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ON CONFORMALLY FLAT GENERALIZED WEAKLY SYMMETRIC MANIFOLDS

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Abstract The present paper deals with a study of conformally flat generalized weakly symmetric manifolds and deduced some geometric properties along with the existence of such notion by a proper example.

1 Introduction

The shape of a space is determined by its curvature and locally symmetric space, introduced by Cartan [7], is such that its curvature is covariantly constant. Later many authors have weakened this notion in different ways such as recurrent manifold by Walker [37], generalized recurrent by Dubey [11], quasi generalized recurrent by Shaikh and Roy [32], weakly generalized recurrent by Shaikh and Roy [33], hyper-generalized recurrent by Shaikh and Patra [31], semisymmetric manifold by Cartan [7], pseudosymmetric manifolds by Chaki [8], pseudosymmetric manifolds by Deszcz [10], [18], weakly symmetric manifold by Tamássy and Binh [35]. Then within very short period of span weak symmetry became an interesting theme of research in the literature of differential geometry and investiged such notion by several authors throughout the globe. For details we refer the reader to see [13], [14], [16], [17], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [30], [34], and also the reference therein. Generalizing the notion of weak symmetry by Tamássy and Binh [35], recently, Baishya [1] introduced a space called generalized weakly symmetric manifold, denoted by $(GWS)_n$, and showed its proper existence. The object of the paper is to study conformally flat $(GWS)_n$. We organized the paper as follows: Section 2 is concerned with preliminaries where the defining conditions of weakly symmetry, genralized weakly symmetry and generalized weakly Ricci-symmetry are given. Section 3 deals with some geometric properties of conformally flat $(GWS)_n$. Finally in the last section the existence of conformally flat $(GWS)_n$ which is not a weakly concircularly symmetric manifold is ensured by a proper example.

2 Preliminaries

Let (M^n, g), $n > 2$, be a semi-Riemannian manifold, i.e. a connected smooth manifold equipped with a semi-Riemannian metric g. We denote by ∇, R, S and r the Levi-Civita connection, the Riemann-Christoffel curvature tensor, Ricci tensor and scalar curvature of (M^n, g) respectively. The Kulkarni-Nomizu product (see, e.g., [12], [29]) $E \wedge F$ of two $(0, 2)$ tensors E and F is defined by

$$\begin{aligned}(E \wedge F)(X_1, X_2, X_3, X_4) &= E(X_1, X_4)F(X_2, X_3) + E(X_2, X_3)F(X_1, X_4)\\ &- E(X_1, X_3)F(X_2, X_4) - E(X_2, X_4)F(X_1, X_3),\end{aligned}$$

$X_i \in \chi(M)$, $i = 1, 2, 3, 4$, where $\chi(M)$ being the Lie algebra of all smooth vector fields on M.

Definition 2.1. A semi-Riemannian manifold $(M, g), n \geq 3$, admitting a tensor field T of type (0,4) is said to be weakly T-symmetric by Tamássy and Binh [35] if

$$\begin{aligned}(\nabla_X T)(Y, U, V, W) = A(X)T(Y, U, V, W) + B(Y)T(X, U, V, W) \\ + \quad C(U)T(Y, X, V, W) + D(V)T(Y, U, X, W) + E(W)T(Y, U, V, X)\end{aligned} \tag{2.1}$$

holds on the set $U = \{x \in M : (\nabla T - \xi \otimes T)_x \neq 0$ for any 1-form $\xi\}$, where A,B,C,D,E, are associated 1-forms and we say that (A, B, C, D, E) is a solution of this weakly T-symmetric manifold.

In particular if $T = R$ the manifold satisfying above condition ia called weakly symmetric manifold according to Tamássy and Binh and is denoted by $(WS)_n$. We note that Prvanović [15] showed that in a $(WS)_n$ $C = B$, $E = D$.

Definition 2.2. A semi-Riemannian manifold $(M, g), n \geq 3$, admitting a tensor field T of type (0,4) is said to be generalized weakly T-symmetric by Baishya [1] if

$$\begin{aligned} &(\nabla_X T)(Y,U,V,W) = A(X)T(Y,U,V,W) + B(Y)T(X,U,V,W) \\ + \;& B(U)T(Y,X,V,W) + D(V)T(Y,U,X,W) + D(W)T(Y,U,V,X) \\ + \;& \alpha(X)G(Y,U,V,W) + \beta(Y)G(X,U,V,W) + \beta(U)G(Y,X,V,W) \\ + \;& \gamma(V)G(Y,U,X,W) + \gamma(W)G(Y,U,V,X) \end{aligned} \tag{2.2}$$

holds on the set $U_J = \{x \in M : (\nabla T - \xi_1 \otimes T - \xi_2 \otimes G)_x \neq 0$ for any 1-forms $\xi_1$and $\xi_2\}$, where $G = \frac{1}{2} g \wedge g$ and A, B, D, α, β and γ are non-zero 1-forms which are defined as $A(X) = g(X, \theta_1)$, $B(X) = g(X, \phi_1)$ and $D(X) = g(X, \pi_1)$, $\alpha(X) = g(X, \theta_2)$, $\beta(X) = g(X, \phi_2)$ and $\gamma(X) = g(X, \pi_2)$. We say that $(A, B, D, \alpha, \beta, \gamma)$ is a solution of this generalized weakly T-symmetric manifold .

In particular if $T = R$ the the manifold satisfying (2.2) ia called a generalized weakly symmetric manifold according to Baishya [1] and is denoted by $(GWS)_n$. Thus defining condition of $(GWS)_n$ is given by

$$\begin{aligned} &(\nabla_X R)(Y,U,V,W) = A(X)R(Y,U,V,W) + B(Y)R(X,U,V,W) \\ + \;& B(U)R(Y,X,V,W) + D(V)R(Y,U,X,W) + D(W)R(Y,U,V,X) \\ + \;& \alpha(X)G(Y,U,V,W) + \beta(Y)G(X,U,V,W) + \beta(U)G(Y,X,V,W) \\ + \;& \gamma(V)G(Y,U,X,W) + \gamma(W)G(Y,U,V,X) \end{aligned} \tag{2.3}$$

on the set $U_{J_1} = \{x \in M : (\nabla R - \xi_1 \otimes R - \xi_2 \otimes G)_x \neq 0$ for any 1-forms $\xi_1 and\ \xi_2\}$, and where $G = \frac{1}{2} g \wedge g$ and A, B, D, α, β and γ are non-zero 1-forms which are defined as $A(X) = g(X, \theta_1)$, $B(X) = g(X, \phi_1)$ and $D(X) = g(X, \pi_1)$, $\alpha(X) = g(X, \theta_2)$, $\beta(X) = g(X, \phi_2)$ and $\gamma(X) = g(X, \pi_2)$.

Definition 2.3. A semi-Riemannian manifold $(M, g), n \geq 3$, admitting a tensor field Z of type (0,2) is said to be generalized weakly Z-symmetric by Baishya [2] if

$$\begin{aligned} &(\nabla_X Z)(X_1, X_2) = \overline{A}(X)Z(X_1, X_2) + \overline{B}(X_1)Z(X, X_2) + \overline{D}(X_2)Z(X_1, X) \\ + \;& \overline{\alpha}(X)g(X_1, X_2) + \overline{\beta}(X_1)g(X, X_2) + \overline{\gamma}(X_2)g(X_1, X) \end{aligned} \tag{2.4}$$

holds on the set $U_Q = \{x \in M : (\nabla Z - \xi_1 \otimes Z - \xi_2 \otimes g)_x \neq 0$ for any 1-forms $\xi_1 and\ \xi_2\}$, where $\overline{A}$, $\overline{B}$, $\overline{D}$, $\overline{\alpha}$, $\overline{\beta}$ and $\overline{\gamma}$ are non-zero 1-forms which are defined as $\overline{A}(X) = g(X, \overline{\theta_1})$, $\overline{B}(X) = g(X, \overline{\phi_1})$, $\overline{D}(X) = g(X, \overline{\pi_1})$, $\overline{\alpha}(X) = g(X, \overline{\theta_2})$, $\overline{\beta}(X) = g(X, \overline{\phi_2})$ and $\overline{\gamma}(X) = g(X, \overline{\pi_2})$.

In particular if $Z = S$ the the manifold satisfying (2.4) ia called generalized weakly Ricci-symmetric manifold according to Baishya [2] and is denoted by $(GWRS)_n$. Thus defining condition of $(GWRS)_n$ becomes

Definition 2.4. A non-flat n-dimensional Riemannian manifold (M^n, g), $(n > 2)$, is said to be a generalized weakly Ricci-symmetric manifold which is the denoted by $(GWRS)_n$, [2] if its Ricci tensor S of type (0,2) is not identically zero and satisfies

$$\begin{aligned} &(\nabla_X S)(Y, W) = \overline{A}(X)S(Y, W) + \overline{B}(Y)S(X, W) + \overline{D}(W)S(Y, X) \\ + \;& \overline{\alpha}(X)g(Y, W) + \overline{\beta}(Y)g(X, W) + \overline{\gamma}(W)g(Y, X) \end{aligned} \tag{2.5}$$

on the set $U_{Q_1} = \{x \in M : (\nabla S - \xi_1 \otimes S - \xi_2 \otimes g)_x \neq 0$ for any 1-forms $\xi_1 and\ \xi_2\}$, where $\overline{A}$,$\overline{B}$, $\overline{D}$, $\overline{\alpha}$, $\overline{\beta}$ and $\overline{\gamma}$ are non-zero 1-forms which are defined as $\overline{A}(X) = g(X, \overline{\theta_1})$, $\overline{B}(X) = g(X, \overline{\phi_1})$, $\overline{D}(X) = g(X, \overline{\pi_1})$, $\overline{\alpha}(X) = g(X, \overline{\theta_2})$, $\overline{\beta}(X) = g(X, \overline{\phi_2})$ and $\overline{\gamma}(X) = g(X, \overline{\pi_2})$.

The local expression of (2.3) is

$$\begin{aligned} R_{mnpq,k} &= A_k R_{mnpq} + B_m R_{knpq} + B_n R_{mkpq} + D_p R_{mnkq} + D_q R_{mnpk} \\ &+ \alpha_k G_{mnpq} + \beta_m G_{knpq} + \beta_n G_{mkpq} + \gamma_p G_{mnkq} + \gamma_q G_{mnpk}, \end{aligned} \tag{2.6}$$

where A_i, B_i, D_i, α_i, β_i and γ_i are non-zero co-vectors. The beauty of such $(GWS)_n$-space is that for suitable choice of 1-forms we get different space such as
(i) symmetric space [7] (for $A = B = D = \alpha = \beta = \gamma = 0$),
(ii) recurrent space [37] (for $A \neq 0$ and $B = D = \alpha = \beta = \gamma = 0$),
(iii) generalized recurrent space [11] (for $A \neq 0, \alpha \neq 0, B = D = \beta = \gamma = 0$),
(iv) pseudo symmetric space [8] (for $\frac{A}{2} = B = D = \delta \neq 0, \alpha = \beta = \gamma = 0$),
(v) generalized pseudo symmetric space [2] (for $\frac{A}{2} = B = D = \delta \neq 0,$ and $\frac{\alpha}{2} = \beta = \gamma = \mu \neq 0$),
(vi) semi-pseudo symmetric space [36] (for $B = D = \delta \neq 0, A = \alpha = \beta = \gamma = 0$),
(vii) generalized semi-pseudo symmetric space [3] (for $B = D = \delta \neq 0, \beta = \gamma = \mu \neq 0$ and $A = \alpha = 0$),
(viii) almost pseudo symmetric space [9] (for $A = E + H, B = D = H$ and $\alpha = \beta = \gamma = 0$),
(ix) almost generalized pseudo symmetric space ([4], [5], [6])(for $A = E + H, B = D = H$ and $\alpha = \lambda + \psi, \beta = \gamma = \lambda$) and
(x) weakly symmetric space [35] (for $A, B, D \neq 0$ and $\alpha = \beta = \gamma = 0$).

3 Some geometric properties of conformally flat $(GWS)_n$

The Weyl Conformal curvature tensor C of type $(0, 4)$ is given by

$$C = R - \frac{1}{n-2} g \wedge S + \frac{r}{(n-1)(n-2)} G.$$

A Riemannian manifold (M^n, g), $n > 3$, is said to be conformally flat if its conformal curvature tensor $C(X, Y, Z, W) = 0$ where $X, Y, Z, W \in \chi(M)$. Then we have

$$R = \frac{1}{n-2} g \wedge S - \frac{r}{(n-1)(n-2)} G. \tag{3.1}$$

Using (3.1) in (2.3) we have

$$\begin{aligned} & (\nabla_X R)(Y, U, V, W) \\ = \ & A(X)[\frac{1}{n-2}(g \wedge S)(Y, U, V, W) - \frac{r}{(n-1)(n-2)} G(Y, U, V, W)] \\ + \ & B(Y)[\frac{1}{n-2}(g \wedge S)(X, U, V, W) - \frac{r}{(n-1)(n-2)} G(X, U, V, W)] \\ + \ & B(U)[\frac{1}{n-2}(g \wedge S)(Y, X, V, W) - \frac{r}{(n-1)(n-2)} G(Y, X, V, W)] \\ + \ & D(V)[\frac{1}{n-2}(g \wedge S)(Y, U, X, W) - \frac{r}{(n-1)(n-2)} G(Y, U, X, W)] \\ + \ & D(W)[\frac{1}{n-2}(g \wedge S)(Y, U, V, X) - \frac{r}{(n-1)(n-2)} G(Y, U, V, X)] \\ + \ & \alpha(X) G(Y, U, V, W) + \beta(Y) G(X, U, V, W) + \beta(U) G(Y, X, V, W) \\ + \ & \gamma(V) G(Y, U, X, W) + \gamma(W) G(Y, U, V, X). \end{aligned} \tag{3.2}$$

Contracting (3.2) we have

$$\begin{aligned}(\nabla_X S)(Y,W) &= S(Y,W)[A(X)+\frac{B(X)}{n-2}+\frac{D(X)}{n-2}] \\ &+ \frac{n-3}{n-2}[S(X,W)B(Y)+S(Y,X)D(W)] \\ &+ \frac{1}{n-2}\{B(QX)g(Y,W)-B(QY)g(X,W)\} \\ &+ \frac{1}{(n-2)}\{D(QX)g(Y,W)-g(Y,X)D(QW)\} \\ &- \frac{r}{(n-1)(n-2)}\{B(X)g(Y,W)-B(Y)g(X,W)\} \\ &- \frac{r}{(n-1)(n-2)}\{D(X)g(Y,W)-g(Y,X)D(W)\} \\ &+ (n-1)[\alpha(X)g(Y,W)+\beta(Y)g(X,W)+\gamma(W)g(Y,X)] \\ &- \beta(Y)g(X,W)+g(Y,W)[\beta(X)+\gamma(X)]-\gamma(W)g(Y,X),\end{aligned} \tag{3.3}$$

where $S(X,\theta_1)=g(QX,\theta_1)=A(QX)$ where Q be the symmetric endomorphism of the tangent bundle corresponding to the Ricci tensor S.
Again contracting (3.3) we have

$$dr(X)=rA(X)+2B(QX)+2D(QX)+(n-1)[n\alpha(X)+2\beta(X)+2\gamma(X)]. \tag{3.4}$$

If we take the scalar curvature of a conformally flat $(GWS)_n$-space is non-zero constant, then we have $dr(X)=0$ and from (3.4) we have

$$rA(X)+2B(QX)+2D(QX)+(n-1)[n\alpha(X)+2\beta(X)+2\gamma(X)]=0. \tag{3.5}$$

This gives the following:

Theorem 3.1. *If the scalar curvature of a conformally flat $(GWS)_n$ is constant, then the 1-forms are related by the expression (3.5).*

Again contraction of (3.3) over X and Y yields

$$\begin{aligned}dr(X) &= 2A(QX)+2B(QX)-2D(QX)+2rD(X) \\ &+ 2(n-1)[\alpha(X)+\beta(X)+(n-1)\gamma(X)].\end{aligned} \tag{3.6}$$

From (3.4) and (3.6) we have

$$r[A(X)-2D(X)]=2A(QX)-4D(QX)+(n-1)(n-2)[2\gamma(X)-\alpha(X)]. \tag{3.7}$$

On simplification we get

$$r=\frac{2A(QX)-4D(QX)+(n-1)(n-2)[2\gamma(X)-\alpha(X)]}{A(X)-2D(X)}. \tag{3.8}$$

Hence we can state the following:

Theorem 3.2. *In a conformally flat $(GWS)_n$ the scalar curvature is given by (3.8) provided $A(X)-2D(X)\neq 0$.*

Theorem 3.3. *In a conformally flat $(GWS)_n$ if $\alpha=2\gamma$ then $\frac{r}{2}$ is the eigenvalue of the Ricci tensor S corresponding to the eigenvector μ defined by $g(X,\mu)=H(X)$ where $H(X)=A(X)-2D(X)$.*

Theorem 3.4. *In a conformally flat $(GWS)_n$ the scalar curvature vanishes if $\alpha=2\gamma$ and $A(QX)=2D(QX)$ provided $A(X)-2D(X)\neq 0$.*

Contracting (3.3) over X and W gives

$$\begin{aligned} dr(X) &= 2A(QX) - 2B(QX) + 2D(QX) + 2rB(X) \\ &+ 2(n-1)[\alpha(X) + (n-1)\beta(X) + \gamma(X)]. \end{aligned} \tag{3.9}$$

From (3.4) and (3.9) we have

$$r = \frac{2A(QX) - 4B(QX) + (n-1)(n-2)[2\beta(X) - \alpha(X)]}{A(X) - 2B(X)}. \tag{3.10}$$

This leads to the following:

Theorem 3.5. *In a conformally flat $(GWS)_n$ the scalar curvature is given by (3.10) provided $A(X) - 2B(X) \neq 0$.*

Theorem 3.6. *In a conformally flat $(GWS)_n$ if $\alpha = 2\beta$ then $\frac{r}{2}$ is the eigenvalue of the Ricci tensor S corresponding to the eigenvector μ_1 defined by $g(X, \mu_1) = H_1(X)$ where $H_1(X) = A(X) - 2B(X)$.*

Theorem 3.7. *In a conformally flat $(GWS)_n$ the scalar curvature vanishes if $\alpha = 2\beta$ and $A(QX) = 2B(QX)$ provided $A(X) - 2B(X) \neq 0$.*

Again from (3.6) and (3.9) we have

$$r = \frac{2[D(QX) - B(QX)] + (n-1)(n-2)[\beta(X) - \gamma(X)]}{D(X) - B(X)}. \tag{3.11}$$

Hence we can state the following:

Theorem 3.8. *In a conformally flat $(GWS)_n$ the scalar curvature is given by (3.11) provided $D(X) - B(X) \neq 0$.*

Theorem 3.9. *In a conformally flat $(GWS)_n$ if $\gamma = \beta$ then $\frac{r}{2}$ is the eigenvalue of the Ricci tensor S corresponding to the eigenvector μ_2 defined by $g(X, \mu_2) = H_2(X)$ where $H_2(X) = D(X) - B(X)$.*

Theorem 3.10. *In a conformally flat $(GWS)_n$ the scalar curvature vanishes if $\gamma = \beta$ and $D(QX) = B(QX)$ provided $D(X) - B(X) \neq 0$.*

From (3.3), it follows that a $(GWS)_n$ is $(GWRS)_n$ if

$$B(QX)g(Y,W) - B(QY)g(X,W) + D(QX)g(Y,W) - D(QW)g(Y,X) = 0. \tag{3.12}$$

Hence we can state the following:

Theorem 3.11. *A conformally flat $(GWS)_n$ satisfying the condition (3.12) is a $(GWRS)_n$.*

4 Existence of conformally flat generalized weakly symmetric space

Let $(\mathbb{R}^4, g)$ be a 4-dimensional Riemannian space endowed with the Riemannian metric g given by

$$\begin{aligned} ds^2 = g_i j dx^i dx^j &= (x^2)^2(e^{x^2})^2(dx^1)^2 + x^2(dx^2)^2 \\ &+ (x^2)^2(e^{x^2})^2(dx^3)^2 + (x^2)^2(e^{x^2})^2(dx^4)^2, \\ &\quad \text{with} \quad x^2 > 0, \quad (i,j = 1,2,3,4). \end{aligned} \tag{4.1}$$

$$\begin{cases} R_{1313} = R_{1414} = R_{3434} = -e^{4x^2}x^2(1+x^2)^2, \\ R_{1212} = R_{2323} = R_{2424} = -\frac{1}{2}e^{2x^2}(-1 + x^2(3 + 2x^2)), \\ r = \frac{3(1+7x^2+4(x^2)^2)}{(x^2)^3}. \end{cases} \tag{4.2}$$

With the help of (4.1), we can find out

$$\begin{cases} G_{1212} = G_{2323} = G_{2424} = -e^{2x^2}(x^2)^3, \\ G_{1313} = G_{1414} = G_{3434} = -e^{4x^2}(x^2)^4. \end{cases} \tag{4.3}$$

The non-vanishing component of covariant derivatives of Riemannian curvature tensors are

$$\begin{cases} R_{2323,2} = R_{1212,2} = R_{2424,2} = \frac{e^{2x^2}(-3+2x^2(3+x^2))}{2x^2}, \\ R_{1213,3} = \frac{1}{2}e^{4x^2}(1+x^2)(3+x^2) = \\ -R_{2334,4} = R_{2434,3} = R_{1214,4} = R_{1323,1} = R_{1424,1}, \\ R_{1414,2} = e^{4x^2}(1+x^2)(3+x^2) = R_{3434,2} = R_{1313,2}. \end{cases} \tag{4.4}$$

Also the metric is conformally flat. We consider the 1-forms as follows:

$$\begin{cases} A(\partial_i) = A_i = \begin{cases} -\frac{15+10x^2+2(x^2)^2}{x^2(3+x^2)} & \text{for } i=2, \\ 0 & \text{otherwise,} \end{cases} \\ \\ B(\partial_i) = B_i = \begin{cases} 1 & \text{for } i=2, \\ 0 & \text{otherwise,} \end{cases} \\ \\ D(\partial_i) = D_i = \begin{cases} -1 & \text{for } i=2, \\ 0 & \text{otherwise,} \end{cases} \end{cases} \tag{4.5}$$

$$\begin{cases} \alpha(\partial_i) = \alpha_i = \begin{cases} \frac{6+25x^2+30(x^2)^2+13(x^2)^3+2(x^2)^4}{(x^2)^4(3+x^2)} & \text{for } i=2, \\ 0 & \text{otherwise,} \end{cases} \\ \\ \beta(\partial_i) = \beta_i = \begin{cases} -\frac{3+6x^2+5(x^2)^2+2(x^2)^3}{2(x^2)^4} & \text{for } i=2, \\ 0 & \text{otherwise,} \end{cases} \\ \gamma(\partial_i) = \gamma_i = \begin{cases} \frac{-3-2x^2+3(x^2)^2+2(x^2)^3}{2(x^2)^4} & \text{for } i=2, \\ 0 & \text{otherwise,} \end{cases} \end{cases} \tag{4.6}$$

where $\partial_i = \frac{\partial}{\partial u^i}$, u^i being the local coordinates of $\mathbb{R}^4$.
In our $\mathbb{R}^4$, (2.6) reduces with these 1-forms to the following equations:

$$\begin{aligned} R_{2323,k} &= A_kR_{2323} + B_1R_{k323} + B_2R_{2k23} + D_1R_{23k3} + D_2R_{232k} \\ &+ \alpha_kG_{2323} + \beta_1G_{k323} + \beta_2G_{2k23} + \gamma_1G_{23k3} + \gamma_2G_{232k} \end{aligned}$$

$$\begin{aligned} R_{2424,k} &= A_kR_{2424} + B_1R_{k424} + B_2R_{2k24} + D_1R_{24k4} + D_2R_{242k} \\ &+ \alpha_kG_{2424} + \beta_1G_{k424} + \beta_2G_{2k24} + \gamma_1G_{24k4} + \gamma_2G_{242k} \end{aligned}$$

$$\begin{aligned} R_{1414,k} &= A_kR_{1414} + B_1R_{k414} + B_2R_{1k14} + D_1R_{14k4} + D_2R_{141k} \\ &+ \alpha_kG_{1414} + \beta_1G_{k414} + \beta_2G_{1k14} + \gamma_1G_{14k4} + \gamma_2G_{141k} \end{aligned}$$

$$\begin{aligned} R_{1212,k} &= A_kR_{1212} + B_1R_{k212} + B_2R_{1k12} + D_1R_{12k2} + D_2R_{121k} \\ &+ \alpha_kG_{1212} + \beta_1G_{k212} + \beta_2G_{1k12} + \gamma_1G_{12k2} + \gamma_2G_{121k} \end{aligned}$$

$$\begin{aligned} R_{1313,k} &= A_kR_{1313} + B_1R_{k313} + B_2R_{1k13} + D_1R_{13k3} + D_2R_{131k} \\ &+ \alpha_kG_{1313} + \beta_1G_{k313} + \beta_2G_{1k13} + \gamma_1G_{13k3} + \gamma_2G_{131k} \end{aligned}$$

$$\begin{aligned} R_{1323,k} &= A_kR_{1323} + B_1R_{k323} + B_2R_{1k23} + D_1R_{13k3} + D_2R_{132k} \\ &+ \alpha_kG_{1323} + \beta_1G_{k323} + \beta_2G_{1k23} + \gamma_1G_{13k3} + \gamma_2G_{132k} \end{aligned}$$

$$\begin{aligned} R_{1424,k} &= A_k R_{1424} + B_1 R_{k424} + B_2 R_{1k24} + D_1 R_{14k4} + D_2 R_{142k} \\ &+ \alpha_k G_{1424} + \beta_1 G_{k424} + \beta_2 G_{1k24} + \gamma_1 G_{14k4} + \gamma_2 G_{142k} \end{aligned}$$

$$\begin{aligned} R_{2434,k} &= A_k R_{2434} + B_1 R_{k434} + B_2 R_{2k34} + D_1 R_{24k4} + D_2 R_{243k} \\ &+ \alpha_k G_{2434} + \beta_1 G_{k434} + \beta_2 G_{2k34} + \gamma_1 G_{24k4} + \gamma_2 G_{243k} \end{aligned}$$

$$\begin{aligned} R_{2334,k} &= A_k R_{2334} + B_1 R_{k334} + B_2 R_{2k34} + D_1 R_{23k4} + D_2 R_{233k} \\ &+ \alpha_k G_{2334} + \beta_1 G_{k334} + \beta_2 G_{2k34} + \gamma_1 G_{23k4} + \gamma_2 G_{233k} \end{aligned}$$

$$\begin{aligned} R_{3434,k} &= A_k R_{3434} + B_1 R_{k434} + B_2 R_{3k34} + D_1 R_{34k4} + D_2 R_{343k} \\ &+ \alpha_k G_{3434} + \beta_1 G_{k434} + \beta_2 G_{3k34} + \gamma_1 G_{34k4} + \gamma_2 G_{343k} \end{aligned}$$

$$\begin{aligned} R_{1213,k} &= A_k R_{1213} + B_1 R_{k213} + B_2 R_{1k13} + D_1 R_{12k3} + D_2 R_{121k} \\ &+ \alpha_k G_{1213} + \beta_1 G_{k213} + \beta_2 G_{1k13} + \gamma_1 G_{12k3} + \gamma_2 G_{121k} \end{aligned}$$

$$\begin{aligned} R_{1214,k} &= A_k R_{1214} + B_1 R_{k214} + B_2 R_{1k14} + D_1 R_{12k4} + D_2 R_{121k} \\ &+ \alpha_k G_{1214} + \beta_1 G_{k214} + \beta_2 G_{1k14} + \gamma_1 G_{12k4} + \gamma_2 G_{121k} \end{aligned}$$

where, $k = 1, 2, 3, 4$. By virtue of (4.2)–(4.6) it can be easily check that the above relations hold and hence we can state the following:

Theorem 4.1. *Let $M = (\mathbb{R}^4, g)$ be a Riemannian manifold equipped with the metric given by (4.1). Then M is a $(GWS)_4$ with non-vanishing and non-constant scalar curvature which is conformally flat.*

It can be easily shown that the manifold under considered metric can not be a weakly concircularly symmetric space. So this exmple support the fact that the converse of the **Theorem 3** in ([1]) is not true.

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