x^3$$
$$+\frac{q_\eta^2}{\eta^2}\Big\{(\mathcal{Z}(1)(7\mathcal{V}''(\frac{\eta x}{q_\eta})-4\mathcal{V}'(\frac{\eta x}{q_\eta}))+2\mathcal{Z}'(1)(d+1)(3(3+d)\mathcal{V}''(\frac{\eta x}{q_\eta})-2(6+3d)\mathcal{V}'(\frac{\eta x}{q_\eta})$$
$$+3(d+1)\mathcal{V}(\frac{\eta x}{q_\eta}))+6\mathcal{Z}''(1)(d+1)^2(\mathcal{V}''(\frac{\eta x}{q_\eta})+2\mathcal{V}'(\frac{\eta x}{q_\eta})+\mathcal{V}(\frac{\eta x}{q_\eta}))\Big\}x^2$$
$$+\frac{q_\eta^3}{\eta^3}\frac{1}{\mathcal{Z}(1)\mathcal{V}(\frac{\eta x}{q_\eta})}\{\mathcal{Z}(1)\mathcal{V}'(\frac{\eta x}{q_\eta})+2(d+1)\mathcal{Z}(1)\left((2d^2+7d+7)\mathcal{V}'(\frac{\eta x}{q_\eta})-2(d+1)\mathcal{V}(\frac{\eta x}{q_\eta})\right)$$
$$+6\mathcal{Z}''(1)(d+1)^2\left((3+2d)\mathcal{V}'(\frac{\eta x}{q_\eta})-2(d+1)\mathcal{V}(\frac{\eta x}{q_\eta})\right)+4\mathcal{Z}'''(1)(d+1)^3\left(\mathcal{V}'(\frac{\eta x}{q_\eta})-\mathcal{V}(\frac{\eta x}{q_\eta})\right)\}x$$
$$+\frac{q_\eta^4}{\eta^4}\frac{(d+1)^4\left(\mathcal{Z}'(1)+7\mathcal{Z}''(1)+6\mathcal{Z}'''(1)+\mathcal{Z}^{(4)}(1)\right)}{\mathcal{Z}'(1)}.$$

Proof. From the linearity of the operators $\mathcal{H}_{\eta}^{(d)}$, and applying Lemma 2.1, we obtain the result of Lemma.

In what follows, we assume that $\lim\limits_{\eta\to\infty}\frac{\eta}{q_\eta}\frac{\mathcal{V}'\left(\frac{\eta x}{q_\eta}\right)-\mathcal{V}\left(\frac{\eta x}{q_\eta}\right)}{\mathcal{V}\left(\frac{\eta x}{q_\eta}\right)}=0$ and
$\lim\limits_{\eta\to\infty}\frac{\eta}{q_\eta}\frac{\mathcal{V}''(\frac{\eta x}{q_\eta})-2\mathcal{V}'(\frac{\eta x}{q_\eta})+\mathcal{V}(\frac{\eta x}{q_\eta})}{\mathcal{V}\left(\frac{\eta x}{q_\eta}\right)}=0.$

Lemma 2.3. *Let $x\in[0,b_n]$. Then using the equality $\lim\limits_{\lambda\to\infty}\frac{\mathcal{V}^{(t)}(\lambda)}{\mathcal{V}(\lambda)}=1$, $t\in\{1,2\}$, we get*

$$\lim_{\eta\to\infty}\frac{\eta}{q_\eta}\mathcal{H}_{\eta}^{(d)}(u-x;x)=\lim_{\eta\to\infty}\frac{\eta}{q_\eta}\frac{\mathcal{V}'(\frac{\eta x}{q_\eta})-\mathcal{V}(\frac{\eta x}{q_\eta})}{\mathcal{V}(\frac{\eta x}{q_\eta})}x+\frac{(d+1)\mathcal{Z}'(1)}{\mathcal{Z}(1)}$$
$$=\frac{(d+1)\mathcal{Z}'(1)}{\mathcal{Z}(1)};$$
$$\lim_{n\to\infty}\frac{\eta}{q_\eta}\mathcal{H}_{\eta}^{(d)}((u-x)^2;x)=x.$$

Further, choosing

$$A(\eta)=max[A_1(\eta).\frac{A_2(\eta)}{2},A_3(\eta)]$$

we can write

$$\mathcal{H}_{\eta}^{(d)}(\Theta_2(s);x)\le A(\eta)(1+x)^2. \tag{2.1}$$

For a fixed $r\in N$ we denote $C_E^r(0,\infty)$ the set of all real-valued r-times continuostly differentiable functions on the $[0,\infty)$, and it is a subspase of $C_E[0,\infty)$.

Let us recall that if $I \subset (-\infty, +\infty)$ is a given interval and τ is a bounded real valued function defined on I, the modulus of continuity $\omega(\tau;\delta)$ of τ defined by

$$\omega(\tau;\delta) = \sup\{\mid \tau(t) - \tau(x) \mid; x, t \in I, \mid t - x \mid < \delta\}$$

for any $\delta > 0$. A continuous function τ definde on I, which statisties the condition $\mid \tau(x) - \tau(y) \mid \leq M_\tau \mid x - y \mid^\alpha$, $(x,y) \in I \times E$ is called locally $Lip(\alpha)$ on E $(0 \leq \alpha \leq 1, E \subset I)$, where M_τ is a constant depending only of τ

Theorem 2.4. *Let $\mathcal{H}_\eta^{(d)}(\tau;x)$ operators are defined by relation (1.3), where $\tau \in C_E^r[0,\infty)$. Then using the equality $\lim\limits_{\lambda\to\infty} \frac{\mathcal{V}^{(t)}(\lambda)}{\mathcal{V}(\lambda)} = 1$, $t \in \{1,2,..,r\}$, we can give*

$$\mathcal{H}_\eta^{(d)}(\tau;x) \xrightarrow{uniformly} \tau$$

on each compact set $[0,b]$ when $\eta \to \infty$.

Proof. Considering Lemma 2.1, it follows that $\lim\limits_{\eta\to\infty} \mathcal{H}_\eta^{(d)}(s^i;x) = s_i(x)$, $i = 0,1,2$ and $s_i(x) = x^i$, and the result follows from the well-known Bohman- Korovkin theorem [7].

Theorem 2.5. *Let $\tau \in C_E[a,b]$. Then for any $0 \leq x \leq b$, the operators $\mathcal{H}_\eta^{(d)}$ defined by (1.3) satisfies the equality*

$$|\mathcal{H}_\eta^{(d)}(\tau;x) - \tau(x)| \leq 2\omega(\tau;\sqrt{\delta_\eta(b)})$$

where

$$\delta_\eta(b) = \frac{\left(\mathcal{V}''(\frac{\eta b}{q_\eta}) - 2\mathcal{V}'(\frac{\eta a}{q_\eta}) - \mathcal{V}(\frac{\eta b}{q_\eta})\right)b^2}{\mathcal{V}(\frac{\eta b}{q_\eta})} + \frac{q_\eta}{\eta\mathcal{Z}(1)\mathcal{V}(\frac{\eta b}{q_\eta})}\Big\{\mathcal{Z}(1)\mathcal{V}'(\frac{\eta b}{q_\eta})$$
$$+ 2(d+1)\mathcal{Z}'(1)(\mathcal{V}'(\frac{\eta b}{q_\eta}) - \mathcal{V}(\frac{\eta b}{q_\eta}))\Big\}b + \frac{q_\eta^2}{\eta^2}\frac{(d+1)^2(\mathcal{Z}'(1) + \mathcal{Z}''(1)}{\mathcal{Z}(1)}.$$

Proof. Using Büyükyazıcı technique [8], from linearity properties of the operators, $\mathcal{H}_\eta^{(d)}$, we get

$$|\mathcal{H}_\eta^{(d)}(\tau;x) - \tau(x)| \leq \frac{1}{\mathcal{Z}(1)\mathcal{V}(\frac{\eta x}{q_\eta})}\sum_{k=0}^{\infty}\mathcal{L}_k\left(\frac{\eta x}{q_\eta}\right)\left|\tau\left(\frac{kq_\eta}{\eta}\right) - \tau(x)\right| \tag{2.2}$$

On the other hand, by using well-known properties of the modulus of continuity $\omega(\tau;\delta)$,

$$|\tau(t) - \tau(x)| \leq \omega(\tau;|t-x|)$$

and

$$\omega(\tau;\lambda\delta) \leq (\lambda^2+1)\omega(\tau;\delta), \lambda > 0$$

we have

$$\left|\tau\left(\frac{kq_\eta}{\eta}\right) - \tau(x)\right| \leq \left(\frac{\left(\frac{kq_\eta}{\eta} - x\right)^2}{\delta^2} + 1\right)\omega(\tau;\delta) \tag{2.3}$$

for every $\delta > 0$. Thus substituting (2.3) in (2.2) and using Lemma 2.2, we obtain

$$|\mathcal{H}_\eta^{(d)}(\tau;x) - \tau(x)| \leq \omega(\tau;\delta)[\frac{1}{\delta^2}\mathcal{H}_\eta^{(d)}(\Theta_2(s);x) + 1]. \tag{2.4}$$

For $0 \leq x \leq b$ we have:

$$\mathcal{H}_\eta^{(d)}(\Theta_2(s);x) \leq \frac{\left(\mathcal{V}''(\frac{\eta b}{q_\eta}) - 2\mathcal{V}'(\frac{\eta b}{q_\eta}) - \mathcal{V}(\frac{\eta b}{q_\eta})\right)b^2}{\mathcal{V}(\frac{\eta b}{q_\eta})} + \frac{q_\eta}{n\mathcal{Z}(1)\mathcal{V}(\frac{\eta b}{q_\eta})}\Big\{\mathcal{Z}(1)\mathcal{V}'(\frac{\eta b}{q_\eta})$$
$$+ 2(d+1)\mathcal{Z}'(1)(\mathcal{V}'(\frac{\eta b}{q_\eta}) - \mathcal{V}(\frac{\eta b}{q_\eta}))\Big\}b + \frac{q_\eta^2}{\eta^2}\frac{(d+1)^2(\mathcal{Z}'(1) + \mathcal{Z}''(1)}{\mathcal{Z}(1)} = \delta_\eta(b). \tag{2.5}$$

Using (2.5) and taking $\delta_\eta = \delta_\eta(b)$ in (2.4), we obtain the desired result.

Theorem 2.6. *If $\omega(\tau', \cdot)$ is the modulus of continuity of function $\tau'(x)$ in $x \in [0,b]$, $b > 0$ and $\tau(x)$ has a continuous bounded derivative, then using the equality $\lim\limits_{\lambda\to\infty} \frac{\mathcal{V}^{(t)}(\lambda)}{\mathcal{V}(\lambda)} = 1$, $t \in \{1,2,..,r\}$, we obtain the approximation*

$$|\mathcal{H}_\eta^{(d)}(\tau;x)-\tau(x)| \le A\left[\left|\frac{\mathcal{V}'\left(\frac{\eta x}{q_\eta}\right)-\mathcal{V}\left(\frac{\eta x}{q_\eta}\right)}{\mathcal{V}\left(\frac{\eta x}{q_\eta}\right)}\right|b+\frac{q_\eta(d+1)\mathcal{Z}'(1)}{\eta\mathcal{Z}(1)}\right]+\left[1+(\delta_\eta(b))^{\frac{1}{2}}\right]\omega\left(\tau',\delta_\eta(b)\right),$$

where $\delta_\eta(b)$ is the same as in Theorem 2.5 and A is a positive constant such that $|\tau'(x)| \le A$.

Proof. Using the mean value theorem, we can write

$$\tau\left(\frac{kq_\eta}{\eta}\right)-\tau(x) = \left(\frac{kq_\eta}{\eta}-x\right)\tau''(x)+\left(\frac{kq_\eta}{\eta}-x\right)\left(\tau'(\psi)-\tau'(x)\right), \tag{2.6}$$

where $x < \psi < \frac{kq_\eta}{\eta}$. Applying $\mathcal{H}_\eta^{(d)}$ on both sides of (2.6), we obtain

$$\begin{aligned}\mathcal{H}_\eta^{(d)}(\tau;x)-\tau(x) &= \tau'(x)\frac{1}{\mathcal{Z}(1)\mathcal{V}(\frac{\eta x}{q_\eta})}\sum_{k=0}^{\infty}\mathcal{L}_k\left(\frac{\eta x}{q_\eta}\right)\left(\frac{kq_\eta}{\eta}-x\right)\\ &+ \frac{1}{\mathcal{Z}(1)\mathcal{V}(\frac{\eta x}{q_\eta})}\sum_{k=0}^{\infty}\mathcal{L}_k\left(\frac{\eta x}{q_\eta}\right)\left(\frac{kq_\eta}{\eta}-x\right)\left(\tau'(\psi)-\tau'(x)\right). \end{aligned}\tag{2.7}$$

Hence,

$$\begin{aligned}|\mathcal{H}_\eta^{(d)}(\tau;x)-\tau(x)| &\le |\tau'(x)||\mathcal{H}_\eta^{(d)}(\Theta_1(s);x)|\\ &+ \frac{1}{\mathcal{Z}(1)\mathcal{V}(\frac{\eta x}{q_\eta})}\sum_{k=0}^{\infty}\mathcal{L}_k\left(\frac{\eta x}{q_\eta}\right)\left|\frac{kq_\eta}{\eta}-x\right|\left|\tau'(\psi)-\tau'(x)\right|\\ &\le A\left[\left|\frac{\mathcal{V}'\left(\frac{\eta x}{q_\eta}\right)-\mathcal{V}\left(\frac{\eta x}{q_\eta}\right)}{\mathcal{V}\left(\frac{\eta x}{q_\eta}\right)}\right|b+\frac{q_\eta(d+1)\mathcal{Z}'(1)}{\eta\mathcal{Z}(1)}\right]\\ &+ \frac{1}{\mathcal{Z}(1)\mathcal{V}(\frac{\eta x}{q_\eta})}\sum_{k=0}^{\infty}\mathcal{L}_k\left(\frac{\eta x}{q_\eta}\right)\left|\frac{kq_\eta}{\eta}-x\right|\omega\left(\tau';\delta\right)\left(\frac{|\frac{kq_\eta}{\eta}-x|}{\delta}+1\right),\end{aligned}$$

since

$$|\psi - x| \le \left|\frac{kq_\eta}{\eta}-x\right|.$$

Thus,

$$\begin{aligned}|\mathcal{H}_\eta^{(d)}(\tau;x)-\tau(x)| &\le A\left[\left|\frac{\mathcal{V}'\left(\frac{\eta x}{q_\eta}\right)-\mathcal{V}\left(\frac{\eta x}{q_\eta}\right)}{\mathcal{V}\left(\frac{\eta x}{q_\eta}\right)}\right|b+\frac{q_\eta(d+1)\mathcal{Z}'(1)}{\eta\mathcal{Z}(1)}\right]\\ &+ \frac{1}{\delta\mathcal{Z}(1)\mathcal{V}(\frac{\eta x}{q_\eta})}\sum_{k=0}^{\infty}\mathcal{L}_k\left(\frac{\eta x}{q_\eta}\right)\left(\frac{kq_\eta}{\eta}-x\right)^2\omega\left(\tau';\delta\right)\\ &+ \frac{1}{\mathcal{Z}(1)\mathcal{V}(\frac{\eta x}{q_\eta})}\sum_{k=0}^{\infty}\mathcal{L}_k\left(\frac{\eta x}{q_\eta}\right)\left|\frac{kq_\eta}{\eta}-x\right|\omega\left(\tau';\delta\right).\end{aligned}\tag{2.8}$$

Now, using Caushy-Schwarz inequality for third term, we get

$$\begin{aligned}|\mathcal{H}_\eta^{(d)}(\tau;x)-\tau(x)| &\le A\left[\left|\frac{\mathcal{V}'\left(\frac{\eta x}{q_\eta}\right)-\mathcal{V}\left(\frac{\eta x}{q_\eta}\right)}{\mathcal{V}\left(\frac{\eta x}{q_\eta}\right)}\right|b+\frac{q_\eta(d+1)\mathcal{Z}'(1)}{\eta\mathcal{Z}(1)}\right]\\ &+ \left[\frac{1}{\delta}\mathcal{H}_\eta^{(d)}((t-x)^2;x)+\left(\mathcal{H}_\eta^{(d)}((t-x)^2;x)\right)^{\frac{1}{2}}\right]\omega\left(\tau';\delta\right).\end{aligned}\tag{2.9}$$

Using relation (15), we obtain

$$\begin{aligned}|\mathcal{H}_\eta^{(d)}(\tau;x)-\tau(x)| &\leq A\left[\left|\frac{\mathcal{V}'\left(\frac{\eta x}{q_\eta}\right)-\mathcal{V}\left(\frac{\eta x}{q_\eta}\right)}{\mathcal{V}\left(\frac{\eta x}{q_\eta}\right)}\right| b+\frac{q_\eta(d+1)\mathcal{Z}'(1)}{\eta\mathcal{Z}(1)}\right] \\ &+ \left[\frac{1}{\delta}\delta_\eta(b)+(\delta_\eta(b))^{\frac{1}{2}}\right]\omega\left(\tau';\delta\right). \qquad (2.10)\end{aligned}$$

Choosing $\delta=\delta_\eta(b)$, we get the desired result.

Theorem 2.7. *Let $\omega\left(\frac{d^r\tau}{dx^r},\cdot\right)$ is the modulus of continuity of r-th derivative of τ where $\tau\in C_E^r[0,\infty)$. Then using the equality $\lim\limits_{\lambda\to\infty}\frac{\mathcal{V}^{(t)}(\lambda)}{\mathcal{V}(\lambda)}=1,\ t\in\{1,2,..,r\}$, we obtain the approximation*

$$\frac{d^r\mathcal{H}_\eta^{(d)}(\tau;x)}{dx^r}\to\tau^{(r)}(x),\ \ n\to\infty.$$

Proof. Using Büyükyazıcı technique [8], considering the relation $\lim\limits_{\lambda\to\infty}\frac{\mathcal{V}^{(t)}(\lambda)}{\mathcal{V}(\lambda)}=1,\ t\in\{1,2,..,r\}$, and by some calculations we obtain the relation

$$\lim_{\eta\to\infty}\frac{d^r}{dx^r}\mathcal{H}_\eta^{(d)}(\tau;x)=\frac{\left(\frac{\eta}{q_\eta}\right)^r\sum_{k=0}^{\infty}\mathcal{L}_k\left(\frac{\eta x}{q_\eta}\right)\Delta^r_{\frac{q_\eta}{\eta}}\tau\left(\frac{k}{\eta}q_\eta\right)}{\mathcal{Z}(1)\mathcal{V}(\frac{\eta x}{q_\eta})} \qquad (2.11)$$

where $\Delta^r_{\frac{q_\eta}{\eta}}f(\frac{k}{\eta}q_\eta)$ denotes the $r-$th difference of τ which corrosponds to increment $\frac{q_\eta}{\eta}$. Combining the properties between finite and divided differences, the $r-$th derivative of the operator $\mathcal{H}_\eta^{(d)}$ is given as:

$$\begin{aligned}\lim_{\eta\to\infty}\frac{d^r}{dx^r}\mathcal{H}_\eta^{(d)}(\tau;x)&=\frac{r!\sum_{k=0}^{\infty}\mathcal{L}_k\left(\frac{\eta x}{q_\eta}\right)\frac{\Delta^r_{\frac{q_\eta}{\eta}}f\left(\frac{k}{\eta}q_\eta\right)}{r!\left(\frac{\eta}{q_\eta}\right)^r}}{\mathcal{Z}(1)\mathcal{V}(\frac{\eta x}{q_\eta})}\\ &=\lim_{\eta\to\infty}\frac{r!\sum_{k=0}^{\infty}\mathcal{L}_k\left(\frac{\eta x}{q_\eta}\right)\left[\frac{k}{\eta}q_\eta,\frac{k+1}{\eta}q_\eta,...,\frac{k+r}{\eta}q_\eta;\tau\right]}{\mathcal{Z}(1)\mathcal{V}(\frac{\eta x}{q_\eta})}\\ &=\lim_{\eta\to\infty}\frac{r!\sum_{k=0}^{\infty}\mathcal{L}_k\left(\frac{\eta x}{q_\eta}\right)\mu\left(\frac{\eta x}{q_\eta}\right)}{\mathcal{Z}(1)\mathcal{V}(\frac{\eta x}{q_\eta})}\\ &=r!\mathcal{H}_\eta^{(d)}(\mu;x). \qquad (2.12)\end{aligned}$$

where $\mu(x)=\left[x,\frac{q_\eta}{\eta}+x,...,r\frac{q_\eta}{\eta}+x;\tau\right]$. From the above theorem, we have

$$\begin{aligned}\lim_{\eta\to\infty}\left|\frac{d^r}{dx^r}\mathcal{H}_\eta^{(d)}(\tau;x)-\tau^{(r)}(x)\right| &\leq \lim_{\eta\to\infty}r!\left|\mathcal{H}_\eta^{(d)}(\mu;x)-\mu(x)\right|+\lim_{\eta\to\infty}\left|r!\mu(x)-\tau^{(r)}(x)\right|\\ &\leq r!\omega(\tau;\delta_\eta(b))+\left|r!\mu(x)-\tau^{(r)}(x)\right|, \qquad (2.13)\end{aligned}$$

where $\delta_\eta(b)$ is defined by Theorem 2.4.

Using some well-known facts of the modulus of continuity and mean theorem, holds:

$$\begin{aligned}|\mu(\delta+x)-\mu(x)| &= \left|\left[\delta+x,\frac{q_\eta}{\eta}+\delta+x,...,r\frac{q_\eta}{\eta}+\delta+x;\tau\right]-\left[x,\frac{q_\eta}{\eta}+x,...,r\frac{q_\eta}{\eta}+x;\tau\right]\right|\\ &=\frac{1}{r!}\left|\frac{d^r}{dx^r}\tau(r\frac{q_\eta}{\eta}\alpha_1+x+\delta)-\frac{d^r}{dx^r}\tau(r\frac{q_\eta}{\eta}\alpha_2+x)\right|\\ &+\frac{1}{r!}\omega(\tau^{(r)};r\frac{q_\eta}{\eta}\left|\alpha_1-\alpha_2\right|+\delta)\\ &\leq\frac{2}{r!}\omega(\tau^{(r)};r\frac{q_\eta}{\eta}+\delta)\end{aligned}$$

where $\alpha_1, \alpha_2 \in (0,1)$, so we get:

$$\omega(\mu,\delta) \leq \frac{1}{r!}\omega(\tau^{(r)}; r\frac{q_\eta}{\eta} + \delta). \tag{2.14}$$

On the other side,

$$\begin{aligned}\left|r!\mu(x) - \frac{d^r}{dx^r}\tau(x)\right| &\leq \left|r!\left[x, \frac{q_\eta}{\eta} + x, ..., r\frac{q_\eta}{\eta} + x; \tau\right] - \tau^{(r)}(x)\right| \\ &\leq \left|\frac{d^r}{dx^r}\tau(x + r\frac{q_\eta}{\eta}\alpha_3) - f^{(r)}(x)\right| \\ &\leq \omega(\tau^{(r)}; \alpha_3 r\frac{q_\eta}{\eta}) \\ &\leq \omega(\tau^{(r)}; r\frac{q_\eta}{\eta}) \end{aligned} \tag{2.15}$$

where $\phi_3 \in (0,1)$. From the initial inequality in the proof, we obtain:

$$\lim_{\eta\to\infty}\left|\frac{d^r}{dx^r}\mathcal{H}_\eta^{(d)}(\tau;x) - \tau^{(r)}(x)\right| \leq \lim_{\eta\to\infty}\frac{2}{r!}\omega(\tau;\delta_\eta(b)) + \lim_{\eta\to\infty}\omega(\tau^{(r)}; r\frac{q_\eta}{\eta}). \tag{2.16}$$

Since $\delta_\eta(b) \to 0$ and $\frac{q_\eta}{\eta} \to 0$, as $\eta \to \infty$ it stems that:

$$\omega(\tau;\delta_\eta(b)) \to 0 \;\; and \;\; \omega(\tau^{(r)}; r\frac{q_\eta}{\eta}) \to 0 \;\; as \; \eta \to \infty.$$

So from last inequality we may conclude that

$$\frac{d^r\mathcal{H}_\eta^{(d)}(\tau;x)}{dx^r} \to \tau^{(r)}(x), \;\; \eta \to \infty.$$

Theorem 2.8. *Let $\mathcal{H}_\eta$ be given (1.3). Then for all $\tau \in C_B^2[0,\infty)$ we have*

$$|\mathcal{H}_\eta(\tau;x) - \tau(x)| \leq \gamma_\eta(x)||\tau||_{C_B^2[0,\infty)}$$

where $\gamma(x) = \gamma_\eta(x) = \alpha_\eta x^2 + \frac{\beta_\eta}{\eta}x + \frac{\gamma_\eta}{\eta^2}$ with

$$\alpha_\eta = \frac{\mathcal{V}''(\frac{\eta x}{q_\eta}) - 2\mathcal{V}'(\frac{\eta x}{q_\eta}) + \mathcal{V}(\frac{\eta x}{q_\eta})}{2\mathcal{V}(\frac{\eta x}{q_\eta})},$$

$$\beta_\eta = \frac{(\mathcal{V}'(\frac{\eta b}{q_\eta}) - \mathcal{V}(\frac{\eta b}{q_\eta}))(\eta\mathcal{Z}(1) + 2(d+1)q_\eta\mathcal{Z}'(1) + q_\eta\mathcal{Z}(1)\mathcal{V}'(\frac{\eta b}{q_\eta})}{\mathcal{Z}(1)\mathcal{V}(\frac{\eta b}{q_\eta})}$$

and

$$\gamma_\eta = q_\eta^2(d+1)^2\frac{\mathcal{Z}'(1) + \mathcal{Z}''(1)}{\mathcal{Z}(1)} + \eta q_\eta(d+1)\frac{\mathcal{Z}'(1)}{\mathcal{Z}(1)}.$$

Proof. Using Mursaleen technique [9], assume that $\tau \in C_B^2[0,\infty)$. Thus from the Taylor's expansion, we can write

$$|\mathcal{H}_\eta(\tau;x) - \tau(x)| = \tau'(\eta)\mathcal{H}_\eta(\Theta_1(s),x) + \frac{1}{2}\tau''(\theta)\mathcal{H}_\eta(\Theta_2(s);x); \theta \in (x,t)$$

Taking into account the fact that

$$\mathcal{H}_\eta(\Theta_1(s);x) = \frac{\mathcal{V}'(\frac{\eta x}{q_\eta}) - \mathcal{V}(\frac{\eta x}{q_\eta})}{\mathcal{V}(\frac{\eta x}{q_\eta})}x + \frac{q_\eta}{\eta}(d+1)\frac{\mathcal{Z}'(1)}{\mathcal{Z}(1)} \geq 0,$$

for $x \leq t$, we can write that

$$|\mathcal{H}_\eta^{(d)}(\tau;x)-\tau(x)| \leq [\frac{\mathcal{V}'(\frac{\eta x}{q_\eta})-\mathcal{V}(\frac{\eta x}{q_\eta})}{\mathcal{V}(\frac{\eta x}{q_\eta})}x+\frac{q_\eta}{\eta}(d+1)\frac{\mathcal{Z}'(1)}{\mathcal{Z}(1)}]||\tau'||_{C_B[0,\infty)}$$
$$+\frac{1}{2}[\frac{\mathcal{V}''(\frac{\eta x}{q_\eta})-2\mathcal{V}'(\frac{\eta x}{q_\eta})+\mathcal{V}(\frac{\eta x}{q_\eta})}{\mathcal{V}(\frac{\eta x}{q_\eta})}x^2+\frac{q_\eta}{\eta}\frac{\mathcal{Z}(1)\mathcal{V}'(\frac{\eta x}{q_\eta})+2(d+1)\mathcal{Z}'(1)(\mathcal{V}'(\frac{\eta x}{q_\eta})-\mathcal{V}(\frac{\eta x}{q_\eta}))}{\mathcal{Z}(1)\mathcal{V}(\frac{\eta x}{q_\eta})}x$$
$$+\frac{q_\eta^2}{\eta^2}(d+1)^2\frac{\mathcal{Z}'(1)+\mathcal{Z}''(1)}{\mathcal{Z}(1)}]||\tau''(x||_{C_B[0,\infty)}$$
$$\leq (\alpha_\eta x^2+\frac{\beta_\eta}{\eta}x+\frac{\gamma_\eta}{\eta^2})||\tau||_{C_B^2[0,\infty)}=\gamma_\eta(x)||\tau||_{C_B^2[0,\infty)} \tag{2.17}$$

which completes the proof.

Theorem 2.9. *Let E be any subset of* $[0,\infty)$. *If is locally Lip*(ϑ) *on E , then we have:*

$$|\mathcal{H}_\eta^{(d)}(\tau;x)-\tau(x)| \leq M_\tau([A(\eta)(1+x^2)]^{\frac{\vartheta}{2}}+2d^\vartheta(x,E)),$$

where, $d(x,E)$ *is the distance between* x *and* E *defined as* $d(x,E)=inf\{|x-y|;y\in E\}$

Proof. Since τ is continuous, then for any $x \leq$ and $y \in E$ satysfies the inequality

$$|\tau(x)-\tau(y)| \leq M_\tau|x-y|^\vartheta,$$

where $\bar{E}$ is closure of E in $(-\infty,\infty)$. Let $(x,z)\in(0,\infty)$ be such that $d(x,E)=|x-z|$. Since $\mathcal{H}_\eta^{(d)}$ is a linear positiv operator, then

$$\begin{aligned}|\mathcal{H}_\eta(\tau;x)-\tau(x)| &\leq \mathcal{H}_\eta(M_\tau|t-x|^\vartheta;x)+M_\tau|z-x|^\vartheta\\ &\leq M_\tau\mathcal{H}_\eta(|(t-x)+(x-z)|^\vartheta;x)+M_\tau|z-x|^\vartheta.\end{aligned} \tag{2.18}$$

Applying inequality $(u+v)^\alpha \leq u^\vartheta+v^\vartheta(u\geq 0, v\geq 0, \vartheta\in[0,1])$, we obtain

$$|\mathcal{H}_n(\tau;x)-\tau(x)| \leq M_\tau(\mathcal{H}_n(|t-x|^\vartheta;x)+|x-z|^\vartheta)+M_g|z-x|^\vartheta.$$

Hence, we get using Hölder inequality which $u=\frac{2}{\vartheta}, v=\frac{2}{2-\vartheta}$, and (2.1) we have

$$\begin{aligned}|\mathcal{H}_n(\tau;x)-\tau(x)| &\leq M_\tau([\mathcal{H}_n((t-x)^2;x)]^{\frac{\vartheta}{2}}+2|x-z|^\vartheta)\\ &\leq M_\tau([A(n)(1+x)^2]^{\frac{\vartheta}{2}}+2d^\vartheta(x,E)]\end{aligned} \tag{2.19}$$

which completes the proof of the Theorem.

Theorem 2.10. *Let* $\tau\in C_B[0,\infty)$. *Then there exists an absolute constant* $c>0$ *independent of the function* τ *and* δ *such that*

$$|\mathcal{H}_n(\tau;x)-\tau(x)| \leq 2C\{min(1,\delta)||\tau||_{C_B})+\omega_2(\tau;\sqrt{\delta})\}$$

where $\delta=\frac{\gamma_\eta(x)}{2}$, *is defined as in Theorem 2.8.*

Proof. Assume that $\tau\in C_B^2[0,\infty)$. From the previous Theorem 2.8, it is clear that

$$|\mathcal{H}_n(\tau;x)-\tau(x)| \leq 2||\tau-g||_{C_B}+\gamma_n(x)||g||_{C_B^2[0,\infty)} \tag{2.20}$$

Now taking the infinum over all $g\in C_B^2[0,\infty)$ sides of (2.20) and using the inequality $K_2(\tau;\delta)\leq c\omega_2(\tau;\sqrt{\delta});\delta>0$, we get the required result.

3 Construct of the bivariate operators

Let $\square = [0,\infty) \times [0,\infty)$ and E be an certain finite. We denote by $C_E(\square) = \{\tau \in C(\square) : |\tau(x,y)| \leq \alpha e^{E(x+y)}\}$ and for $i = 0,1,2,3$ then test function defined by $e_0(x,y) = 1, e_1(x,y) = x, e_2(x,y) = y$ and $e_3(x,y) = x^2 + y^2$.
We define the following generalited Chlodowsky variate of Szasz for two variables with the help of d-Brenke type polinomials as follows:

$$\mathcal{H}^{(d)}_{\eta,m}(\tau;xy) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \frac{\mathcal{L}_{k_1}\left(\frac{\eta x}{q_\eta}\right)\mathcal{L}_{k_2}\left(\frac{\sigma y}{l_\sigma}\right)}{\mathcal{Z}^2(1)\mathcal{V}_1(\frac{\eta x}{q_\eta})\mathcal{V}_2(\frac{\sigma y}{l_\sigma})}\tau\left(\frac{k_1 q_\eta}{\eta}, \frac{k_2 l_\sigma}{\sigma}\right) \tag{3.1}$$

where $\mathcal{Z}(\cdot), \mathcal{V}(\cdot)$ and $\mathcal{L}_k(\cdot)$ have same properties given by (1.1), and $(q_\eta), (l_\sigma)$ is a positive increasing sequences such that:

$$\lim_{\eta\to\infty} q_\eta = \infty, \lim_{\sigma\to\infty} l_\sigma = \infty \text{ and } \lim_{\eta\to\infty} \frac{q_\eta}{\eta} = 0, \lim_{\sigma\to\infty} \frac{l_\sigma}{\sigma} = 0.$$

Lemma 3.1. *Let*

$$\mathcal{Z}_0(s_1^{d+1})\mathcal{Z}_0(s_2^{d+1})\mathcal{V}_1(s_1)\mathcal{V}_2(s_2) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \mathcal{L}_{k_1}(x)\mathcal{L}_{k_2}(y)s_1^{k_1}s_2^{k_2}. \tag{3.2}$$

Then we have

$$\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} \mathcal{L}_{k_1}(\frac{\eta x}{q_\eta})\mathcal{L}_{k_2}(\frac{\sigma y}{l_\sigma}) = \mathcal{Z}^2(1)\mathcal{V}_1(\frac{\eta x}{q_\eta})\mathcal{V}_2(\frac{\sigma y}{l_\sigma})$$

$$\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} k_1\mathcal{L}_{k_1}(\frac{\eta x}{q_\eta})\mathcal{L}_{k_2}(\frac{\sigma y}{l_\sigma}) = \mathcal{Z}'(1)\mathcal{Z}(1)(d+1)\mathcal{V}_1(\frac{\eta x}{q_\eta})\mathcal{V}_2(\frac{\sigma y}{l_\sigma}) + \mathcal{Z}^2(1)\frac{\eta x}{q_\eta}\mathcal{V}_1'(\frac{\eta x}{q_\eta})\mathcal{V}_2(\frac{\sigma y}{l_\sigma})$$

$$\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} k_2\mathcal{L}_{k_1}(\frac{\eta x}{q_\eta})\mathcal{L}_{k_2}(\frac{\sigma y}{l_\sigma}) = \mathcal{Z}'(1)\mathcal{Z}(1)(d+1)\mathcal{V}_1(\frac{\eta x}{q_\eta})\mathcal{V}_2(\frac{\sigma y}{l_\sigma}) + \mathcal{Z}^2(1)\mathcal{V}_1(\frac{\eta x}{q_\eta})\mathcal{V}_2(\frac{\sigma y}{l_\sigma});$$

$$\begin{aligned}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} k_1^2\mathcal{L}_{k_1}(\frac{\eta x}{q_\eta})\mathcal{L}_{k_2}(\frac{\sigma y}{l_\sigma}) = {}& \frac{\eta^2 x^2}{q_\eta^2}\mathcal{Z}^2(1)\mathcal{V}_1''(\frac{\eta x}{q_\eta})\mathcal{V}_2(\frac{\sigma y}{l_\sigma}) \\ &+ \frac{\eta x}{q_\eta}\mathcal{V}_1'(\frac{\eta x}{q_\eta})\mathcal{V}_2(\frac{\sigma y}{l_\sigma})\mathcal{Z}(1)\cdot(\mathcal{Z}(1) + 2\mathcal{Z}'(d+1) \\ &+ \mathcal{Z}(1)\mathcal{V}_1(\frac{\eta x}{q_\eta})\cdot\mathcal{V}_2(\frac{\sigma t}{l_\sigma})((d+1)^2\mathcal{Z}' + \mathcal{Z}''(1)(d+1)^2).\end{aligned}$$

$$\begin{aligned}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} k_2^2\mathcal{L}_{k_1}(\frac{\eta x}{q_\eta})\mathcal{L}_{k_2}(\frac{\sigma y}{l_\sigma}) = {}& \frac{\sigma^2 y^2}{l_\sigma^2}\mathcal{Z}_1^2(1)\mathcal{V}_2''(\frac{\sigma y}{l_\sigma})\mathcal{V}_1(\frac{\eta x}{q_\eta}) \\ &+ \frac{\sigma y}{l_\sigma}\mathcal{V}_1(\frac{\eta x}{q_\eta})\mathcal{V}_2'(\frac{\sigma y}{l_\sigma})\mathcal{Z}(1)\cdot(\mathcal{Z}(1) + 2\mathcal{Z}'(1)(d+1) \\ &+ \mathcal{Z}(1)\mathcal{V}_1(\frac{\eta x}{q_\eta})\cdot\mathcal{V}_2(\frac{\sigma y}{l_\sigma})((d+1)^2\mathcal{Z}' + \mathcal{Z}''(1)(d+1)^2).\end{aligned}$$

Proof. By using the generating function (3.2) for the d-Brenke polynomials, one can easily find the above equalities.

Lemma 3.2. *By using Lemma 3.1, we have*
$\mathcal{H}_{\eta,\sigma}(1;x,y) = 1;$
$\mathcal{H}^{(d)}_{\eta,\sigma}(e_1;x,y) = \frac{\mathcal{V}_1'(\frac{\eta x}{q_\eta})}{\mathcal{V}(\frac{\eta x}{q_\eta})}x + \frac{q_\eta}{\eta}(d+1)\cdot\frac{\mathcal{Z}'(1)}{\mathcal{Z}(1)};$
$\mathcal{H}^{(d)}_{\eta,\sigma}(e_2;x,y) = \frac{\mathcal{V}_2\prime(\frac{\eta x}{q_\eta})}{\mathcal{V}_2(\frac{\eta x}{q_\eta})}y + \frac{l_\sigma}{\sigma}(d+1)\cdot\frac{\mathcal{Z}'(1)}{\mathcal{Z}(1)};$

$\mathcal{H}^{(d)}_{\eta,\sigma}(e_3;x,y) = \frac{\mathcal{V}_1''(\frac{\eta x}{q_\eta})}{\mathcal{V}_1\left(\frac{\eta x}{q_\eta}\right)}x^2 + \frac{\mathcal{V}_2''\left(\frac{\sigma y}{l_\sigma}\right)}{\mathcal{V}_2\left(\frac{\sigma y}{l_\sigma}\right)}y^2 + \frac{q_\eta}{\eta}\frac{\mathcal{V}_1'(\frac{\eta x}{q_\eta})(\mathcal{Z}(1)+2\mathcal{Z}'(1)(d+1))}{\mathcal{Z}(1)\mathcal{V}_1(\frac{\eta x}{q_\eta})}x + \frac{l_\sigma}{\sigma}\frac{\mathcal{V}_2(\frac{my}{l_\sigma})(\mathcal{Z}(1)+2\mathcal{Z}'(1)(d+1))}{\mathcal{Z}(1)\mathcal{V}_2(\frac{\sigma y}{l_\sigma})}y + \frac{(d+1)^2(\mathcal{Z}'(1)+\mathcal{Z}''(1))}{\mathcal{Z}(1)}(\frac{q_\eta^1}{\eta^2} + \frac{l_\sigma^2}{\sigma^2}).$

Theorem 3.3. *If $\tau \in C_E(\Box)$ and and assume that the $\lim\limits_{\lambda\to\infty}\frac{\mathcal{V}_i^{(t)}(\lambda)}{\mathcal{V}_i(\lambda)} = 1$, $t \in \{1,2\}$, $i \in \{1,2\}$, hold for $k = 1,2$, then*

$$\lim_{\eta,\sigma\to\infty}\mathcal{H}^{(d)}_{\eta,m}(\tau;x,y) = \tau(x,y),$$

and the operators $\mathcal{H}^{(d)}_{\eta,\sigma}$ converge uniformuly on $\Box_{ab} = [0,a]*[0,b]$.

Proof. From Theorem 3.2, we have $\lim\limits_{\eta,m\to\infty}\mathcal{H}^{(d)}_{\eta,\sigma}(e_i;x,y) = e_i(x,y)$ $i \in \{0,1,2,3\}$ uniformly on $\Box_{ab}$. Applying Korovkin's theorem, we obtain the required result.

Theorem 3.4. *If $f \in C_E(\Box)$, then for all $(x,y) \in \Box$ we have:*

$$|\mathcal{H}^{(d)}_{\eta,\sigma}(\tau;x,y) - \tau(x,y)| \le 2\omega^{(1)}(\tau;\sqrt{\delta_\eta(b)}) + 2\omega^{(2)}(\tau;\sqrt{\delta_\sigma(c)})$$

where $\delta_\eta(b)$ and $\delta_\sigma(c)$ is defined

Proof. Suppose that $\tau \in C_E(\Box)$. By using Lemma 1.2, we obtain following inequality:

$$\begin{aligned}
|\mathcal{H}^{(d)}_{\eta,\sigma}(\tau;x,y) - \tau(x,y)| &\le \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{\mathcal{L}_{k_1}\left(\frac{\eta x}{q_\eta}\right)\mathcal{L}_{k_2}\left(\frac{\sigma y}{l_\sigma}\right)}{\mathcal{Z}^2(1)\mathcal{V}_1(\frac{\eta x}{q_\eta})\mathcal{V}_2(\frac{\sigma y}{l_\sigma})}|\tau(\frac{k_1 q_\eta}{\eta},\frac{k_2 l_\sigma}{\sigma}) - \tau(x,y)| \\
&\le \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{\mathcal{L}_{k_1}\left(\frac{\eta x}{q_\eta}\right)\mathcal{L}_{k_2}\left(\frac{\sigma y}{l_\sigma}\right)}{\mathcal{Z}^2(1)\mathcal{V}_1(\frac{\eta x}{q_\eta})\mathcal{V}_2(\frac{\sigma y}{l_\sigma})}|\tau(\frac{k_1 q_\eta}{\eta},\frac{k_2 l_\sigma}{\sigma}) - \tau(x,\frac{k_2 l_\sigma}{\sigma})| \\
&+ \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{\mathcal{L}_{k_1}\left(\frac{\eta x}{q_\eta}\right)\mathcal{L}_{k_2}\left(\frac{\sigma y}{l_\sigma}\right)}{\mathcal{Z}^2(1)\mathcal{V}_1(\frac{\eta x}{q_\eta})\mathcal{V}_2(\frac{\sigma y}{l_\sigma})}|\tau(x,\frac{k_2 l_\sigma}{\sigma}) - \tau(x,y)| \\
&= \Delta_1 + \Delta_2.
\end{aligned} \tag{3.3}$$

Now, we consider Δ_1. By using well-known properties of the modulus of continuity, we obtain the formula,

$$\begin{aligned}
\Delta_1 &= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{\mathcal{L}_{k_1}\left(\frac{\eta x}{q_\eta}\right)\mathcal{L}_{k_2}\left(\frac{\sigma y}{l_\sigma}\right)}{\mathcal{Z}^2(1)\mathcal{V}_1(\frac{\eta x}{q_\eta})\mathcal{V}_2(\frac{\sigma y}{l_\sigma})}|\tau(\frac{k_1 q_\eta}{\eta},\frac{k_2 l_\sigma}{\sigma}) - \tau(x,\frac{k_2 l_\sigma}{\sigma})| \\
&\le \{1 + \frac{1}{\delta_\eta}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\frac{\mathcal{L}_{K_1}\left(\frac{\eta x}{q_\eta}\right)\mathcal{L}_{K_2}\left(\frac{\sigma y}{l_\sigma}\right)}{\mathcal{Z}^2(1)\mathcal{V}_1(\frac{\eta x}{q_\eta})\mathcal{V}_2(\frac{\sigma y}{l_\sigma})}|\frac{k_1 q_\eta}{\eta} - x|\}.
\end{aligned} \tag{3.4}$$

By using Theorem 2.5, we obtain:

$$\Delta_1 \le 2\omega^{(1)}(f,\sqrt{\delta_\eta(b)}) \tag{3.5}$$

where $\delta_\eta(b)$ is defined by Theorem 2.5.
In a similarly way, we have

$$\Delta_2 \le 2\omega^{(2)}(f,\sqrt{\delta_\sigma(c)}) \tag{3.6}$$

Putting (3.5) and (3.6) in (3.3), we obtain the desired result.

Theorem 3.5. *Let $\omega^{(1)}\left(\frac{d^r\tau}{dx^r},\cdot\right)$ and $\omega^{(2)}\left(\frac{d^r\tau}{dx^r},\cdot\right)$ is the partial continuity modulus of r-th derivative of τ where $\tau\in C_E^r(\square)$. Then using the equality $\lim\limits_{\lambda\to\infty}\frac{\mathcal{V}_i^{(t)}(\lambda)}{\mathcal{V}_i(\lambda)}=1,\ i=1,2,\ t\in\{1,2,..,r\}$, we obtain the approximation*

(i) $\frac{d^r}{dx^r}\mathcal{H}_\eta^{(d)}(\tau;x,y)\to\frac{d^r}{dx^r}\tau(x,y),\ \ \eta,\sigma\to\infty;$

(ii) $\frac{d^r}{dy^r}\mathcal{H}_\eta^{(d)}(\tau;x,y)\to\frac{d^r}{dy^r}\tau(x,y),\ \ \eta,\sigma\to\infty;$

Proof. By simple calculations, the following formula is obtained

$$\begin{aligned}\lim_{\eta,\sigma\to\infty}\frac{d^s}{dx^s}\mathcal{H}_{\eta,\sigma}^{(d)}(\tau;x,y) &= \lim_{\eta,\sigma\to\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}r!\frac{\mathcal{L}_{K_1}\left(\frac{\eta x}{q_\eta}\right)\mathcal{L}_{K_2}\left(\frac{\sigma y}{l_\sigma}\right)}{\mathcal{Z}^2(1)\mathcal{V}_1(\frac{\eta x}{q_\eta})\mathcal{V}_2(\frac{\sigma y}{l_\sigma})}\frac{\Delta^r_{\frac{q_\eta}{\eta}}\tau(\frac{k_1q_\eta}{\eta},\frac{k_2l_\sigma}{\sigma})}{(\frac{q_\eta}{\eta})^r}\\ &= \lim_{\eta,\sigma\to\infty}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}r!\frac{\mathcal{L}_{k_1}\left(\frac{\eta x}{q_\eta}\right)\mathcal{L}_{k_2}\left(\frac{\sigma y}{l_\sigma}\right)}{\mathcal{Z}^2(1)\mathcal{V}_1(\frac{\eta x}{q_\eta})\mathcal{V}_2(\frac{\sigma y}{l_\sigma})}[\frac{k_1}{\eta}q_\eta,\frac{k_1+1}{\eta}q_\eta,...,\frac{k_1+r}{\eta}q_\eta,\tau;\frac{k_2l_\sigma}{\sigma}]\\ &= \lim_{\eta,\sigma\to\infty}\frac{r!}{\mathcal{Z}^2(1)\mathcal{V}_1(\frac{\eta x}{q_\eta})\mathcal{V}_2(\frac{\sigma y}{l_\sigma})}\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\mathcal{L}_{k_1}(\frac{\eta x}{q_\eta})\mathcal{L}_{k_2}(\frac{\sigma y}{l_\sigma})\varphi\left(\frac{k_1\eta}{q_\eta},\frac{k_2l_\sigma}{\sigma}\right)\\ &= \lim_{\eta,\sigma\to\infty}r!\mathcal{H}_{\eta,\sigma}^{(d)}(\varphi;x,y)\end{aligned}$$

where $\varphi(x,y)=[x,x+\frac{q_\eta}{\eta},...,x+r\frac{q_\eta}{\eta};\tau;y]$ and so we obtain

$$\lim_{\eta,\sigma\to\infty}\frac{d^r}{dx^r}\mathcal{H}_{\eta,\sigma}^{(d)}(\tau;x,y)=\lim_{\eta,\sigma\to\infty}r!\mathcal{H}_{\eta,\sigma}^{(d)}(\varphi;x,y).$$

Then, using Theorem 3.4, we get

$$\begin{aligned}\lim_{\eta,\sigma\to\infty}|\frac{d^r}{dx^r}\tau(x,y)\mathcal{H}_{\eta,\sigma}^{(d)}(\varphi;x,y)-\frac{d}{dx^r}| &\le \lim_{\eta,\sigma\to\infty}r!|\mathcal{H}_{\eta,\sigma}^{(d)}(\varphi;x,y)-\varphi(x,y)|\\ &+\lim_{\eta,\sigma\to\infty}|r!\varphi(x,y)-\frac{d^r}{dx}\tau(x,y)|\le\lim_{\eta,\sigma\to\infty}2r!\omega^{(1)}(\varphi;\sqrt{\delta_\eta(a)})\\ &+\lim_{\eta,\sigma\to\infty}2r!\omega^{(2)}(\varphi;\sqrt{\delta_\sigma(b)})+\lim_{\eta,\sigma\to\infty}|r!\varphi(x,y)-\frac{d^r}{dx}\tau(x,y)| \quad (3.8)\end{aligned}$$

where $\delta=\delta_\eta(b)$ and $\delta_\sigma(c)$ is the same as in the Theorem 2.4. On other hand, we write

$$\begin{aligned}|\varphi(x+\delta_1,y+\delta_2)-\varphi(x,y)| &\le |\varphi(x+\delta_1,y+\delta_2)-\varphi(x+\delta_1,y)|\\ &+|\varphi(x+\delta_1,y)-\varphi(x,y)|=A+B. \quad (3.9)\end{aligned}$$

In first we estimate A:

$$\begin{aligned}A=|\varphi(x+\delta_1,y+\delta_2)-\varphi(x+\delta_1,y)| &= |[y+\delta_2,y+\delta_2+\frac{l_\sigma}{\sigma},...,y+\delta_2+\frac{rl_\sigma}{\sigma};\tau,x+\delta_1]\\ &-[y,y+\frac{l_\sigma}{\sigma},...,y+\frac{rl_\sigma}{\sigma};\tau;x+\delta_1]|\\ =\frac{1}{r!}|\frac{d^r}{dy^r}\tau(x+\delta_1,y+\delta_2+\Theta_1\frac{ra_m}{m})-\frac{d^r}{dy^r}\tau(x+\delta_1,y+\delta_2+\Theta_2\frac{rl_\sigma}{\sigma})| \quad (3.10)\end{aligned}$$

where $0<\Theta_1,\Theta_2<1$, therefore:

$$A_1\le\frac{1}{r!}\omega^{(2)}(\frac{d^r}{dy^r}\tau;\delta_2+|\Theta_1-\Theta_2|\frac{rl_\sigma}{\sigma})\le\frac{1}{r!}\omega^{(2)}(\frac{d^r}{dy^r}\tau;\delta_2+\frac{rl_\sigma}{\sigma})$$

Hence, we obtain

$$\omega^{(2)}(\varphi;\delta_2)\le\frac{1}{r!}\omega^{(2)}(\frac{d^r}{dy^r}\tau;\delta_2+\frac{rl_\sigma}{\sigma}). \quad (3.11)$$

Similarly,

$$B_1 \leq \frac{1}{r!}\omega^{(1)}(\frac{d^r}{dx^r}\tau;\delta_1+\frac{rq_\eta}{\eta})$$

and

$$\omega(\varphi;\delta_1) \leq \frac{1}{r!}\omega^{(1)}(\frac{d^r}{dx^r}\tau;\delta_1+\frac{rq_\eta}{\eta}). \tag{3.12}$$

Putting (3.11) and (3.12) in (3.10), we get

$$|\varphi(x+\delta_1,y+\delta_2-\varphi(x,y)| \leq \frac{1}{r!}\omega^{(1)}(\frac{d^r}{dx^r}\tau;\delta_1+\frac{rq_\eta}{\eta})+\omega^{(2)}(\frac{d^r}{dx^r}\tau;\delta_2+\frac{ra_m}{m})$$

On the other hand, we may write

$$\begin{aligned}|r!\varphi(x,y)-\frac{d^r}{dx^r}\tau(x,y)| &= |r![x,x+\frac{q_\eta)}{\eta},x+\frac{2}{\eta}q_\eta,...,x+\frac{r}{\eta}q_\eta;\tau;y]-\frac{\partial^r}{\partial x^r}\tau(x,y)| \\ &\leq |\frac{\partial^r}{\partial x^r}\tau(x+\frac{r}{\eta}q_\eta\theta_3)-\frac{d^r}{dx^r}\tau(x,y)| \\ &\leq \omega^{(1)}(\frac{\partial^r}{\partial x^r}\tau;\theta_3\frac{r}{\eta}q_\eta) \\ &\leq \omega^{(1)}(\frac{d^r}{dx^r}\tau;\frac{r}{\eta}q_\eta)\end{aligned}$$

where $\theta_3 \in (0,1)$, therefore

$$\begin{aligned}&\lim_{\eta,\sigma\to\infty}|\frac{d^r}{dx^r}\mathcal{H}^{(d)}_{\eta,\sigma}(\varphi;x,y)-\frac{d}{dx^r}\tau(x,y)| \leq \lim_{\eta,\sigma\to\infty}r!|\mathcal{H}^{(d)}_{\eta,\sigma}(\varphi;x,y)-\varphi(x,y)|+\lim_{\eta,\sigma\to\infty}|r!\varphi(x,y)-\frac{d^r}{dx}\tau(x,y)| \\ &\leq \lim_{\eta,\sigma\to\infty}2r!\omega^{(1)}(\varphi;\sqrt{\delta_\eta(a)})+\lim_{\eta,\sigma\to\infty}2r!\omega^{(2)}(\varphi;\sqrt{\delta_\sigma(b)})+\lim_{\eta,\sigma\to\infty}\omega^{(1)}(\frac{d^r}{dx^r}\tau;\frac{r}{\eta}q_\eta).\end{aligned}$$

Since $\delta_\eta(b), \delta_\sigma(c)$ and $\frac{r}{\eta}q_\eta$ tend to zero as $\eta,\sigma\to\infty$, it follows that $\omega^{(1)}(\varphi;\sqrt{\delta_\eta(a)}), \omega^{(2)}(\varphi;\sqrt{\delta_\sigma(b)})$ and $\omega^{(1)}(\frac{d^r}{dx^r}\tau;\frac{r}{\eta}q_\eta)$ tend to zero. Since

$$\frac{d^r}{dx^r}\mathcal{H}^{(d)}_\eta(\tau;x,y) \to \frac{d^r}{dx^r}\tau(x,y), \ \ \eta,\sigma\to\infty;$$

The proof ii) can be made in a similar way.

Theorem 3.6. *For every $\tau \in C_E(\square)$ such that τ_x, τ_y, τ_{xx}, $\tau_{yy} \in C_E(\square)$, by using the equality $\lim_{\lambda\to\infty}\frac{\mathcal{V}^{(t)}(\lambda)}{\mathcal{V}(\lambda)}=1, t\in\{1,2,..,r\}$, we obtain*

$$\lim_{\eta\to\infty}\frac{\eta}{q_\eta}\mathcal{H}^{(d)}_{\eta,\eta}(\tau;x,y)=\frac{(d+1)\mathcal{Z}'(1)}{\mathcal{Z}(1)}\left[\tau'_x(x,y)+\tau'_y(x,y)\right]+\frac{x}{2}\tau''_{x^2}(x,y)+\frac{y}{2}\tau''_{y^2}(x,y)$$

uniformly with respect to $(x,y)\in\square_{ab}$.

Proof. For a fixed $(x,y)\in\square$ and for all $(u,v)\in\square$, by the Taylor formula we have

$$\begin{aligned}\tau(u,v)&=\tau(x,y)+\tau'_x(x,y)(u-x)+\tau'_y(x,y)(v-y)+\frac{1}{2}\{\tau''_{x^2}(x,y)(u-x)^2 \\ &+\tau''_{yy}(x,y)(u-x)(v-y)+\tau''_{y^2}(x,y)(v-y)^2\}+\psi(u,v)((u-x)^2+(v-y)^2)\end{aligned} \tag{3.13}$$

where $\psi(u,v)$ is a function belong to the space $C_E(\Box)$ and $\lim\limits_{(u,v)\to(x,y)} \psi(u,v) = 0$.
From the linearity of $\mathcal{H}^{(d)}_{\eta,\eta}$ and by (3.13) we can write

$$\begin{aligned}\frac{\eta}{q_\eta}\mathcal{H}^{(d)}_{\eta,\eta}(\tau(u,v);x,y) &= \frac{\eta}{q_\eta}\tau(x,y) + \frac{\eta}{q_\eta}\tau'_x(x,y)\mathcal{H}^{(d)}_{\eta,\eta}((u-x);x,y)\\ &+ \frac{\eta}{q_\eta}\mathcal{H}^{(d)}_{\eta,\eta}((u-y));x,y) + \frac{1}{2}\{\tau''_{x^2}(x,y)\frac{\eta}{q_\eta}\mathcal{H}^{(d)}_{\eta,\eta}((u-x)^2;x,y) + \frac{\eta}{q_\eta}\mathcal{H}^{(d)}_{\eta,\eta}((u-x,y)\\ &+ \tau''_{y^2}(x,y)\frac{\eta}{q_\eta}\mathcal{H}^{(d)}_{\eta,\eta}((v-y)^2;x,y)\} + \frac{\eta}{q_\eta}\mathcal{H}^{(d)}_{\eta,\eta}(\psi(u,v)((u-x)^2+(v-y)^2),x,y)).\end{aligned} \tag{3.14}$$

From Lemma 3.2, by using linearity of operators $\mathcal{H}^{(d)}_{\eta,\eta}$ we deduce

$$\begin{aligned}\lim_{\eta\to\infty}\frac{\eta}{q_\eta}\mathcal{H}^{(d)}_{\eta,\eta}(u-x;x,y) &= \lim_{\eta\to\infty}\frac{\eta}{q_\eta}\frac{\mathcal{V}'(\frac{\eta x}{q_\eta})-\mathcal{V}(\frac{\eta x}{q_\eta})}{\mathcal{V}(\frac{\eta x}{q_\eta})}x + \frac{(d+1)\mathcal{V}'(1)}{\mathcal{V}(1)}\\ &= \frac{(d+1)\mathcal{V}'(1)}{\mathcal{V}(1)};\\ \lim_{\eta\to\infty}\frac{\eta}{q_\eta}\mathcal{H}^{(d)}_{\eta,\eta}((u-x)^2;x,y) &= x;\\ \lim_{\eta\to\infty}\frac{\eta}{q_\eta}\mathcal{H}^{(d)}_{\eta,\eta}(v-y;y) &= \frac{(d+1)\mathcal{Z}'(1)}{\mathcal{Z}(1)};\\ \lim_{\eta\to\infty}\frac{\eta}{q_\eta}\mathcal{H}^{(d)}_{\eta,\eta}((v-y)^2;y) &= y.\end{aligned} \tag{3.15}$$

Applying the Hölder inequality for the last term on the right side of (3.14), we obtain

$$\begin{aligned}&\frac{\eta}{q_\eta}\mathcal{H}^{(d)}_{\eta,\eta}(\psi(u,v)((u-x)^2+(v-y)^2),x,y))\\ &\qquad\leq \{\mathcal{H}^{(d)}_{\eta,\eta}(\psi^2(u,v),x,y)\}^{\frac{1}{2}}\cdot\{\mathcal{H}^{(d)}_{\eta,\eta}((u-x)^2+(v-y)^2,x,y)\}^{\frac{1}{2}}\\ &\leq \sqrt{2}\{\mathcal{H}^{(d)}_{\eta,\eta}(\psi^2(u,v),x,y)\}^{\frac{1}{2}}\{\frac{\eta^2}{q_\eta^2}\mathcal{H}^{(d)}_{\eta,\eta}((u-x_0)^4,x,y) + \frac{\eta^2}{q_\eta^2}\mathcal{H}^{(d)}_{\eta,\eta}((v-y)^4,x,y)\}^{\frac{1}{2}}.\end{aligned} \tag{3.16}$$

By Theorem 3.3 we get

$$\lim_{n\to\infty}\mathcal{H}^{(d)}_{\eta,\eta})\psi(u,v;x,y) = \psi^2(x,y) = 0, \tag{3.17}$$

Using (3.17) and Lemma 3.2, we obtain from (3.14)

$$\lim_{n\to\infty}\frac{\eta}{q_\eta}\mathcal{H}^{(d)}_{\eta,\eta})(\psi(u,v)(u-x)^2+(v-y)^2),x,y) = 0 \tag{3.18}$$

Now, taking the lim as $\eta\to\infty$ in (3.14) and using (3.18) and (3.16) we have

$$\lim_{\eta\to\infty}\frac{\eta}{q_\eta}\mathcal{H}^{(d)}_{\eta,\eta}(\tau;x,y) = \frac{(d+1)\mathcal{Z}'(1)}{\mathcal{Z}(1)}\left[\tau'_x(x,y)+\tau'_y(x,y)\right] + \frac{x}{2}\tau''_{x^2}(x,y) + \frac{y}{2}\tau''_{y^2}(x,y)$$

which completes the proof.

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