Palestine Journal of Mathematics

Vol. 11(2)(2022) , 129–135

On some properties of $\mathcal{I}^{\mathcal{K}}-$convergence

Shyamal Debnath and Chiranjib Choudhury

Communicated by V. Lokesha

MSC 2020 Classifications: Primary 40A35; Secondary 40A05.

Keywords and phrases: Ideal, Filter, $\mathcal{I}-$convergence, $\mathcal{I}^{\mathcal{K}}-$convergence.

The authors are thankful to the referee for his/her useful comments. The second author is grateful to ***University Grants Commission, India*** *for their fellowships funding under the* ***UGC-JRF*** *scheme during the preparation of this paper.*

Abstract. In 2011 M. Macaj and M. Sleziak first introduced the concept of $\mathcal{I}^{\mathcal{K}}$ convergence mainly as a generalization of statistical convergence. In this paper, we introduce and investigate a few interesting properties of $\mathcal{I}^{\mathcal{K}}$ convergence of real sequences. We also study some implication relations between $\mathcal{I}$, $\mathcal{I}^*$, and $\mathcal{I}^{\mathcal{K}}-$convergence of such sequences. Further, we find the conditions under which (i) $\mathcal{I}^{\mathcal{K}}-$convergence implies $\mathcal{K}-$convergence, (ii) $\mathcal{I}^{\mathcal{K}}-$convergence implies $\mathcal{I}-$convergence, (iii) $\mathcal{I}-$convergence implies $\mathcal{I}^{\mathcal{K}}-$convergence, and (iv) $\mathcal{K}^{\mathcal{I}}-$convergence implies $\mathcal{I}^{\mathcal{K}}-$convergence.

1 Introduction

The main objective of this paper is to deal with various properties of $\mathcal{I}^{\mathcal{K}}-$convergence which is a generalization of $\mathcal{I}^*-$convergence.

In 2000 the idea of ideal convergence was first introduced by Kostrkyo and Salat[11]. They studied various fundamental properties of $\mathcal{I}$ and $\mathcal{I}^*-$convergence and found that their idea is a generalization of so many important convergence concepts introduced earlier. Based on $\mathcal{I}-$convergence several generalizations was made by researchers and several analytical and topological properties has been investigated (see [6],[7],[9],[10],[14],[15]) and this area becomes one of the most active areas of research.

On the other hand in 2011, M. Macaj and M. Sleziak extended the idea of $\mathcal{I}^*-$convergence to $\mathcal{I}^{\mathcal{K}}-$convergence(see [12]) where $\mathcal{I}$ and $\mathcal{K}$ both are ideals. Later on, several investigations have been done in this area and some fruitful outcome has been reflected in ([1],[2],[3],[4],[5],[13],[16]).

2 Definitions and Preliminaries

Definition 2.1. A family $\mathcal{I} \subset 2^X$ of subsets of a nonempty set X is said to be an ideal in X if and only if (i) $\emptyset \in \mathcal{I}$ (ii) $A, B \in \mathcal{I}$ implies $A \cup B \in \mathcal{I}$ (Additive) and (iii) $A \in \mathcal{I}, B \subset A$ implies $B \in \mathcal{I}$ (Hereditary).

If $\forall x \in X$, $\{x\} \in \mathcal{I}$ then $\mathcal{I}$ is said to be admissible. Also, $\mathcal{I}$ is said to be non-trivial if $X \notin \mathcal{I}$ and $\mathcal{I} \neq \{\emptyset\}$.

Some standard examples of ideal are given below:
(i) The set $\mathcal{I}_f$ of all finite subsets of $\mathbb{N}$ is an admissible ideal in $\mathbb{N}$.
(ii) The set $\mathcal{I}_d$ of all subsets of natural numbers having natural density 0 is an admissible ideal in $\mathbb{N}$.
(iii) The set $\mathcal{I}_c = \{A \subseteq \mathbb{N} : \sum_{a\in A} a^{-1} < \infty\}$ is an admissible ideal in $\mathbb{N}$.
(iv) Suppose $\mathbb{N} = \bigcup\limits_{p=1}^{\infty} D_p$ be a decomposition of $\mathbb{N}$ (for $i \neq j$, $D_i \cap D_j = \emptyset$). Then the set $\mathcal{I}$ of all subsets of $\mathbb{N}$ which intersects finitely many D_p's forms an ideal in $\mathbb{N}$.

More important examples can be found in [9] and [10].

Definition 2.2. A family $\mathcal{F} \subset 2^X$ of subsets of a nonempty set X is said to be a filter in X if and only if (i) $\emptyset \notin \mathcal{F}$ (ii) $M, N \in \mathcal{F}$ implies $M \cap N \in \mathcal{F}$ and (iii) $M \in \mathcal{F}, N \supset M$ implies $N \in \mathcal{F}$.

If $\mathcal{I}$ is a proper non-trivial ideal in X, then $\mathcal{F}(\mathcal{I}) = \{M \subset X : \exists\, A \in \mathcal{I}\; s.t\; M = X \setminus A\}$ is a filter in X. It is called the filter associated with the ideal $\mathcal{I}$.

Definition 2.3. [11] A sequence $x = (x_k)$ is said to be $\mathcal{I}-$convergent to l if and only if for every $\varepsilon > 0$, the set $\{k \in \mathbb{N} : |x_k - l| \geq \varepsilon\}$ belongs to $\mathcal{I}$. The real number l is called the $\mathcal{I}-$limit of the sequence $x = (x_k)$. Symbolically, $\mathcal{I} - \lim x = l$.

Definition 2.4. [11] Let $\mathcal{I}$ be an admissible ideal in $\mathbb{N}$. A sequence $x = (x_k)$ is said to be $\mathcal{I}^*-$convergent to l, if there exists a set $M = \{m_1 < m_2 < ... < m_k < ...\}$ in the associated filter $\mathcal{F}(\mathcal{I})$ such that $\lim\limits_{k} x_{m_k} = l$.

Definition 2.5. [5] Let $\mathcal{I}$ and $\mathcal{K}$ be two ideals on the same set X. Then $\mathcal{I}$ is said to have the additive property with respect to $\mathcal{K}$ or the condition $AP(\mathcal{I}, \mathcal{K})$ holds if for every countable family of mutually disjoint sets $(A_1, A_2, ...)$ from $\mathcal{I}$ there exists a countable family of sets $(B_1, B_2, ...)$ in $\mathcal{I}$ such that the symmetric differences $A_j \triangle B_j \in \mathcal{K}$ for every $j \in \mathbb{N}$ and $\bigcup\limits_{j=1}^{\infty} B_j \in \mathcal{I}$.

In particular, if we consider $X = \mathbb{N}$ and $\mathcal{K} = \mathcal{I}_f$ then we obtain the condition AP which was introduced by Kostyrko et al. in [11].

Definition 2.6. [12] Let $\mathcal{I}$ and $\mathcal{K}$ be two ideals in $\mathbb{N}$. A sequence $x = (x_k)$ is said to be $\mathcal{I}^{\mathcal{K}}-$convergent to l if there exists $M \in \mathcal{F}(\mathcal{I})$ such that the sequence $y = (y_k)$ defined by $y_k = \begin{cases} x_k, & k \in M \\ l, & k \notin M \end{cases}$ is $\mathcal{K}-$convergent to l.

If we consider $\mathcal{K} = \mathcal{I}_f$ then $\mathcal{I}^{\mathcal{K}}-$convergence concept coincides with $\mathcal{I}^*-$convergence [11]. Further, if we take $\mathcal{K} = \mathcal{I}_d$ then we get $\mathcal{I}^*-$statistical convergence which was introduced by Debnath and Rakshit in [6].

Note that $\mathcal{I}^{\mathcal{I}_d}-$convergence implies $\mathcal{I}-$statistical convergence (Theorem 3.9 of [6]).

Example 2.7. Consider the decomposition of $\mathbb{N}$ given by $\mathbb{N} = \bigcup\limits_{p=1}^{\infty} D_p$ where $D_p = \{2^{p-1}(2s - 1) : s = 1, 2, 3, ..\}$. Let $\mathcal{I}$ be the ideal consisting of all subsets of $\mathbb{N}$ which intersects finite number of D_p's. Consider the sequence $x = (x_k)$ defined by $x_k = \frac{1}{p}$ if $k \in D_p$. Then the sequence is $\mathcal{I}^{\mathcal{I}}-$convergent to 0.
Justification: Let $M = \mathbb{N} \setminus D_1$. Then $M \in \mathcal{F}(\mathcal{I})$ and it is easy to verify that the sequence $y = (y_k)$ defined by $y_k = \begin{cases} x_k, & k \in M \\ 0, & k \notin M \end{cases}$ is $\mathcal{I}-$convergent to 0. Thus $\mathcal{I}^{\mathcal{I}} - \lim x = 0$.

Remark 2.8. [1] If $\mathcal{I}$ and $\mathcal{K}$ are two ideals in $\mathbb{N}$ then the set $\mathcal{I} \vee \mathcal{K} = \{A \cup B : A \in \mathcal{I}, B \in \mathcal{K}\}$ forms an ideal in $\mathbb{N}$. Further if $\mathcal{I} \vee \mathcal{K}$ is non-trivial then the dual filter of $\mathcal{I} \vee \mathcal{K}$ is denoted and defined by $\mathcal{F}(\mathcal{I} \vee \mathcal{K}) = \{M \cap N : M \in \mathcal{F}(\mathcal{I}), N \in \mathcal{F}(\mathcal{K})\}$.

Throughout the paper, unless stated, the symbols $\mathcal{I}, \mathcal{K}, \mathcal{I} \vee \mathcal{K}, \mathcal{I}_1, \mathcal{I}_2, \mathcal{K}_1$, and $\mathcal{K}_2$ stands for non-trivial admissible ideal in $\mathbb{N}$, and the sequences that we have considered are real sequences.

3 Main Results

Theorem 3.1. *Suppose $x = (x_k)$ be a sequence such that $\mathcal{I}^{\mathcal{K}} - \lim x = l$. Then l is unique.*

Proof. If possible suppose there exists $l_1, l_2 \in \mathbb{R}, l_1 \neq l_2$ such that

$$\mathcal{I}^{\mathcal{K}} - \lim x = l_1 \;\; and \;\; \mathcal{I}^{\mathcal{K}} - \lim x = l_2.$$

So there exists $M, N \in \mathcal{F}(\mathcal{I})$ such that the sequences $y = (y_k)$ defined as $y_k = \begin{cases} x_k, & k \in M \\ l_1, & k \notin M \end{cases}$ and $z = (z_k)$ defined as $z_k = \begin{cases} x_k, & k \in N \\ l_2, & k \notin N \end{cases}$ are $\mathcal{K}-$convergent to l_1 and l_2, respectively.

Therefore by Theorem 2.1 of [10], we can say that the sequence $y-z=(y_k-z_k)$ defined as $y_k-z_k=\begin{cases}0, & k\in M\cap N\\ x_k-l_2, & k\in M\setminus N\\ l_1-x_k, & k\in N\setminus M\\ l_1-l_2, & k\in M^c\cap N^c\end{cases}$ is $\mathcal{K}-$convergent to l_1-l_2. Thus by definition

$$\forall\varepsilon>0,\ \{k\in\mathbb{N}:|(y_k-z_k)-(l_1-l_2)|\geq\varepsilon\}\in\mathcal{K}. \tag{3.1}$$

Choose $\varepsilon=\frac{|l_1-l_2|}{2}$. Then from Eq. (3.1) we get

$$\{k\in\mathbb{N}:|(y_k-z_k)-(l_1-l_2)|\geq\frac{|l_1-l_2|}{2}\}\in\mathcal{K}.$$

Now as the inclusion $M\cap N\subseteq\{k\in\mathbb{N}:|(y_k-z_k)-(l_1-l_2)|\geq\frac{|l_1-l_2|}{2}\}$ holds, so by hereditary of $\mathcal{K}$, $M\cap N\in\mathcal{K}$ which implies $\mathbb{N}\setminus(M\cap N)\in\mathcal{F}(\mathcal{K})$. Again as $M,N\in\mathcal{F}(\mathcal{I})$, so $M\cap N\in\mathcal{F}(\mathcal{I})$. Now $\mathbb{N}\setminus(M\cap N)\in\mathcal{F}(\mathcal{K})$ and $M\cap N\in\mathcal{F}(\mathcal{I})$ implies $(\mathbb{N}\setminus(M\cap N))\cap(M\cap N)\in\mathcal{F}(\mathcal{I}\vee\mathcal{K})$ i.e $\emptyset\in\mathcal{F}(\mathcal{I}\vee\mathcal{K})$, a contradiction. □

Theorem 3.2. *Let $\mathcal{I}$, $\mathcal{K}$ and $\mathcal{I}\vee\mathcal{K}$ be non-trivial ideal in $\mathbb{N}$ such that $\mathcal{I}^{\mathcal{K}}-\lim x=l_1$ and $\mathcal{I}^{\mathcal{K}}-\lim y=l_2$. Then-*
(i) $\mathcal{I}^{\mathcal{K}}-\lim(x+y)=l_1+l_2$ (ii) $\mathcal{I}^{\mathcal{K}}-\lim(xy)=l_1l_2$.

Proof. (i) Suppose $\mathcal{I}^{\mathcal{K}}-\lim x=l_1$ and $\mathcal{I}^{\mathcal{K}}-\lim y=l_2$. Then by definition, there exists $M,N\in\mathcal{F}(\mathcal{I})$ such that the sequences $u=(u_k)$ defined by $u_k=\begin{cases}x_k, & k\in M\\ l_1, & k\notin M\end{cases}$ and $v=(v_k)$ defined by $v_k=\begin{cases}y_k, & k\in N\\ l_2, & k\notin N\end{cases}$ are $\mathcal{K}-$convergent to l_1 and l_2, respectively. By Theorem 2.1(ii) of [10], the sequence $u+v=(u_k+v_k)$ defined by $u_k+v_k=\begin{cases}x_k+y_k, & k\in M\cap N\\ x_k+l_2, & k\in M\setminus N\\ y_k+l_1, & k\in N\setminus M\\ l_1+l_2, & k\in M^c\cap N^c\end{cases}$ is $\mathcal{K}-$convergent to l_1+l_2. In other words

$$\forall\,\varepsilon>0,\ \{k\in\mathbb{N}:|(u_k+v_k)-(l_1+l_2)|\geq\varepsilon\}\in\mathcal{K}. \tag{3.2}$$

By definition of $u+v$ we have

$$\begin{aligned}\{k\in\mathbb{N}:|(u_k+v_k)-(l_1+l_2)|\geq\varepsilon\}=&\{k\in M\cap N:|(x_k+y_k)-(l_1+l_2)|\geq\varepsilon\}\\ &\cup\{k\in M\setminus N:|x_k-l_1|\geq\varepsilon\}\\ &\cup\{k\in N\setminus M:|y_k-l_2|\geq\varepsilon\}.\end{aligned} \tag{3.3}$$

Clearly $M\cap N\in\mathcal{F}(\mathcal{I})$. Now consider the sequence $w=(w_k)$ defined as $w_k=\begin{cases}x_k+y_k, & k\in M\cap N\\ l_1+l_2, & k\notin M\cap N\end{cases}$. Then from Eq. (3.2), (3.3) and by definition of w

$$\begin{aligned}\{k\in\mathbb{N}:|w_k-(l_1+l_2)|\geq\varepsilon\}=&\{k\in M\cap N:|w_k-(l_1+l_2)|\geq\varepsilon\}\\ &\cup\{k\in(M\cap N)^c:|w_k-(l_1+l_2)|\geq\varepsilon\}\\ =&\{k\in M\cap N:|(x_k+y_k)-(l_1+l_2)|\geq\varepsilon\}\\ &\subseteq\{k\in\mathbb{N}:|(u_k+v_k)-(l_1+l_2)|\geq\varepsilon\}\in\mathcal{K}.\end{aligned} \tag{3.4}$$

From Eq. (3.4), it is clear that w is $\mathcal{K}-$convergent to l_1+l_2. Hence $x+y$ is $\mathcal{I}^{\mathcal{K}}-$convergent to l_1+l_2.
(ii) We omitted the proof as it can be obtained by applying similar technique. □

Theorem 3.3. *Let $\mathcal{I}^*-\lim x=l$ then $\mathcal{I}^{\mathcal{K}}-\lim x=l$.*

Proof. Let $\mathcal{I}^* - \lim x = l$. Then there exists a set $M = \{m_1 < m_2 < ... < m_k < ...\} \in \mathcal{F}(\mathcal{I})$ such that $\lim_k x_{m_k} = l$. Which implies that the sequence $y = (y_k)$ defined as $y_k = \begin{cases} x_k, & k \in M \\ l, & k \notin M \end{cases}$ is ordinary convergent to l. Now by Theorem 2.1 of [10], we can say that for any ideal $\mathcal{K}$, the sequence $y = (y_k)$ is $\mathcal{K}-$convergent to l. Hence $x = (x_k)$ is $\mathcal{I}^{\mathcal{K}}-$convergent to l. □

Remark 3.4. Converse of the above theorem is not necessarily true. Consider Example 2.7. The sequence is $\mathcal{I}^{\mathcal{K}}-$convergent to 0 for $\mathcal{K} = \mathcal{I}$. But it is not $\mathcal{I}^*-$convergent to 0 (Example 2.1 of [10]).

Theorem 3.5. *Let* $\mathcal{K} - \lim x = l$ *then* $\mathcal{I}^{\mathcal{K}} - \lim x = l$.

Proof. Since $\mathcal{K} - \lim x = l$, so for every $\varepsilon > 0$,

$$\{k \in \mathbb{N} : |x_k - l| \geq \varepsilon\} \in \mathcal{K}. \tag{3.5}$$

Choose $M = \mathbb{N}$ from $\mathcal{F}(\mathcal{I})$. Consider the sequence $y = (y_k)$ defined by $y_k = x_k, k \in M$. Then using Eq. (3.5), we get for every $\varepsilon > 0$, $\{k \in \mathbb{N} : |y_k - l| \geq \varepsilon\} \in \mathcal{K}$ i.e $y = (y_k)$ is $\mathcal{K}-$convergent to l. Hence $\mathcal{I}^{\mathcal{K}} - \lim x = l$. □

Remark 3.6. Converse of Theorem 3.5 is not necessarily true.

Example 3.7. Consider the ideals $\mathcal{I}_c = \{A \subseteq \mathbb{N} : \sum_{a \in A} a^{-1} < \infty\}$ and $\mathcal{I}_d = \{A \subseteq \mathbb{N} : d(A) = 0\}$. Let $x = (x_k)$ be the sequence defined as $x_k = \begin{cases} 1, & k \text{ is prime} \\ 0, & k \text{ is not prime} \end{cases}$. Then there exists $M =$ set of all non-prime numbers $\in \mathcal{F}(\mathcal{I}_{\mathrm{d}})$ such that the sequence $y = (y_k)$ defined as $y_k = \begin{cases} x_k, & k \in M \\ 0, & k \notin M \end{cases}$ is the null sequence and therefore $\mathcal{I}_c-$convergent to 0. Hence $x = (x_k)$ is ${\mathcal{I}_d}^{\mathcal{I}_c}-$convergent to 0.
But we claim that $x = (x_k)$ is not $\mathcal{I}_c-$convergent to 0. For if $\mathcal{I}_c - \lim x = 0$, then for $\varepsilon = \frac{1}{2}$, the set $\{k \in \mathbb{N} : |x_k - 0| \geq \frac{1}{2}\} =$ set of all prime numbers $\in \mathcal{I}_c$, a contradiction.

From the above example naturally question arises under what condition a sequence $\mathcal{I}^{\mathcal{K}}-$converging to l will also $\mathcal{K}-$convergent to l. The next theorem is regarding such a condition.

Theorem 3.8. *Let $\mathcal{I}$ and $\mathcal{K}$ be two ideals in $\mathbb{N}$ such that $\mathcal{I} \subseteq \mathcal{K}$. Let $x = (x_k)$ be a real sequence such that $\mathcal{I}^{\mathcal{K}} - \lim x = l$. Then $\mathcal{K} - \lim x = l$.*

Proof. Let $\mathcal{I} \subseteq \mathcal{K}$ holds and the sequence $x = (x_k)$ is $\mathcal{I}^{\mathcal{K}}-$convergent to l. So by definition, there exists $M \in \mathcal{F}(\mathcal{I})$ such that the sequence $y = (y_k)$ defined as $y_k = \begin{cases} x_k, & k \in M \\ l, & k \notin M \end{cases}$ is $\mathcal{K}-$convergent to l, which immediately implies

$$\forall \varepsilon > 0, \{k \in M : |x_k - l| \geq \varepsilon\} \in \mathcal{K}. \tag{3.6}$$

Thus $\{k \in \mathbb{N} : |x_k - l| \geq \varepsilon\} \subseteq \{k \in M : |x_k - l| \geq \varepsilon\} \cup (\mathbb{N} \setminus M) \in \mathcal{K}$ by Eq. (3.6) and since as per our assumption $\mathcal{I} \subseteq \mathcal{K}$. Hence $\mathcal{K} - \lim x = l$. □

Theorem 3.9. *If every subsequence of $x = (x_k)$ is $\mathcal{I}^{\mathcal{K}}-$convergent to l, then x is $\mathcal{I}^{\mathcal{K}}-$convergent to l.*

Proof. If possible let us assume the contrary. Then for every $M \in \mathcal{F}(\mathcal{I})$, the sequence $y = (y_k)$ defined as $y_k = \begin{cases} x_k, & k \in M \\ l, & k \notin M \end{cases}$ is not $\mathcal{K}-$convergent to l. Thus for every $M \in \mathcal{F}(\mathcal{I})$ there exists an $\varepsilon_M > 0$ such that

$$A = M \cap \{k \in \mathbb{N} : |x_k - l| \geq \varepsilon_M\} \notin \mathcal{K}.$$

Since $\mathcal{K}$ is admissible, so A is infinite. Let $A = \{a_1 < a_2 < .... < a_k < ...\}$. Construct a subsequence $z = (z_k)$ defined as $z_k = x_{a_k}$ for $k \in \mathbb{N}$. Then $\mathcal{I}^{\mathcal{K}} - \lim z \neq l$, a contradiction to our hypothesis. □

Theorem 3.10. *Let $x = (x_k)$ be a sequence such that $\mathcal{I}^{\mathcal{K}} - \lim x = l$. Then every subsequence of x is $\mathcal{I}^{\mathcal{K}}-$convergent to l if and only if both $\mathcal{I}$ and $\mathcal{K}$ does not contain infinite sets.*

Proof. There are two possible cases.
Case-I: When $\mathcal{K}$ contain an infinite set.
Suppose A be an infinite set and $A \in \mathcal{K}$. Then $\mathbb{N} \setminus A \in \mathcal{F}(\mathcal{K})$ and $\mathbb{N} \setminus A$ is also infinite. Let $\varepsilon > 0$ be arbitrary. Choose $l_1 \in \mathbb{R}$ such that $l_1 \neq l$. Define a sequence $x = (x_k)$ as $x_k = \begin{cases} l_1, & k \in A \\ l, & k \in \mathbb{N} \setminus A \end{cases}$. Then $\{k \in \mathbb{N} : |x_k - l| \geq \varepsilon\} \subseteq A \in \mathcal{K}$ which means that x is $\mathcal{K}-$convergent to l. Therefore by Theorem 3.5, x is $\mathcal{I}^{\mathcal{K}}-$convergent to l. But clearly the subsequence $(x_k)_{k\in A}$ of x is $\mathcal{I}^{\mathcal{K}}-$convergent to l_1 not to l.
Case-II: When $\mathcal{K}$ does not contain an infinite set.
If $\mathcal{K}$ does not contains an infinite set, then $\mathcal{K} = \mathcal{I}_f$ and $\mathcal{I}^{\mathcal{K}}-$convergence concept coincides with $\mathcal{I}^*-$convergence.
Subcase-I: If $\mathcal{I}$ contain an infinite set.
Let B be any infinite set such that $B \in \mathcal{I}$. Then $\mathbb{N} \setminus B \in \mathcal{F}(\mathcal{I})$ and $\mathbb{N} \setminus B$ is also infinite. Define a sequence $x = (x_k)$ as $x_k = \begin{cases} \xi, & k \in B \\ l, & k \in \mathbb{N} \setminus B \end{cases}$. Clearly x is $\mathcal{I}^*-$convergent to l. But clearly the subsequence $(x_k)_{k\in B}$ of x is not $\mathcal{I}^*-$convergent to l.
Subcase-II: If $\mathcal{I}$ does not contain an infinite set.
In this subcase, $\mathcal{I}^{\mathcal{K}}-$convergence coincides with ordinary convergence (as Proposition 3.1 of [1] is true for $\mathcal{I}^{\mathcal{K}}-$convergence) so any subsequence of x is convergent to l. □

Remark 3.11. If a sequence is $\mathcal{I}^{\mathcal{K}}-$convergent then it may not be $\mathcal{I}-$convergent.

Example 3.12. Let us consider the ideal $\mathcal{I}$ which is defined in Example 2.7 and the ideal $\mathcal{I}_c = \{A \subseteq \mathbb{N} : \sum_{a\in A} a^{-1} < \infty\}$. Let $M = \{k \in \mathbb{N} : k = 2^p$ for some non-negative integer $p\}$. Consider the sequence $x = (x_k)$ defined by $x_k = \begin{cases} 1, & k \in M \\ 0, & k \notin M \end{cases}$. Then it is easy to check that x is $\mathcal{I}^{\mathcal{I}_c}-$convergent to 0 but x is not $\mathcal{I}-$convergent to 0.

Theorem 3.13. *Let $\mathcal{I}$ and $\mathcal{K}$ be two ideals in $\mathbb{N}$. Let $x = (x_k)$ be any real sequence. Then $\mathcal{I}^{\mathcal{K}} - \lim x = l$ implies $\mathcal{I} - \lim x = l$ if and only if $\mathcal{K} \subseteq \mathcal{I}$.*

Proof. Let us assume the contrary. Then there exists a set, say $A \in \mathcal{K} \setminus \mathcal{I}$. Let l_1 and l_2 be two real numbers such that $l_1 \neq l_2$. Define a sequence $x = (x_k)$ as $x_k = \begin{cases} l_1, & k \in A \\ l_2, & k \in \mathbb{N} \setminus A \end{cases}$. Let $\varepsilon > 0$ be arbitrary. Clearly

$$\{k \in \mathbb{N} : |x_k - l_2| \geq \varepsilon\} \subseteq A \in \mathcal{K}$$

which means that x is $\mathcal{K}-$convergent to l_2. Therefore by Theorem 3.5, x is $\mathcal{I}^{\mathcal{K}}-$convergent to l_2. By hypothesis x is $\mathcal{I}-$convergent to l_2. Therefore for $\varepsilon = |l_1 - l_2|$, $\{k \in \mathbb{N} : |x_k - l_2| \geq |l_1 - l_2|\} = A \in \mathcal{I}$, a contradiction. Hence we must have $\mathcal{K} \subseteq \mathcal{I}$.

Proof of converse part is straightforward (similar to Theorem 3.1 (ii) of [1]) so omitted. □

Remark 3.14. If a sequence is $\mathcal{I}-$convergent then it may not be $\mathcal{I}^{\mathcal{K}}-$convergent. Consider the ideal $\mathcal{I}$ and the sequence $x = (x_k)$ defined in Example 2.7. Then it is proven in Example 2.1 of [10] that $\mathcal{I}^{\mathcal{I}_f} - \lim x \neq 0$ although $\mathcal{I} - \lim x = 0$.

Theorem 3.15. *Let $\mathcal{I}$ and $\mathcal{K}$ be two ideals in $\mathbb{N}$. Then $\mathcal{I} - \lim x = l$ implies $\mathcal{I}^{\mathcal{K}} - \lim x = l$ if and only if the condition $AP(\mathcal{I}, \mathcal{K})$ holds.*

Proof. The proof is similar to the proof of Theorem 3.4 and 3.5 in [1]. □

Remark 3.16. If a sequence is $\mathcal{K}^{\mathcal{I}}-$convergent then it may not be $\mathcal{I}^{\mathcal{K}}-$convergent. Consider Example 2.7. Clearly, x is $\mathcal{I}_f^{\mathcal{I}}-$convergent to 0 but not $\mathcal{I}^{\mathcal{I}_f}-$convergent to 0.

Theorem 3.17. *Let $\mathcal{I}$ and $\mathcal{K}$ be two ideals in $\mathbb{N}$. Then $\mathcal{K}^{\mathcal{I}} - \lim x = l$ implies $\mathcal{I}^{\mathcal{K}} - \lim x = l$ if $\mathcal{I} \subseteq \mathcal{K}$.*

Proof. Proof is trivial so omitted. □

Theorem 3.18. *Let $\mathcal{I}$, $\mathcal{K}$, $\mathcal{I}_1$, $\mathcal{I}_2$, $\mathcal{K}_1$, $\mathcal{K}_2$ are ideals in $\mathbb{N}$. Then-*
(i) If $\mathcal{I}^{\mathcal{K}_1} - \lim x = l = \mathcal{I}^{\mathcal{K}_2} - \lim x$ then $\mathcal{I}^{\mathcal{K}_1 \vee \mathcal{K}_2} - \lim x = l$.
(ii) If $\mathcal{I}_1^{\mathcal{K}} - \lim x = l = \mathcal{I}_2^{\mathcal{K}} - \lim x$ then $(\mathcal{I}_1 \vee \mathcal{I}_2)^K - \lim x = l$.

Proof. (i) Since $\mathcal{I}^{\mathcal{K}_1} - \lim x = l$ and $\mathcal{I}^{\mathcal{K}_2} - \lim x = l$, so there exists $M, N \in \mathcal{F}(\mathcal{I})$ such that

$$\forall\, \varepsilon, \delta > 0,\ \{k \in M : |x_k - l| \geq \varepsilon\} \in \mathcal{K}_1 \text{ and } \{k \in N : |x_k - l| \geq \delta\} \in \mathcal{K}_2.$$

By the hereditary property of $\mathcal{K}_1$ and $\mathcal{K}_2$, we have

$$\forall\, \varepsilon, \delta > 0,\ \{k \in M \cap N : |x_k - l| \geq \varepsilon\} \in \mathcal{K}_1 \text{ and } \{k \in M \cap N : |x_k - l| \geq \delta\} \in \mathcal{K}_2. \quad (3.7)$$

Let $\eta > 0$ be arbitrary. Then from Eq. (3.7), choosing $\varepsilon = \delta = \eta$ we get

$$\{k \in M \cap N : |x_k - l| \geq \eta\} \in \mathcal{K}_1 \vee \mathcal{K}_2.$$

Now as $M \cap N \in \mathcal{F}(\mathcal{I})$, so the sequence $w = (w_k)$ defined as $w_k = \begin{cases} x_k, & k \in M \cap N \\ l, & k \notin M \cap N \end{cases}$ is $\mathcal{K}_1 \vee \mathcal{K}_2-$convergent to l. Hence $(\mathcal{I}_1 \vee \mathcal{I}_2)^K - \lim x = l$.
(ii) Since $\mathcal{I}_1^{\mathcal{K}} - \lim x = l$ and $\mathcal{I}_2^{\mathcal{K}} - \lim x = l$, so there exists $M \in \mathcal{F}(\mathcal{I}_1)$ and $N \in \mathcal{F}(\mathcal{I}_2)$ such that

$$\forall\, \varepsilon, \delta > 0,\ \{k \in M : |x_k - l| \geq \varepsilon\} \in \mathcal{K} \text{ and } \{k \in N : |x_k - l| \geq \delta\} \in \mathcal{K}.$$

By hereditary property of $\mathcal{K}$, we have

$$\forall\, \varepsilon, \delta > 0,\ \{k \in M \cap N : x_k \geq l + \varepsilon\} \in \mathcal{K} \text{ and } \{k \in M \cap N : x_k \leq l - \delta\} \in \mathcal{K}.$$

Which implies

$$\forall\, \varepsilon, \delta > 0,\ \{k \in M \cap N : x_k \geq l + \varepsilon \text{ or } x_k \leq l - \delta\} \in \mathcal{K}. \quad (3.8)$$

Let $\eta > 0$ be arbitrary. Choosing $\varepsilon = \delta = \eta$ we get from Eq. (3.8),

$$\forall\, \eta > 0,\ \{k \in M \cap N : |x_k - l| \geq \eta\} \in \mathcal{K}.$$

As $M \cap N \in \mathcal{F}(\mathcal{I}_1 \vee \mathcal{I}_2)$, so we can conclude that $(\mathcal{I}_1 \vee \mathcal{I}_2)^K - \lim x = l$. □

Theorem 3.19. *For any sequence $x = (x_k)$ if $\mathcal{I}^{\mathcal{K}} - \lim x = l = \mathcal{K}^{\mathcal{I}} - \lim x$ then- (i) $(\mathcal{I} \vee \mathcal{K})^{\mathcal{I} \vee \mathcal{K}} - \lim x = l$ and (ii) $\mathcal{I} \vee \mathcal{K} - \lim x = l$.*

Proof. (i) From the given conditions, there exist $M \in \mathcal{F}(\mathcal{I})$ and $N \in \mathcal{F}(\mathcal{K})$ such that

$$\forall \varepsilon, \delta > 0, \{k \in M : |x_k - l| \geq \varepsilon\} \in \mathcal{K} \text{ and } \{k \in N : |x_k - l| \geq \delta\} \in \mathcal{I}$$

which implies

$$\forall \varepsilon, \delta > 0, \{k \in M : |x_k - l| \geq \varepsilon\} \cup \{k \in N : |x_k - l| \geq \delta\} \in \mathcal{I} \vee \mathcal{K}.$$

Let $\eta > 0$ be given. Then choosing $\varepsilon = \delta = \eta$ we get

$$\forall \eta > 0, \{k \in M \cap N : |x_k - l| \geq \eta\} \subseteq \{k \in M \cup N : |x_k - l| \geq \eta\} \in \mathcal{I} \vee \mathcal{K}.$$

Now it is a routine work to prove that $(\mathcal{I} \vee \mathcal{K})^{\mathcal{I} \vee \mathcal{K}} - \lim x = l$.
(ii) The proof is parallel to that of Proposition 3.1 in [1]. □

References

[1] A. K. Banerjee and M. Paul, Weak and weak* $\mathcal{I}^{\mathcal{K}}-$convergence in normed spaces, *arXiv.org/abs/1811.06707* (2018).

[2] A. K. Banerjee and M. Paul, Strong-$\mathcal{I}^{\mathcal{K}}-$convergence in probabilistic metric spaces, *arXiv.org/abs/1808.03268v1* (2018).

[3] A. K. Banerjee and M. Paul, A note on $\mathcal{I}^{\mathcal{K}}$ nd $\mathcal{I}^{\mathcal{K}^*}-$convergence in topological spaces, *arXiv.org/abs/1807.11772v1* (2018).

[4] P. Das, S. Sengupta and J. Supina, I^K-convergence of sequence of functions, *Math. Slovaca.* **69(5)**, 1137–1148 (2019).

[5] P. Das, M. Sleziak and V.Toma, I^K-Cauchy functions, *Topology Appl.* **173**, 9–27 (2014).

[6] S. Debnath and D. Rakshit, On $I-$statistical convergence, *Iran. J. Math. Sci. Inform.* **13(2)**, 101–109 (2018).

[7] K. Demirci, $I-$limit superior and limit inferior, *Math. Commun.* **6(2)**, 165–172 (2001).

[8] A. R. Freedman and J. J. Sember, Densities and summability, *Pacific J. Math.* **95(2)**, 293–305 (1981).

[9] J. Gogola, M. Macaj and T. Visnyai, On $I_c^{(q)}-$convergence, *Ann. Math. Inform.* **38**, 27–36 (2011).

[10] P. Kostyrko, M. Macaj, T. Salat and M. Sleziak, I-convergence and extremal I-limit points, *Math. Slovaca.* **55(4)**, 443–464 (2005).

[11] P. Kostyrko, T. Salat and W. Wilczynski, I-convergence, *Real Anal. Exch.* **26(2)**, 669–686 (2000-2001).

[12] M. Macaj and M. Sleziak, I^K-convergence, *Real Anal. Exch.* **36(1)**, 177–194 (2010-2011).

[13] M. Mursaleen, S. Debnath and D. Rakshit, On $I-$statistical limit superior and $I-$statistical limit inferior, *Filomat* **31(7)**, 2103–2108 (2017).

[14] A. Nabiev, S. Pehlivan and M. Gurdal, On $I-$Cauchy sequences, *Taiwanese J. Math.* **11(2)**, 569–576 (2007).

[15] T. Salat, B. C. Tripathy and M. Ziman, On some properties of $I-$convergence, *Tatra Mt. Math. Publ.* **28(2)**, 274–286 (2004).

[16] E. Savas, S. Debnath and D. Rakshit, On $I-$statistically rough convergence, *Publ. Inst. Math.(Beograd) (N.S.)* **105(119)**, 145–150 (2019).

Author information

Shyamal Debnath, Department of Mathematics, Tripura University (A Central University), Suryamaninagar-799022, Agartala, India.
E-mail: `shyamalnitamath@gmail.com`

Chiranjib Choudhury, Department of Mathematics, Tripura University (A Central University), Suryamaninagar-799022, Agartala, India.
E-mail: `chiranjibchoudhury123@gmail.com`

Received: October 5, 2020
Accepted: May 29, 2021