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Selberg Type Inequalities in 2-*-semi inner product space and its applications

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Abstract. In this paper, we prove a type of Selberg type inequality in a $2-^*-$semi inner product $A-$module over a C^*-algebra A.

1 Introduction and Preliminaries

The theory of 2-metric space and linear 2-normed space were first introduced by Gahler in 1963 [13]. Since then, many authors, Freese et al. Gahler, Cho et al., and Gunamwan et al., have developed extensively topogical and geometric structures of 2-inner product spaces, 2-normed spaces, 2-metric spaces, semi-2-normed spaces, semi-2-metric spaces (see[7, 14, 17, 20, 21, 22]).
The finitely generated modules equipped with inner products over a C^*-algebra was first considered by Mallios [19]. Recently, meany researchers have studied geometric properties of Hilbert C^*-modules and 2^*-semi inner product A-module spaces. For example, Dragomir, Khorsavi and Moslehian [5], K. Kubo, F. Kubo and Y. Seo [18] showed several variants of the Selberg inequality and these generalizations in the framework of a Hilbert C^*-modules. B. Mohebbi Najmabadi and T. L. Shateri [21] showed several variants of the Cauchy Schwarz inequality in the framework of a $2-^*-$semi inner product $A-$module over C^*-algebra. We showed in [2, 16] the Selberg inequality and its generalisation in a Hilbert C^*-modules. The aim of the paper is to extend the Selberg inequality from Hilbert spaces and Hilbert C^*-module spaces to 2^*-semi inner product A-module spaces over a C^*-algebra A. Which is a simultaneous extensions of the Cauchy-Schwartz inequality, the Bessel inequality, the Bombieri inequality and the Boas-Bellman inequality in a $2-^*-$ inner product $A-$module over a C^*-algebra A. Moreover we gave a $2-^*-$semi inner product $A-$module over a C^*-algebra version of a refinement of the Selberg inequality.
First we recall some definitions and we review some inequalities.

Definition 1.1. Let A be a C^*-algebra with unit. An element $a \in A$ is positive and we write $a \geq 0$, if $a = a^*$ and $Sp(a) = \{\lambda | a - \lambda I \text{ is not invertible}\} \subseteq \mathbb{R}_+$. The set of all positive elements of A is denoted by A^+. If $a, b \in A$ then $a \leq b$ means that $b - a \in A^+$.
For every $a \in A$, we denoted the absolute value of a by $|a| = (a^*a)^{\frac{1}{2}}$.

Definition 1.2. [19]A complex linear space X is said to be an inner product $A-$module (or pre-Hilbert $A-$module) if X is a right $A-$module together with a C^*-valued map $(x, y) \to \langle x, y\rangle :$ $X \times X \to A$ such that
$(i)\langle x, \alpha y + \beta z\rangle = \alpha\langle x, y\rangle + \beta\langle x, z\rangle$ for all $x, y, z \in X, \alpha, \beta \in \mathbb{C}$
$(ii)\langle x, ya\rangle = \langle x, y\rangle a$ for all $x, y, \in X, a \in A$
$(iii)\langle x, y\rangle = \langle y, x\rangle^*$ for all $x, y, \in X,$
$(iv)\langle x, x\rangle \geq 0$ for all $x \in X$, and $\langle x, x\rangle = 0$ then $x = 0$.
We alwas assume that the linear structures of A and X are compatible. We write $||x|| =$

$||\langle x, x\rangle||^{\frac{1}{2}}$, where the latter norm denotes the C^*-norm of A. If an linear product $A-$module X is complete with respect to its norm, then X is called C^*-module.

Definition 1.3. Let X be a right A-module were A is a C^*-algebra. An A-combination of $x_1, x_2, ...x_n$ in X is written as follows:

$$\sum_1^n x_i a_i = x_1 a_1 + x_2 a_2 + x_n a_n, a_i \in A,$$

and $x_1, x_2, ...x_n$ are called A-independent if the equation $x_1a_1 + x_2a_2 + x_na_n = 0$ has exactly one solution, namely $a_1 = a_2 = ... = a_n = 0$; otherwise, we say $x_1, x_2, ...x_n$ are $A-$ dependent. The maximum number of element in X, that are $A-$independent, is called $A-$rank of X.

Definition 1.4. [21] Let A be a C^*-algebra and X be a linear space by A-rank greater than 1, which is also a right A-module. We define a function $\langle ., .|.\rangle : X \times X \times X \to A$, which satisfies the following properties:
$(T_1)\langle x, x|y\rangle = 0$ if only if $x = ya$ for $a \in A$;
$(T_2)\langle x, x|y\rangle \geq 0$ for all $x, y \in X$;
$(T_3)\langle x, x|y\rangle = \langle y, y|x\rangle$ for all $x, y \in X$:
$(T_4)\langle x, y|z\rangle = \langle y, x|z\rangle^*$ for all $x, y, z \in X$:
$(T_5)\langle xa, yb|z\rangle = a^*\langle x, y|z\rangle b$ for all $x, y, z \in X$ and $a, b \in A$:
$(T_6)\langle \alpha x, y|z\rangle = \overline{\alpha}\langle x, y|z\rangle$ for all $x, y, z \in X$ and $\alpha \in \mathbb{C}$;
$(T_7)\langle x + y, z|w\rangle = \langle x, z|w\rangle + \langle y, z|w\rangle$ for all $x, y, z, w \in X$.
Then the fonction $\langle ., .|.\rangle$ is called $2 -^* -$inner product and $(X, \langle ., .|.\rangle)$ is called $2 -^* -$inner product space. If X satisfies all conditions for a $2 -^* -$inner product except the second part of condition (T_1), then we call X is 2^*-semi inner product space.

Example 1.5. [21] Let A be an unital commutative C^*-algebra and X be a pre-Hilbert $A-$module with inner product $\langle ., .|.\rangle$. Define
$\langle ., .|.\rangle : X \times X \times X \to A$ by $\langle x, y|z\rangle = \langle x, y\rangle\langle z, z\rangle - \langle x, z\rangle\langle z, y\rangle$.
Then $(X, \langle ., .|.\rangle)$ is a 2^*-semi inner product space.

Sine $\langle x, x|z\rangle$ is positive element in A, there is a positive square root of $\langle x, x|z\rangle$ denoted by $|x, z|$ and $||x, z|| = ||\langle x, x|z\rangle||^{\frac{1}{2}}$.

The Selberg type inequality. Let $y_1, ..., y_n$ be nonzero vectors in a Hilbert space X with inner product $\langle, \rangle$. Then, for all $x \in X$,

$$\sum_{j=1}^n \frac{|\langle x, y_j\rangle|^2}{\sum_{k=1}^n |\langle y_j, y_k\rangle|} \leq ||x||^2. \tag{1.1}$$

In [8], the Selberg inequality is refined as follows: if $\langle y, y_j\rangle = 0$ for given $\{y_j\}$, then

$$|\langle x, y\rangle|^2 + \sum_{j=1}^n \frac{|\langle x, y_j\rangle|^2}{\sum_{k=1}^n |\langle y_j, y_k\rangle|}||y||^2 \leq ||x||^2||y||^2, \tag{1.2}$$

holds for all $x \in X$.
It might be useful to observe that, out of (1.1), one may get the following inequality
1. For $n = 1$ and $y = y_1$ the Cauchy-Schwarz inequality

$$\langle x, y\rangle \leq ||x||||y||. \tag{1.3}$$

2. For $y_1, ..., y_n$, be orthogonal sequence of vectors, the Bessel inequality

$$\sum_{j=1}^n |\langle x, y_j\rangle|^2 \leq ||x||^2. \tag{1.4}$$

3 The Bonbieri inequality

$$\sum_{j=1}^n |\langle x, y_j\rangle|^2 \leq ||x||^2 \max_{1\leq j\leq n} \sum_{k=1}^n |\langle y_j, y_k\rangle|. \tag{1.5}$$

4 The Boas-Bellman inequality in [4]

$$\sum_{j=1}^{n} |\langle x, y_j\rangle|^2 \leq ||x||^2 (\max_{1\leq j\leq n} ||y_j||^2 + (n-1)\max_{j\neq k} |\langle y_j, y_k\rangle|). \tag{1.6}$$

The following lemma is useful to prove the Selberg inequality in a $2-^{*}-$semi inner product $A-$module over a C^*-algebra A.

Lemma 1.6. *[18]*
If $a \in A$, then the operator matrix on $A \oplus A$

$$B = \begin{pmatrix} |a^*| & -a \\ -a^* & |a| \end{pmatrix}$$

is positive, and $\begin{pmatrix} \xi \\ \nu \end{pmatrix} \in N(B)$ *if only if $|a^*|\xi = a\nu$ where $N(B)$ is the kernel of B.*

2 MAIN RESULT

Lemma 2.1. *Let be X a $2-^{*}-$semi inner product over a C^*-algebra A. If $x, y_1, ..., y_n, z \in X$ then*

$$\begin{pmatrix} \langle y_1, y_1|z\rangle & \cdots & \langle y_1, y_n|z\rangle \\ & \ddots & \\ \langle y_n, y_1|z\rangle & \cdots & \langle y_n, y_n|z\rangle \end{pmatrix} \leq \begin{pmatrix} \sum_{j=1}^{n} |\langle y_j, y_1|z\rangle| & & 0 \\ & \ddots & \\ 0 & & \sum_{j=1}^{n} |\langle y_j, y_n|z\rangle|. \end{pmatrix} \tag{2.1}$$

Proof. We put $N = \begin{pmatrix} \langle y_1, y_1|z\rangle & \cdots & \langle y_1, y_n|z\rangle \\ & \ddots & \\ \langle y_n, y_1|z\rangle & \cdots & \langle y_n, y_n|z\rangle \end{pmatrix}$ and

$$M = \begin{pmatrix} \sum_{j=1}^{n} |\langle y_j, y_1|z\rangle| & & 0 \\ & \ddots & \\ 0 & & \sum_{j=1}^{n} |\langle y_j, y_n|z\rangle| \end{pmatrix}.$$

We have

$$M - N = \begin{pmatrix} \sum_{j=1}^{n} |\langle y_j, y_1|z\rangle| - \langle y_1, y_1|z\rangle & & -\langle y_1, y_n|z\rangle \\ & \ddots & \\ -\langle y_n, y_1|z\rangle & & \sum_{j=1}^{n} |\langle y_j, y_n|z\rangle| - \langle y_n, y_n|z\rangle \end{pmatrix}$$

then $M - N$ is the following form:

$$\sum_{i,j=1}^{n} \begin{pmatrix} 0 & & & 0 \\ & |\langle y_j, y_i|z\rangle| & -\langle y_i, y_j|z\rangle & \\ & -\langle y_j, y_i|z\rangle & |\langle y_i, y_j|z\rangle| & \\ 0 & & & 0 \end{pmatrix}$$

and for each pair i, j, $M - N$ it positive by lemma (1.6). □

Now, we show the following Selberg type inequality in a $2-^{*}-$semi inner product over a C^*-algebra.

Theorem 2.2. *Let A be a C^*-algebra and X be a $2-^{*}-$semi inner product over the C^*-algebra A. If $x, y_1, ..., y_n, z$ are nonzero vectors in X such that $|y_1, z|, ..., |y_n, z|$ are invertible, then*

$$\sum_{i=1}^{n} \langle x, y_i|z\rangle (\sum_{j=1}^{n} |\langle y_j, y_i|z\rangle|)^{-1} \langle y_i, x|z\rangle \leq |x, z|^2. \tag{2.2}$$

The equality in(2.2) holds if only if $x = \sum_{i=1}^{n} y_i a_i$ for some $a_i \in A$ and $i = 1, ..., n$ such that for arbitrary $i \neq j$, $\langle y_i, y_j|z\rangle = 0$ or $|\langle y_j, y_i|z\rangle| a_i = \langle y_i, y_j|z\rangle a_j$.

Proof. We put $a_i = \sum_{j=1}^n |\langle y_j, y_i|z\rangle|$ for $i = 1, ..., n$. Since $|y_1, z|, ..., |y_n, z|$ are invertible, it follows that a_i is invertible in A. It follows from lemma (2.1) that
$\sum_{1\leq i,j\leq n}\langle x, y_i|z\rangle a_i^{-1}\langle y_i, y_j|z\rangle a_j^{-1}\langle y_j, x|z\rangle$
$= (\langle x, y_1|z\rangle a_1^{-1} ... \langle x, y_n|z\rangle a_n^{-1}) \begin{pmatrix} \langle y_1, y_1|z\rangle & \cdots & \langle y_1, y_n|z\rangle \\ & \ddots & \\ \langle y_n, y_1|z\rangle & \cdots & \langle y_n, y_n|z\rangle \end{pmatrix} \begin{pmatrix} a_1^{-1}\langle y_1, x|z\rangle \\ \vdots \\ a_n^{-1}\langle y_n, x|z\rangle \end{pmatrix}$
$\leq (\langle x, y_1|z\rangle a_1^{-1} ... \langle x, y_n|z\rangle a_n^{-1}) \begin{pmatrix} a_1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & a_n \end{pmatrix} \begin{pmatrix} a_1^{-1}\langle y_1, x|z\rangle \\ \vdots \\ a_n^{-1}\langle y_n, x|z\rangle \end{pmatrix}$
$= \sum_{i=1}^n \langle x, y_i|z\rangle a_i^{-1}\langle y_i, x|z\rangle,$
and this implies
$0 \leq \langle x - \sum_{i=1}^n y_i a_i^{-1}\langle y_i, x|z\rangle, [x - \sum_{i=1}^n y_i a_i^{-1}\langle y_i, x|z\rangle]|z\rangle$
$= \langle x, x|z\rangle - 2\sum_{i=1}^n \langle x, y_i|z\rangle a_i^{-1}\langle y_i, x|z\rangle + \sum_{i,j=1}^n \langle x, y_i|z\rangle a_i^{-1}\langle y_i, y_j|z\rangle a_j^{-1}\langle y_j, x|z\rangle$
$\leq \langle x, x|z\rangle - \sum_{i=1}^n \langle x, y_i|z\rangle a_i^{-1}\langle y_j, x|z\rangle.$
Hence we have the desired inequality (2.2).
The equality in (2.2) holds if only if the following equations are satisfied

$$x = \sum_{i=1}^n y_i a_i^{-1}\langle y_i, x|z\rangle \tag{2.3}$$

and for arbitrary $i \neq j$

$$(\langle x, y_i|z\rangle a_i^{-1}\langle x, y_j|z\rangle a_j^{-1}) \begin{pmatrix} |\langle y_j, y_i|z\rangle| & -\langle y_i, y_j|z\rangle \\ -\langle y_j, y_i|z\rangle & |\langle y_i, y_j|z\rangle| \end{pmatrix} \begin{pmatrix} a_i^{-1}\langle y_i, x|z\rangle \\ a_j^{-1}\langle y_j, x|z\rangle \end{pmatrix} = 0. \tag{2.4}$$

$$\Leftrightarrow \begin{pmatrix} |\langle y_j, y_i|z\rangle| & -\langle y_i, y_j|z\rangle \\ -\langle y_j, y_i|z\rangle & |\langle y_i, y_j|z\rangle| \end{pmatrix}^{\frac{1}{2}} \begin{pmatrix} a_i^{-1}\langle y_i, x|z\rangle \\ a_j^{-1}\langle y_j, x|z\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

$$\Leftrightarrow \begin{pmatrix} |\langle y_j, y_i|z\rangle| & -\langle y_i, y_j|z\rangle \\ -\langle y_j, y_i|z\rangle & |\langle y_i, y_j|z\rangle| \end{pmatrix} \begin{pmatrix} a_i^{-1}\langle y_i, x|z\rangle \\ a_j^{-1}\langle y_j, x|z\rangle \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Hence it follows from lemma (1.6) the condition(2.6) is equivalent to the following (2.5) and (2.6): For arbitrary $i \neq j$

$$\langle y_i, y_j|z\rangle = 0 \tag{2.5}$$

or

$$\langle y_j, y_i|z\rangle a_i^{-1}\langle y_j, x|z\rangle = \langle y_i, y_j|z\rangle a_j^{-1}\langle y_j, x|z\rangle. \tag{2.6}$$

Conversely, suppose that $x = \sum_{i=1}^n y_i b_i$ for some $b_i \in A$ and for $i \neq j, \langle y_i, y_j|z\rangle = 0$ or $|\langle y_j, y_i|z\rangle|b_i = \langle y_i, y_j|z\rangle b_j$. Then
$\sum_{i=1}^n \langle x, y_i|z\rangle(\sum_{j=1}^n |\langle y_j, y_i|z\rangle|)^{-1}\langle y_i, x|z\rangle = \sum_{i=1}^n \langle x, y_i|z\rangle(\sum_{j=1}^n |\langle y_j, y_i|z\rangle|)^{-1}\sum_{j=1}^n \langle y_i, y_j|z\rangle b_j$
$= \sum_{i=1}^n \langle x, y_i|z\rangle(\sum_{j=1}^n |\langle y_j, y_i|z\rangle|)^{-1}\sum_{j=1}^n |\langle y_j, y_i|z\rangle|b_i$
$= \sum_{i=1}^n \langle x, y_i|z\rangle(\sum_{j=1}^n |\langle y_j, y_i|z\rangle|)^{-1}(\sum_{j=1}^n |\langle y_j, y_i|)z\rangle|b_i$
$= \sum_{i=1}^n \langle x, y_i|z\rangle b_i$
$= \langle x, x|z\rangle.$
Whence the proof is complete. □

B. Mohebbi Najmabadi and T.I.Shateri in [21], Theorem (2.1), showed if X is an $2-^*-$ semi inner product over a C^*-algebra , $x, y, z \in X$ and $|x, z| \in Z(A)$, then

$$|\langle x, y|z\rangle|^2 \leq |x, z|^2|y, z|^2. \tag{2.7}$$

By Theorem (2.2), we have the following corollary, which is improvement of (2.2).

Corollary 2.3. *Let X be a $2-^*-$inner product over a C^*-algebra $A, x, y, z \in X$ such that $|y, z|$ is invertible in A then we have the Cauchy Schwarz inequality in $2-^*-$inner product over a C^*-algebra A as follow*

$$\langle x, y|z\rangle(|y, z|^2)^{-1}\langle y, x|z\rangle \leq |x, z|^2. \tag{2.8}$$

Proof. By taking $n = 1$ and $y = y_1$ in (2.2), we obtain the result. □

N.S. Barnett, Y.J. Cho, S.S. Dragomir, S.M. Kang, And S.S. Kimg in [1] showed a version for $2-$inner product space of the Selberg inequality: If X is a 2-inner product space and $x, y_1, ..., y_n, z \in X$ such that $\sum_{i=1}^{n} ||\langle y_i, y_j|z\rangle|| \neq 0$ then

$$\sum_{j=1}^{n} \frac{|\langle x, y_j|z\rangle|^2}{\sum_{k=1}^{n} |\langle y_j, y_k|z\rangle|} \leq \langle x, x|z\rangle. \tag{2.9}$$

By Theorem (2.2), we have the following corollary.

Corollary 2.4. *Let X be a $2-^*$-semi inner product space. If $x, y, y_1...y_n, z \in X$ such that $\sum_{i=1}^{n} |\langle y_i, y_j|z\rangle| \neq 0$, then*

$$\sum_{j=1}^{n} \frac{|\langle x, y_j|z\rangle|^2}{\sum_{k=1}^{n} ||\langle y_j, y_k|z\rangle||} \leq \langle x, x|z\rangle.$$

Proof. By assumption it follows that $\sum_{k=1}^{n} |\langle y_j, y_k|z\rangle|$ is invertible in A and hence

$$(\sum_{k=1}^{n} |\langle y_j, y_k|z\rangle|)^{-1} \geq (\sum_{k=1}^{n} ||\langle y_j, y_k|z\rangle||)^{-1}.$$

Therefore, Theorem (2.2) implies Corollary (2.4).

□

Moreover, in ([18]) Kyoko Kubo, fumio Kubo and Yuki Seo showed a Hilbert C^*-module version of fujii-Nakamoto type (1.2), wich is a refinement of (1.1) in a inner product C^*-module over a unital C^*-algebra: If X a inner product C^*-module over a unital C^*-algebra, $x, y, y_1...y_n$ are nonzero vectors in X such that $y_1...y_n$ are nonsingular, $\langle y, y_i\rangle$=0 for $i = 1, ..., n$ and $\langle x, y\rangle = u|\langle x, y\rangle|$ is a polar decomposition in$A, i, e, u \in A$ is a partial isometry, then

$$|\langle y, x\rangle| \leq u^*\langle y, y\rangle u\sharp(\langle x, x\rangle - \sum_{i=1}^{n}\langle x, y_i\rangle(\sum_{j=1}^{n} |\langle y_j, y_i\rangle|)^{-1}\langle y_i, x\rangle) \tag{2.10}$$

were $\sharp$ is the operator geometric defined by $a\sharp b := a^{\frac{1}{2}}(a^{\frac{-1}{2}} b a^{\frac{-1}{2}})a^{\frac{1}{2}}$ for a invertible.
We show a $2-^*-$semi inner product $A-$module over a C^*-algebra version of a refinement of the Selberg inequality due to fujii and Nakamoto, which is another version of (2.2).

Theorem 2.5. *Let X be a $2-^*-$semi inner product over a C^*-algebra $A, x, y, y_1, ..., y_n, z$ in X such that $|y, z|, |y_1, z|, ..., |y_n, z|$ are invertible such $\langle y, y_i|z\rangle = 0$ for $i = 1, ..., n$ then*

$$\langle x, y|z\rangle(|y, z|^2)^{-1}\langle y, x|z\rangle + \sum_{i=1}^{n}\langle x, y_i|z\rangle(\sum_{j=1}^{n} |\langle y_j, y_i|z\rangle|)^{-1}\langle y_i, x|z\rangle \leq |x, z|^2. \tag{2.11}$$

Proof. We put

$$u = x - \sum_{i=1}^{n} y_i(\sum_{j=1}^{n}\langle y_j, y_i|z\rangle)^{-1}\langle y_i, x|z\rangle.$$

We have from proof of theorem (2.2)

$$|u, z|^2 = |x-\sum_{i=1}^{n} y_i(\sum_{i=1}^{n}\langle y_j, y_i|z\rangle)^{-1}\langle y_i, x|z\rangle|^2 \leq |x, z|^2-\sum_{i=1}^{n}\langle x, y_i|z\rangle(\sum_{j=1}^{n} |\langle y_j, y_i|z\rangle|)^{-1}\langle y_i, x|z\rangle.$$

Since $\langle y, u|z\rangle = \langle y, x|z\rangle$ it follows that
$\langle x, y|z\rangle(|y, z|^2)^{-1}\langle y, x|z\rangle = \langle u, y|z\rangle(|y, z|^2)^{-1}\langle y, u|z\rangle \leq |u, z|^2$ by the Cauchy-Schwarz inequality (2.8), then
$\langle x, y|z\rangle(|y, z|^2)^{-1}\langle y, x|z\rangle + \sum_{i=1}^{n}\langle x, y_j|z\rangle(\sum_{j=1}^{n} |\langle y_j, y_i|z\rangle|)^{-1}\langle y_i, x|z\rangle \leq |x, z|^2$. □

From Theorem (2.2) the following result of Bessel in a $2-^*-$inner product over a C^*-algebra A can be obtained.

Corollary 2.6. *Let X be a $2-^*-$inner product over a C^*-algebra. If $y_1...y_n$ be a sequence of unit vectors in X such that $\langle y_j, y_i|z\rangle = 0$ for $1 \leq j \neq i \leq n$ then*

$$\sum_{j=1}^{n} |\langle y_j, x|z\rangle|)^2 \leq |x,z|^2. \tag{2.12}$$

Proof. We have $(\sum_{j=1}^{n} |\langle y_j, y_i|z\rangle|)^{-1} = 1_A$; Thus the result follows immediately from inequality (2.2). □

In [1] Theorem 7 N.S. Barnett, Y.J. Chof, S.S. Dragomir, S.M. Kang, And S.S. Kimg showed a $2-$inner product space version of Bombieri type (1.5):If $x, y_1...y_n, z$ are vectors in a 2-inner product space X such that $||y_1, z||, ..., ||y_n, z||$ are nonzero then

$$\sum_{i=1}^{n} |\langle x, y_i|z\rangle|^2 \leq |x,z|^2 \max_{1\leq j\leq n} \sum_{k=1}^{n} ||\langle y_j, y_k|z\rangle||. \tag{2.13}$$

We show a 2^*--semi inner product version of Bombieri type inequality.

Corollary 2.7. *Let X be a $2-^*-$inner product over a C^*-algebra . If $x, y_1...y_n, z$ are nonzero vectors in X such that $|y_1, z|, ..., |y_n, z|$ are invertible then*

$$\sum_{i=1}^{n} \langle x, y_i|z\rangle\langle y_i, x|z\rangle \leq |x,z|^2 \max_{1\leq j\leq n} \sum_{k=1}^{n} ||\langle y_j, y_k|z\rangle||. \tag{2.14}$$

Proof. Since for $j = 1, ..., n$, we observe that

$$\sum_{k=1}^{n} |\langle y_j, y_k|z\rangle| \leq \sum_{k=1}^{n} ||\langle y_j, y_k|z\rangle|| \leq \max_{1\leq j\leq n} \sum_{k=1}^{n} ||\langle y_j, y_k|z\rangle||$$

then

$$\frac{1}{\max_{1\leq j\leq n} \sum_{k=1}^{n} ||\langle y_j, y_k|z\rangle||} \leq (\sum_{k=1}^{n} |\langle y_j, y_k|z\rangle|)^{-1}.$$

We also have

$$\frac{\sum_{i=1}^{n} \langle x, y_i|z\rangle\langle y_i, x|z\rangle}{\max_{1\leq j\leq n} \sum_{k=1}^{n} ||\langle y_j, y_k|z\rangle||} \leq \sum_{i=1}^{n} \langle x, y_i|z\rangle (\sum_{k=1}^{n} |\langle y_j, y_k|z\rangle|)^{-1} \langle y_i, x|z\rangle.$$

Then by using theorem (2.2) we get

$$\sum_{i=1}^{n} \langle x, y_i|z\rangle\langle y_i, x|z\rangle \leq |x,z|^2 \max_{1\leq j\leq n} \sum_{k=1}^{n} ||\langle y_j, y_k|z\rangle||.$$

Wich complete the proof of corollary

□

In a similar way we show a $2-^*-$semi inner product version of Boas-Bellmann type inequality.

Corollary 2.8. *Let X be a $2-^*-$inner product over a C^*-algebra . If $x, y_1, ..., y_n, z$ are nonzero vectors in X such that $|y_1, z|, ..., |y_n, z|$ are invertible, then*

$$\sum_{i=1}^{n} \langle x, y_i|z\rangle\langle y_i, x|z\rangle \leq |x,z|^2 (\max_{1\leq j\leq n} |y_j|^2 + (n-1) \max_{k\neq j} ||\langle y_j, y_k|z\rangle||) \tag{2.15}$$

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