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DIVISIBILITY AND FILTERS IN DISTRIBUTIVE LATTICES

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Abstract: The notion of divisibility is introduced in a distributive lattice with respect to a filter and it is proved that the set of all multipliers of an element is a filter. A congruence is defined on a distributive lattice with respect to these multiplier filters and established a set of equivalent conditions for the corresponding quotient lattice to become a Boolean algebra. The concepts of D-prime elements and D-irreducible elements are introduced and characterized in terms of corresponding multiplier filters.

1 Introduction

G. Birkhoff [1], George Grätzer, G. Szász and many authors have studied about various types of ideals and congruences all intimated to some extent the behavior of ideals in a distributive lattice. Mandelker [4] introduced the notion of annihilators and characterized distributive lattices with the help of these annihilators. Later many authors like W.H. Cornish [3] and T.P. Speed [11, 12] made an exclusive study on annihilators and characterized many algebraic structures like normal lattices and quasi-complemented lattices with the help of these annihilators. In 2013, Rao [9] studied the properties of D-filters of MS-algebras. Later in 2016, Rao and Badawy [10] studied the properties of co-annihilator filters of distributive lattices.

The main objective of this paper is to study the divisibility theory in a bounded distributive lattice. The major emphasis is given to the intersection of divisibility concepts of number theory and the theory of filters of a distributive lattice. In this paper, the concept of divisibility is introduced with respect to dense elements of distributive lattices and obtained that the set of all multipliers of an element forms a filter. Later it is proved that the class of all these filters forms a complete Boolean algebra. The notions of D-prime elements and D-irreducible elements are also introduced in a distributive lattice and established a relation between these elements and the corresponding filters formed by their multipliers. Finally, it is proved that every D-irreducible element is a D-prime element.

2 Preliminaries

The reader is referred to [1] and [2] for the elementary notions and notations of distributive lattices. However some of the preliminary definitions and results of [11] are presented for the ready reference of the reader.

Definition 2.1. [1] An algebra $(L, \wedge, \vee)$ of type $(2, 2)$ is called a distributive lattice if for all $x, y, z \in L$, it satisfies the following properties $(1), (2), (3)$ and (4) along with (5) or $(5')$

(1) $x \wedge x = x, x \vee x = x$
(2) $x \wedge y = y \wedge x, x \vee y = y \vee x$
(3) $(x \wedge y) \wedge z = x \wedge (y \wedge z), (x \vee y) \vee z = x \vee (y \vee z)$
(4) $(x \wedge y) \vee x = x, (x \vee y) \wedge x = x$
(5) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$
(5′) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$

A non-empty subset A of a lattice L is called an ideal(filter) of L if $a \vee b \in A(a \wedge b \in A)$ and $a \wedge x \in A(a \vee x \in A)$ whenever $a, b \in A$ and $x \in L$. The set $\mathcal{I}(L)$ of all ideals of $(L, \vee, \wedge, 0)$ forms a complete distributive lattice as well as the set $\mathcal{F}(L)$ of all filters of $(L, \vee, \wedge, 1)$ forms a complete distributive lattice. A proper ideal (filter) M of a lattice is called maximal if there exists no proper ideal (filter) N such that $M \subset N$.

Definition 2.2. [2] Let $(L, \wedge, \vee)$ be a lattice. A partial ordering relation $\leq$ is defined on L by $x \leq y$ if and only if $x \wedge y = x$ and $x \vee y = y$. In this case, the pair $(L, \leq)$ is called a partially ordered set. If $x \leq y$ or $y \leq x$ for all $x, y \in L$, then $(L, \leq)$ is called totally ordered.

The set $(a] = \{x \in L \mid x \leq a\}$ is called a principal ideal generated by a and the set of all principal ideals is a sublattice of $\mathcal{I}(L)$. Dually the set $[a) = \{x \in L \mid a \leq x\}$ is called a principal filter generated by a and the set of all principal filters is a sublattice of $\mathcal{F}(L)$. A proper ideal (proper filter) P of a lattice L is called prime if for all $a, b \in L$, $a \wedge b \in P$ $(a \vee b \in P)$ then $a \in P$ or $b \in P$. Every maximal ideal (maximal filter) is prime.

Definition 2.3. [12] Let L be a lattice with 0. For any non-empty subset A of L, the annihilator of A is defined as $A^* = \{x \in L \mid a \wedge x = 0 \text{ for all } a \in A\}$.

If $A = \{a\}$, then we denote $(\{a\})^*$ by $(a)^*$.

Lemma 2.4. [11] *For any two elements a, b of a distributive lattice L with 0, we have*

(1) $a \leq b$ *implies* $(b)^* \subseteq (a)^*$
(2) $(a \vee b)^* = (a)^* \cap (b)^*$
(3) $(a \wedge b)^{**} = (a)^{**} \cap (b)^{**}$
(4) $(a)^* = L$ *if and only if* $a = 0$

An element a of a lattice L is called a *dense element* if $(a)^* = \{0\}$. The set D of all dense elements of a distributive lattice L forms a filter of L. Throughout the rest of this note, L stands for a bounded distributive lattice with bounds 0 and 1, unless otherwise mentioned.

3 Divisibility and filters

In this section, the concept of D-divisibility is introduced in lattices and then some properties of the D-filters are observed. A congruence θ is defined on L and established a set of equivalent conditions for the quotient lattice $L_{/\theta}$ to become a Boolean algebra.

Definition 3.1. For any nonempty subset A of a lattice L, define $A^\circ = \{x \in L \mid a \vee x \in D \text{ for all } a \in A\}$.

It is clear that $L^\circ = D$ and $D^\circ = L$. It can be also observed that $D \subseteq A^\circ$ for any subset A of a lattice L. For any $a \in L$, we simply represent $(\{a\})^\circ$ by $(a)^\circ$. Then clearly $1^\circ = L$.

Proposition 3.2. *For any nonempty subset A of a lattice L, A° is a filter of L such that $D \subseteq A^\circ$.*

Proof. Clearly $1 \in A^\circ$. Let $x, y \in A^\circ$. Then $a \vee x \in D$ and $a \vee y \in D$ for all $a \in A$. Now $a \vee (x \wedge y) = (a \vee x) \wedge (a \vee y) \in D$. Hence $x \wedge y \in A^\circ$. Let $x \in A^\circ$ and $y \in L$ with $x \leq y$. Then $a \vee x \leq a \vee y$ for all $a \in A$. Thus $a \vee y \in D$. Hence $y \in A^\circ$ for all $a \in A$. Therefore A° is a filter of L. Form the definition of A°, we get $D \subseteq A^\circ$. □

Lemma 3.3. *For any two subsets A and B of a lattice L, we have*

(1) $A \subseteq B$ *implies* $B^\circ \subseteq A^\circ$
(2) $A \subseteq A^{\circ\circ}$
(3) $A^{\circ\circ\circ} = A^\circ$
(4) $A^\circ = L$ *if and only if* $A \subseteq D$

Proof. (1) Assume that $A \subseteq B$. Let $x \in B^\circ$. Then $a \vee x \in D$ for all $a \in B$. Since $A \subseteq B$, we get $a \vee x \in D$ for all $a \in A$. Therefore $x \in A^\circ$. Thus $B^\circ \subseteq A^\circ$.
(2) Let $a \in A$. Then for every $x \in A^\circ$, we get $a \vee x \in D$. Hence $a \vee x \in D$ for all $x \in A^\circ$.

Thus $a \in A^{\circ\circ}$. Therefore $A \subseteq A^{\circ\circ}$.
(3) From condition (2), $A \subseteq A^{\circ\circ}$. Then $A^{\circ\circ\circ} \subseteq A^{\circ}$. Again $A^{\circ} \subseteq (A^{\circ})^{\circ\circ} = A^{\circ\circ\circ}$. Therefore $A^{\circ\circ\circ} = A^{\circ}$.
(4) Suppose $A^{\circ} = L$. Then $0 \in A^{\circ}$. Now $a = a \vee 0 \in D$ for all $a \in A$. Hence $a \in D$ for all $a \in A$. Therefore $A \subseteq D$. Conversely assume that $A \subseteq D$. Let $x \in L$. Then $a \vee x \in D$ for all $a \in A \subseteq D$. Hence $x \in A^{\circ}$. Therefore $A^{\circ} = L$. □

In case of filters, we have the following result:

Proposition 3.4. *For any two filters F and G of a lattice L, the following conditions hold:*

(1) $F^{\circ} \cap F^{\circ\circ} = D$
(2) $(F \vee G)^{\circ} = F^{\circ} \cap G^{\circ}$
(3) $(F \cap G)^{\circ\circ} = F^{\circ\circ} \cap G^{\circ\circ}$

Proof. (1) Clearly $D \subseteq F^{\circ} \cap F^{\circ\circ}$. Conversely, let $x \in F^{\circ} \cap F^{\circ\circ}$. Then $x = x \vee x \in D$. Hence, it yields $F^{\circ} \cap F^{\circ\circ} \subseteq D$. Therefore $F^{\circ} \cap F^{\circ\circ} = D$.
(2) Clearly $(F \vee G)^{\circ} \subseteq F^{\circ} \cap G^{\circ}$. Conversely, let $x \in F^{\circ} \cap G^{\circ}$. Let $c \in F \vee G$ be an arbitrary element. Then we can write $c = i \wedge j$ for some $i \in F$ and $j \in G$. Since $x \in F^{\circ} \cap G^{\circ}$, we get $x \vee i \in D$ and $x \vee J \in D$ Now $x \vee c = x \vee (i \wedge j) = (x \vee i) \wedge (x \vee j) \in D$. Hence $x \in (F \vee G)^{\circ}$. Thus $F^{\circ} \cap G^{\circ} \subseteq (F \vee G)^{\circ}$. Therefore $(F \vee G)^{\circ} = F^{\circ} \cap G^{\circ}$.
(3) Clearly $(F \cap G)^{\circ\circ} \subseteq F^{\circ\circ} \cap G^{\circ\circ}$. Conversely, let $x \in F^{\circ\circ} \cap G^{\circ\circ}$, $y \in (F \cap G)^{\circ}$, $f \in F$ and $g \in G$. Clearly $f \vee g \in F \cap G$. Now

$$\begin{aligned}
y \in (F \cap G)^{\circ} &\Rightarrow y \vee (f \vee g) \in D && \text{since } f \vee g \in F \cap G \\
&\Rightarrow (y \vee f) \vee g \in D && \text{for all } g \in G \\
&\Rightarrow y \vee f \in G^{\circ} \\
&\Rightarrow x \vee (y \vee f) \in D && \text{since } x \in G^{\circ\circ} \\
&\Rightarrow (x \vee y) \vee f \in D && \text{for all } f \in F \\
&\Rightarrow x \vee y \in F^{\circ} \\
&\Rightarrow x \vee y \in F^{\circ} \cap F^{\circ\circ} = D && \text{since } x \in F^{\circ\circ} \\
&\Rightarrow x \in (y)^{\circ} && \text{for all } y \in (F \cap G)^{\circ} \\
&\Rightarrow x \in (F \cap G)^{\circ\circ}
\end{aligned}$$

Hence $F^{\circ\circ} \cap G^{\circ\circ} \subseteq (F \cap G)^{\circ\circ}$. Therefore $(F \cap G)^{\circ\circ} = F^{\circ\circ} \cap G^{\circ\circ}$. □

The following corollary is a direct consequence of the above results.

Corollary 3.5. *Let L be a lattice. For any $a, b \in L$, we have the following:*

(1) $([a))^{\circ} = (a)^{\circ}$
(2) $a \leq b$ *implies* $(a)^{\circ} \subseteq (b)^{\circ}$
(3) $(a \wedge b)^{\circ} = (a)^{\circ} \cap (b)^{\circ}$
(4) $(a \vee b)^{\circ\circ} = (a)^{\circ\circ} \cap (b)^{\circ\circ}$
(5) $(a)^{\circ} = L$ *if and only if a is dense*

The following result is a simple observation in a lattice.

Lemma 3.6. *Let L be a lattice and $a, b \in L$. Then we have*

(1) $(a)^{\circ} = (b)^{\circ}$ *implies* $(a \wedge x)^{\circ} = (b \wedge x)^{\circ}$ *for all* $x \in L$
(2) $(a)^{\circ} = (b)^{\circ}$ *implies* $(a \vee x)^{\circ} = (b \vee x)^{\circ}$ *for all* $x \in L$

Proof. (1) Let $(a)^{\circ} = (b)^{\circ}$. Then $(a \wedge x)^{\circ} = (a)^{\circ} \cap (x)^{\circ} = (b)^{\circ} \cap (x)^{\circ} = (b \wedge x)^{\circ}$.
(2) Let $(a)^{\circ} = (b)^{\circ}$. Then $(a \vee x)^{\circ\circ} = (a)^{\circ\circ} \cap (x)^{\circ\circ} = (b)^{\circ\circ} \cap (x)^{\circ\circ} = (b \vee x)^{\circ\circ}$. Hence $(a \vee x)^{\circ\circ\circ} = (b \vee x)^{\circ\circ\circ}$. Therefore $(a \vee x)^{\circ} = (b \vee x)^{\circ}$. □

Definition 3.7. Let $a, b \in L$ and D be the set of all dense elements of L. Then we say that a is a divisor of b or a divides b with respect to D. If $(b)^\circ = (a \vee c)^\circ$ for some $c \in L$. In this case we write it as $(a|b)_D$.

The following lemma reveals some facts about divisibility in lattices.

Lemma 3.8. *Let L be a lattice. For any $a, b, c \in L$, we have the following properties:*

(1) $(a|a)_D$
(2) *If $a \leq b$ then $(a|b)_D$*
(3) *If $(a)^\circ = (b)^\circ$ then $(a|b)_D$ and $(b|a)_D$*
(4) *If $(a|b)_D$ and $(b|c)_D$ then $(a|c)_D$*
(5) *If $(a|b)_D$ then $(a|b \vee x)_D$ for all $x \in L$*
(6) *If $(a|b)_D$ then $(a \vee x|b \vee x)_D$ and $(a \wedge x|b \wedge x)_D$ for all $x \in L$.*

Proof. (1) Let $a \in L$. Since $a = a \vee a$, we get $(a)^\circ = (a \vee a)^\circ$ for some $a \in L$. Hence $(a|a)_D$.
(2) Let $a \leq b$. Then $b = a \vee b$. Hence $(b)^\circ = (b \vee a)^\circ$. Therefore $(a|b)_D$.
(3) Let $(a)^\circ = (b)^\circ$. Then we get $(a \vee b)^\circ = (b \vee b)^\circ = (b)^\circ$. Thus $(a \vee b)^\circ = (b)^\circ$. Therefore $(a|b)_D$. Similarly we get $(b|a)_D$.
(4) Let $(a|b)_D$ and $(b|c)_D$. Then $(b)^\circ = (a \vee x)^\circ$ for some $x \in L$ and $(c)^\circ = (b \vee y)^\circ$ for some $y \in L$. Now $(c)^\circ = (b \vee y)^\circ = (a \vee x \vee y)^\circ$. Therefore $(a|c)_D$.
(5) Let $(a|b)_D$. Then $(b)^\circ = (a \vee y)^\circ$ for some $y \in L$. Hence $(b \vee x)^\circ = (a \vee y \vee x)^\circ = (a \vee (x \vee y))^\circ$. Therefore $(a|b \vee x)_D$.
(6) Let $(a|b)_D$. Then $(b)^\circ = (a \vee y)^\circ$ for some $y \in L$. Hence $(b \vee x)^\circ = (a \vee x \vee y)^\circ$ for some $x \in L$. Therefore $(a \vee x|b \vee x)_D$. Similarly we can prove that $(a \wedge x|b \wedge x)_D$. □

Definition 3.9. Let L be a lattice and $a \in L$. Then define $(a)^\perp$ as the set of all multipliers of a. That is $(a)^\perp = \{x \in L \mid (a|x)_D\}$.

It is clear that $(1)^\perp = D$ and $(0)^\perp = L$.

Lemma 3.10. *Let L be a lattice. For any $a \in L$, $(a)^\perp$ is a filter of L such that $D \subseteq (a)^\perp$.*

Proof. Clearly $1 \in L$. Hence $(a)^\perp$ is non-empty. Let $x, y \in (a)^\perp$. Then $(a|x)_D$ and $(a|y)_D$. Since $(a|x)_D$, we get $(x)^\circ = (a \vee t)^\circ$ for some $t \in L$. Since $(a|y)_D$, we get $(y)^\circ = (a \vee s)^\circ$ for some $s \in L$. Now $(x \wedge y)^\circ = (x)^\circ \cap (y)^\circ = (a \vee t)^\circ \cap (a \vee s)^\circ = ((a \vee t) \wedge (a \vee s))^\circ = (a \vee (t \wedge s))^\circ$. Hence $(a|x \wedge y)_D$. Therefore $x \wedge y \in (a)^\perp$. Let $x \in (a)^\perp$ and $x \leq y$. Then $(a|x)_D$. Hence $(x)^\circ = (a \vee t)^\circ$ for some $t \in L$. Since $x \leq y$, we get $(x)^\circ \subseteq (y)^\circ$. Since $(x)^\circ = (a \vee t)^\circ$, we get $(x \vee y)^\circ = (a \vee t \vee y)^\circ$. Hence $(y)^\circ = (a \vee t \vee y)^\circ$. It yields $(a|y)_D$. Thus $y \in (a)^\perp$. Therefore $(a)^\perp$ is a filter of L. Let $x \in D$. Then $(x)^\circ = L$. For any $c \in D$, we have clearly $a \vee c \in D$. Hence $(a \vee c)^\circ = L$. Thus $(x)^\circ = (a \vee c)^\circ = L$. Thus $(a|x)_D$, which means $x \in (a)^\perp$. Therefore $D \subseteq (a)^\perp$. □

Lemma 3.11. *Let L be a lattice. For any $a, b \in L$, we have the following properties:*

(1) $a \in (a)^\perp$
(2) $a \in (b)^\perp$ *implies* $(a)^\perp \subseteq (b)^\perp$
(3) $a \leq b$ *implies* $(b)^\perp \subseteq (a)^\perp$
(4) $(a)^\circ = (b)^\circ$ *implies* $(a)^\perp = (b)^\perp$
(5) $(a)^\perp \cap (b)^\perp = (a \vee b)^\perp$

Proof. (1) Let $a \in L$. Since $a = a \vee a$, we get $(a)^\circ = (a \vee a)^\circ$ for this $a \in L$. Hence $(a|a)_D$. Therefore $a \in (a)^\perp$.
(2) Let $a \in (b)^\perp$. Then $(b|a)_D$. Hence $(a)^\circ = (b \vee t)^\circ$ for some $t \in L$. Let $x \in (a)^\perp$. Then $(a|x)_D$. Hence $(x)^\circ = (a \vee s)^\circ$ for some $s \in L$. Thus $(x)^\circ = (a \vee s)^\circ = (b \vee t \vee s)^\circ$. Hence $(b|x)_D$. Thus $x \in (b)^\perp$. Therefore $(a)^\perp \subseteq (b)^\perp$.
(3) Let $a \leq b$ and $x \in (b)^\perp$. Then $(b|x)_D$. Hence $(x)^\circ = (b \vee t)^\circ$ for some $t \in L$. Thus $(x)^\circ = (a \vee b \vee t)^\circ$ (because of $a \leq b$). Hence $(a|x)_D$. Therefore $x \in (a)^\perp$.

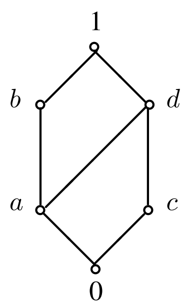
(4) Assume that $(a)^\circ = (b)^\circ$. Let $x \in (a)^\perp$. Then $(a|x)_D$. Hence $(x)^\circ = (a \vee t)^\circ$ for some $t \in L$. Thus $(x)^\circ = (b \vee t)^\circ$. Hence $(b|x)_D$. Therefore $x \in (b)^\perp$, which means $(a)^\perp \subseteq (b)^\perp$. Similarly, we can prove that $(b)^\perp \subseteq (a)^\perp$.

(5) Clearly $(a \vee b)^\perp \subseteq (a)^\perp \cap (b)^\perp$. Conversely, let $x \in (a)^\perp \cap (b)^\perp$. Then $x \in (a)^\perp$ and $x \in (b)^\perp$ imply that $(a|x)_D$ and $(b|x)_D$. Hence $(x)^\circ = (a \vee c)^\circ$ and $(x)^\circ = (b \vee d)^\circ$ for some $c, d \in L$. Now $(x)^{\circ\circ} = (x)^{\circ\circ} \cap (x)^{\circ\circ} = (a \vee c)^{\circ\circ} \cap (b \vee d)^{\circ\circ} = (a \vee c \vee b \vee d)^{\circ\circ}$. It yields $(x)^\circ = (x)^{\circ\circ\circ} = (a \vee c \vee b \vee d)^{\circ\circ\circ} = (a \vee c \vee b \vee d)^\circ$. Hence $(a \vee b|x)_D$. Thus $x \in (a \vee b)^\perp$. Therefore $(a)^\perp \cap (b)^\perp = (a \vee b)^\perp$. □

Proposition 3.12. *Let L be a lattice and $d \in L$. Then $(d)^\circ = D$ if and only if $(d)^\perp = L$.*

Proof. Assume that $(d)^\circ = D$. Then $(d)^\circ = D = (0)^\circ$. Hence $(0)^\circ = (d)^\circ = (0 \vee d)^\circ$. Thus $(d|0)_D$. Therefore $0 \in (d)^\perp$. Since $0 \in (d)^\perp$, we get $(0)^\perp \subseteq (d)^\perp$. Therefore $(d)^\perp = L$. Conversely, let $(d)^\perp = L$. Then $0 \in (d)^\perp$. Hence $(d|0)_D$. Thus $(0)^\circ = (d \vee s)^\circ$ for some $s \in L$. Therefore $(d)^\circ = (0 \vee d)^\circ = (d \vee d \vee s)^\circ = (d \vee s)^\circ = (0)^\circ = D$. □

Let us denote the set of all filters of the form $(x)^\perp$, $x \in L$ by $\mathcal{F}^\perp(L)$. In general, $\mathcal{F}^\perp(L)$ is not a sublattice of $\mathcal{F}(L)$ of all filters of L. For, consider the following distributive lattice $L = \{0, a, b, c, d, 1\}$ whose Hasse diagram is given by:



Then clearly $(b)^\perp = \{b, d, 1\}$ and $(d)^\perp = \{d, 1\}$. Hence $(b)^\perp \vee (d)^\perp = \{b, d, 1\} \vee \{d, 1\} = \{b, d, 1\}$. But $(b \wedge d)^\perp = (a)^\perp = \{a, b, d, 1\}$. Thus $(b)^\perp \vee (d)^\perp \neq (b \wedge d)^\perp$. Therefore $\mathcal{F}^\perp(L)$ is not a sublattice of $\mathcal{F}(L)$.

However, we have the following theorem.

Theorem 3.13. *For any lattice L, the set $\mathcal{F}^\perp(L)$ forms a complete distributive lattice on its own.*

Proof. For any $(a)^\perp, (b)^\perp \in \mathcal{F}^\perp(L)$ (where $a, b \in L$), define as follows:

$$(a)^\perp \cap (b)^\perp = (a \vee b)^\perp \text{ and } (a)^\perp \sqcup (b)^\perp = (a \wedge b)^\perp$$

From Lemma 3.11(5), $(a \vee b)^\perp$ is the infimum of both $(a)^\perp$ and $(b)^\perp$ in $\mathcal{F}^\perp$. Since $a \wedge b \leq a$ and $a \wedge b \leq b$, we get $(a)^\perp \subseteq (a \wedge b)^\perp$, $(b)^\perp \subseteq (a \wedge b)^\perp$. Therefore $(a \wedge b)^\perp$ is an upper bound of $(a)^\perp$ and $(b)^\perp$. Let $(c)^\perp$ be an upper bound of $(a)^\perp$ and $(b)^\perp$. Then $(a)^\perp \subseteq (c)^\perp$ and $(b)^\perp \subseteq (c)^\perp$. Hence $a \in (c)^\perp$ and $b \in (c)^\perp$. Since $(c)^\perp$ is a filter, we get $a \wedge b \in (c)^\perp$, which implies that $(a \wedge b)^\perp \subseteq (c)^\perp$. Thus $(a \wedge b)^\perp$ is the least upper bound of $(a)^\perp$ and $(b)^\perp$. Hence $(a)^\perp \sqcup (b)^\perp = (a \wedge b)^\perp$. Therefore $\mathcal{F}^\perp(L)$ is a lattice. We now prove the distributivity of these filters. For any $(a)^\perp, (b)^\perp, (c)^\perp \in \mathcal{F}^\perp(L)$, we have

$$\begin{aligned}
(a)^\perp \sqcup \{(b)^\perp \cap (c)^\perp\} &= (a)^\perp \sqcup (b \vee c)^\perp \\
&= (a \wedge (b \vee c))^\perp \\
&= ((a \wedge b) \vee (a \wedge c))^\perp \\
&= (a \wedge b)^\perp \cap (a \wedge c)^\perp \\
&= ((a)^\perp \sqcup (b)^\perp) \cap ((a)^\perp \sqcup (c)^\perp).
\end{aligned}$$

Therefore $(\mathcal{F}^\perp(L), \cap, \sqcup)$ is a distributive lattice. Let $a, b \in L$. Then $(a)^\perp, (b)^\perp \in \mathcal{F}^\perp(L)$. Define $(a)^\perp \leq (b)^\perp \Leftrightarrow (a)^\perp \subseteq (b)^\perp$. Clearly $(\mathcal{F}^\perp(L), \leq)$ is a partially ordered set. Clearly L and $\{1\}$

are the bounds for $\mathcal{F}^{\perp}(L)$. Also by the extension of the property of Lemma 3.11(5), we get $\mathcal{F}^{\perp}(L)$ is a bounded and complete distributive lattice. □

We now introduce a congruence θ on L in the following:

Definition 3.14. Let L be a lattice. For any $a, b \in L$, define a relation θ on L as follows:

$$(a, b) \in \theta \text{ if and only if } (a)^{\perp} = (b)^{\perp}.$$

Lemma 3.15. *Let L be a lattice. Then the relation θ defined above is a congruence on L.*

Proof. Clearly θ is an equivalence relation on L. Assume that $(a, b), (c, d) \in \theta$. Then we get $(a)^{\perp} = (b)^{\perp}$ and $(c)^{\perp} = (d)^{\perp}$. Now $(a \vee c)^{\perp} = (a)^{\perp} \cap (c)^{\perp} = (b)^{\perp} \cap (d)^{\perp} = (b \vee d)^{\perp}$. Therefore $(a \vee c, b \vee d) \in \theta$. Dually $(a \wedge c)^{\perp} = (a)^{\perp} \sqcup (c)^{\perp} = (b)^{\perp} \sqcup (d)^{\perp} = (b \wedge d)^{\perp}$. Hence $(a \wedge c)^{\perp} = (b \wedge d)^{\perp}$. Thus $(a \wedge c, b \wedge d) \in \theta$. Therefore θ is congruence on L. □

We know that, for any congruence θ on L, $L_{/\theta}$ = $\{ [x]_{\theta} \mid x \in L\}$, (where $[x]_{\theta}$ is the congruence class of x modulo θ) is a distributive lattices with respect to the operations

$$[x]_{\theta} \wedge [y]_{\theta} = [x \wedge y]_{\theta} \quad \text{and} \quad [x]_{\theta} \vee [y]_{\theta} = [x \vee y]_{\theta}$$

for all $x, y \in L$. Now, define a mapping $\rho : L \to L_{/\theta}$ by $\rho(x) = [x]_{\theta}$ for all $x \in L$. Then clearly ρ is a natural homomorphism. Also ρ is a surjective mapping.

Definition 3.16. An element a of a lattice L is called *D-dense* if $(a)^{\circ} = D$.

Lemma 3.17. *Let L be a lattice. Then $D'(L) = \{x \in L \mid (x)^{\circ} = D\}$ is an ideal of L.*

Proof. Clearly $0 \in D'(L)$. Let $x, y \in D'(L)$. Then $(x)^{\circ} = D = (y)^{\circ}$. Now $(x \vee y)^{\circ\circ} = (x)^{\circ\circ} \cap (y)^{\circ\circ} = D^{\circ} \cap D^{\circ} = L \cap L = L$. Hence $(x \vee y)^{\circ} = (x \vee y)^{\circ\circ\circ} = L^{\circ} = D$. Therefore $x \vee y \in D'(L)$. Let $x \in D'(L)$ and $y \in L$. Since $x \in D'(L)$, we get $(x)^{\circ} = D$. Now $(x \wedge y)^{\circ} \subseteq (x)^{\circ} = D$. Hence $(x \wedge y)^{\circ} = D$ (since $D \subseteq (x \wedge y)^{\circ}$). Thus $x \wedge y \in D'(L)$. Therefore $D'(L)$ is an ideal of L. □

Lemma 3.18. *$D'(L)$ is the zero congruence class of $L_{/\theta}$ and D is the unit congruence of $L_{/\theta}$.*

Proof. For any $a, b \in D'(L)$, we have $(a)^{\circ} = D = (b)^{\circ}$. Hence $(a)^{\perp} = (b)^{\perp}$. Thus $(a, b) \in \theta$. Therefore $D'(L)$ is a congruence class of $L_{/\theta}$. For any $a \in D'(L)$ and $x \in L$, we get $[a]_{\theta} \wedge [x]_{\theta} = [a \wedge x]_{\theta} = D'(L)$ (because of $a \wedge x \in D'(L)$). That is $D'(L) \cap [x]_{\theta} = [a]_{\theta} \cap [x]_{\theta} = [a \wedge x]_{\theta} = D'(L)$ (because of $a \in D'(L)$). Hence $D'(L) \subseteq [x]_{\theta}$ for all $x \in L$, which means $D'(L)$ is the zero congruence of $L_{/\theta}$. For any $a, b \in D$, we have $(a)^{\circ} = L = (b)^{\circ}$. Hence $(a)^{\perp} = (b)^{\perp}$. Thus $(a, b) \in \theta$. Therefore D is a congruence class of $L_{/\theta}$. For any $a \in D$ and $x \in L$, we get $[a]_{\theta} \vee [x]_{\theta} = [a \vee x]_{\theta} = D$ (because of $a \vee x \in D$). That is $D \vee [x]_{\theta} = [a]_{\theta} \vee [x]_{\theta} = [a \vee x]_{\theta} = D$ (since $a \in D$). Hence $[x]_{\theta} \subseteq D$ for all $x \in L$. Thus D is the unit congruence of $L_{/\theta}$. □

We now establish a set of equivalent conditions for $L_{/\theta}$ to become a Boolean algebra.

Theorem 3.19. *The following assertions are equivalent in a lattice L:*

(1) *$L_{/\theta}$ is a Boolean algebra;*
(2) *$\mathcal{F}^{\perp}(L)$ is a Boolean algebra;*
(3) *For each $x \in L$ there exist $y \in L$ such that $x \wedge y \in D'$ and $x \vee y \in D$.*

Proof. (1) $\Rightarrow$ (2): Assume that $L_{/\theta}$ is a Boolean algebra. Define a mapping $\phi : L_{/\theta} \to \mathcal{F}^{\perp}(L)$ by $\phi([x]_{\theta}) = (x)^{\perp}$ for all $[x]_{\theta} \in L_{/\theta}$. Clearly ϕ is well defined. For any $[x]_{\theta}, [y]_{\theta} \in L_{/\theta}$, let us suppose that $\phi([x]_{\theta}) = \phi([y]_{\theta})$. Then we have $(x)^{\perp} = (y)^{\perp}$. Hence we get $(x, y) \in \theta$. Thus $[x]_{\theta} = [y]_{\theta}$. Therefore ϕ is injective. Let $x \in \mathcal{F}^{\perp}(L)$, where $x \in L$. Now for this x, $\rho(x) = [x]_{\theta} \in L_{/\theta}$ such that $\phi([x]_{\theta}) = (x)^{\perp}$. Therefore ϕ is surjective and hence it is bijective. Now, let $[x]_{\theta}, [y]_{\theta} \in L_{/\theta}$ where $x, y \in L$. Then $\phi([x]_{\theta} \cap [y]_{\theta}) = \phi([x \wedge y]_{\theta}) = (x \wedge y)^{\perp} = (x)^{\perp} \sqcup (y)^{\perp} = \phi([x]_{\theta}) \sqcup \phi([y]_{\theta})$. Again $\phi([x]_{\theta} \vee [y]_{\theta}) = \phi([x \vee y]_{\theta}) = (x \vee y)^{\perp} = (x)^{\perp} \cap (y)^{\perp} = \phi([x]_{\theta}) \cap \phi([y]_{\theta})$. Thus $L_{/\theta}$ is dual isomorphic to $\mathcal{F}^{\perp}(L)$. Therefore $\mathcal{F}^{\perp}(L)$ is a Boolean algebra.

(2) $\Rightarrow$ (3): Assume that $\mathcal{F}^{\perp}(L)$ is a Boolean algebra. Let $x \in L$. Then $(x)^{\perp} \in \mathcal{F}^{\perp}(L)$. Since $\mathcal{F}^{\perp}(L)$ is a Boolean algebra, there exists $(y)^{\perp} \in \mathcal{F}^{\perp}(L)$ such that $(x \vee y)^{\perp} = (x)^{\perp} \cap (y)^{\perp} = (1)^{\perp} = D$ and $(x \wedge y)^{\perp} = (x)^{\perp} \sqcup (y)^{\perp} = (0)^{\perp} = L$. Hence $x \vee y \in D$ and $(x \wedge y)^{\circ} = D$ from Proposition 3.12. Therefore $x \vee y \in D$ and $x \wedge y \in D'(L)$.
(3) $\Rightarrow$ (1): Assume the condition (3). Let $[x]_{\theta} \in L_{/\theta}$ where $x \in L$. Then there exists $x' \in L$ such that $x \wedge x' \in D'(L)$ and $x \vee x' \in D$. Hence $[x]_{\theta} \cap [x']_{\theta} = [x \wedge x']_{\theta} = D'(L)$ also $[x]_{\theta} \vee [x']_{\theta} = [x \vee x']_{\theta} = D$. Therefore $L_{/\theta}$ is a Boolean algebra. □

4 D-prime and D-irreducible elements

In this section, the concepts of D-prime elements and D-irreducible elements are introduced, in terms of divisibility given in lattices. Finally, these elements are characterized in term of prime filters and maximal filters of lattices respectively.

Definition 4.1. An element $0 \neq a \in L$ is called *D-prime* if it satisfies the following property:

$$(a\,|\,b \vee c)_D \;\; \textit{implies that} \;\; (a\,|\,b)_D \;\text{ or }\; (a\,|\,c)_D$$

In the following theorem, D-prime elements are characterized.

Theorem 4.2. *Let a be a non-dense element of a lattice L. Then a is a D-prime element of L if and only if $(a)^{\perp}$ is a prime filters of L.*

Proof. Let a be a non-dense element of L. Assume that a is a D-prime element of L. Let $x, y \in L$ be such that $x \vee y \in (a)^{\perp}$. Then $(a\,|\,x \vee y)_D$. Since a is a D-prime element of L, we get $(a\,|\,x)_D$ or $(a\,|\,y)_D$. Hence $x \in (a)^{\perp}$ or $y \in (a)^{\perp}$. Therefore $(a)^{\perp}$ is a prime filter of L. Conversely suppose that $(a)^{\perp}$ is a prime filter of L. Let $x, y \in L$ and $(a\,|\,x \vee y)_D$. Then $x \vee y \in (a)^{\perp}$. Since $(a)^{\perp}$ is prime, we get either $x \in (a)^{\perp}$ or $y \in (a)^{\perp}$. Hence $(a\,|\,x)_D$ or $(a\,|\,y)_D$. Therefore a is a D-prime element of L. □

Next, the concept of D-irreducible elements is introduced.

Definition 4.3. A non-dense element a of a lattice L is called *D-irreducible* if $(a)^{\circ} = (b \vee c)^{\circ}$ then either $(b)^{\circ} = D$ or $(c)^{\circ} = D$.

Lemma 4.4. *Every D-dense element of a lattice L is a D-irreducible element.*

Proof. Let d be a D-dense element of L. Suppose $(d)^{\circ} = (b \vee c)^{\circ}$ for some $b, c \in L$. Now $(b)^{\circ\circ} \cap (c)^{\circ\circ} = (b \vee c)^{\circ\circ} = (d)^{\circ\circ} = D^{\circ} = L$. Hence $L \subseteq (b)^{\circ\circ}$ and $L \subseteq (c)^{\circ\circ}$, which produces $(b)^{\circ\circ\circ} = D$ and $(c)^{\circ\circ\circ} = D$. Thus $(b)^{\circ} = D$ and $(c)^{\circ} = D$. Therefore d is a D-irreducible element. □

D-irreducible elements are now characterized in the following:

Theorem 4.5. *Let L be a lattice. Suppose $a \in L$ be such that $(a)^{\circ} \neq D$. Then the following assertions are equivalent:*

(1) *a is D-irreducible;*
(2) (i) *$(a)^{\perp}$ is maximal among all proper filters of the form $(x)^{\perp}$,*
(ii) *For any $x \in L$, $(a)^{\circ} = (a \vee x)^{\circ}$ implies $(x)^{\circ} = D$.*

Proof. (1) $\Rightarrow$ (2) (i): Assume that a is D-irreducible. Suppose $(a)^{\perp} \subseteq (b)^{\perp} \neq L$ for some $b \in L$. We have $a \in (a)^{\perp} \subseteq (b)^{\perp}$. Then $(b|a)_D$. Hence there exists $c \in L$ such that $(a)^{\circ} = (b \vee c)^{\circ}$. Since a is D-irreducible, we get either $(b)^{\circ} = D$ or $(c)^{\circ} = D$. Since $(b)^{\perp} \neq L$, by Proposition 3.12, we get $(b)^{\circ} \neq D$. Hence $(c)^{\circ} = D$. Now

$$\begin{aligned} (c)^{\circ} = D = (0)^{\circ} \quad &\Rightarrow \quad (b \vee c)^{\circ} = (b \vee 0)^{\circ} \\ &\Rightarrow \quad (a)^{\circ} = (b)^{\circ} \\ &\Rightarrow \quad (a)^{\perp} = (b)^{\perp}. \end{aligned}$$

Therefore $(a)^{\perp}$ is maximal among all filters of the form $(x)^{\perp}$.
$(1) \Rightarrow (2)$ (ii): Suppose $(a)^{\circ} = (a \vee x)^{\circ}$ for some $x \in L$. Since a is a D-irreducible, we get either $(a)^{\circ} = D$ or $(x)^{\circ} = D$. Since $(a)^{\circ} \neq D$, we must have $(x)^{\circ} = D$.
$(2) \Rightarrow (1)$: Assume the conditions $(2)(i)$ and $(2)(ii)$. Let $a \in L$ be such that $(a)^{\circ} \neq D$. Suppose $(a)^{\circ} = (b \vee c)^{\circ}$ for some $b, c \in L$. Hence $(c|a)_D$. So we get $a \in (c)^{\perp}$ and hence $(a)^{\perp} \subseteq (c)^{\perp}$. Since $(a)^{\perp}$ is maximal, we get either $(a)^{\perp} = (c)^{\perp}$ or $(c)^{\perp} = L$. Suppose $(a)^{\perp} = (c)^{\perp}$. Then

$$\begin{aligned} c \in (a)^{\perp} &\Rightarrow (a|c)_D \\ &\Rightarrow (c)^{\circ} = (r \vee a)^{\circ} \qquad \text{for some } r \in L \\ &\Rightarrow (c \vee b)^{\circ} = (r \vee a \vee b)^{\circ} \\ &\Rightarrow (a)^{\circ} = (r \vee a \vee b)^{\circ} \\ &\Rightarrow (r \vee b)^{\circ} = D \qquad \text{form condition } 2(ii) \\ &\Rightarrow (b)^{\circ} = D. \qquad \text{since } (b)^{\circ} \subseteq (r \vee b)^{\circ} \end{aligned}$$

Suppose $(c)^{\perp} = L$. Then we get the following consequence:

$$\begin{aligned} (c)^{\perp} = L &\Rightarrow 0 \in (c)^{\perp} \\ &\Rightarrow (c|0)_D \\ &\Rightarrow (0)^{\circ} = (c \vee r)^{\circ} \qquad \text{for some } r \in L \\ &\Rightarrow D = (c \vee r)^{\circ}. \end{aligned}$$

Hence $(c)^{\circ} \subseteq (c \vee r)^{\circ} = D$. Thus $(c)^{\circ} = D$. Therefore a is D-irreducible. □

Theorem 4.6. *Every D-irreducible element of a lattice L is a D-prime element.*

Proof. Let $a \in L$. Suppose a is a D-dense element of L. That is $(a)^{\circ} = D$. Then by Lemma 4.4, a is D-irreducible. Suppose a is not a D-dense element of L. That is $(a)^{\circ} \neq D$. Assume that a is a D-irreducible element of L. Then by Theorem 4.5, we get that $(a)^{\perp}$ is a maximal among all proper filters of the form $(r)^{\perp}$, where $r \in L$. Choose $x, y \in L$ such that $x \notin (a)^{\perp}$ and $y \notin (a)^{\perp}$. Hence $(a)^{\perp} \subset (a)^{\perp} \sqcup (x)^{\perp} \subseteq (a \wedge x)^{\perp}$ and also $(a)^{\perp} \subset (a \wedge y)^{\perp}$. By the maximality of $(a)^{\perp}$, we get $(a \wedge x)^{\perp} = L$ and $(a \wedge y)^{\perp} = L$. Now

$$\begin{aligned} L = L \cap L &= (a \wedge x)^{\perp} \cap (a \wedge y)^{\perp} \\ &= \{(a)^{\perp} \sqcup (x)^{\perp}\} \cap \{(a)^{\perp} \sqcup (y)^{\perp}\} \\ &= (a)^{\perp} \sqcup \{(x)^{\perp} \cap (y)^{\perp}\} \\ &= (a)^{\perp} \sqcup (x \wedge y)^{\perp}. \end{aligned}$$

If $x \vee y \in (a)^{\perp}$, then $(x \vee y)^{\perp} \subseteq (a)^{\perp}$. Hence $(a)^{\perp} = L$, which is a contradiction. Thus $(a)^{\perp}$ is a prime filter of L. Therefore by Theorem 4.2, a is a D-prime element of L. □

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