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SENSITIVITY ANALYSIS OF A CLASS OF INTEREST RATE DERIVATIVES IN A VARIANCE GAMMA LÉVY MARKET

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Abstract Interest rate derivatives are often characterized by jumps of different magnitudes due to abnormal occurrences from certain sudden or rare events. Adequate pricing of such derivatives require the use of Lévy processes. In order to manage risks, it is necessary to study the effects of changes in a model's parameters. In this paper, we extend an existing Vasicek interest rate model with a Brownian motion as an underlying to a subordinated Lévy market with a variance gamma process as an underlying. We derive an expression for the price of an interest rate derivative called a 'zero-coupon bond' in the market. We employ Malliavin calculus to determine the sensitivity of the model to changes in some critical parameters of the associated interest rate derivatives by computing the greeks: *delta* and *gamma*. These are important risk-measuring quantities in a financial market, especially a variance gamma Lévy market.

1 Introduction

Price paths and derivative securities, such as interest rate derivatives, generally experience jumps, whose number may be finite or infinite. These jumps may lead to fat tails and thus, influence excess kurtosis and skewness [17]. The jumps arise due to factors such as changes in government policy, weather disruption, natural disaster and presence of pandemic.

A survey of the literature shows that not much has been done in providing a model that captures the jumps in interest rate derivative markets and their associated sensitivity analysis. Hence, in this paper, we fill the gap by extending the Vasicek interest rate model to a subordinated Lévy market, use the expression to derive the price of an interest rate derivative called a *zero-coupon bond* and then carry out a sensitivity analysis using the Malliavin calculus. This has never been done in the literature. An *interest rate derivative* is a derivative security in which the underlying asset is the right to pay or receive a notional amount of money at a given interest rate while *sensitivity analysis* involves considering the effect of possible changes in the parameters of a given model on the latter's behaviour.

A subordinated Lévy process called *variance gamma (VG)* is a suitable underlying process for markets with jumps, excess kurtosis and skewness since it takes care of the left and right symmetric increase in tails' probabilities of return distribution (kurtosis), and the left and right assymetric of the tails of return density ([13], [6], [17]). Examples of VG markets include (i) currency option market: here, jumps in exchange rates occur in order to reflect lumpiness in the influx of information relevant to pricing [6]; (ii) equity-linked insurance products [8]; and (iii) share return markets ([17], [13]). Moreover, the importance of a VG process in the modelling of turbulent periods in engineering, chemical, coal and petroleum manufacturing, and modelling financial and insurance sectors during financial disasters has been discussed by Rathgeber et al. [17].

The VG process was first introduced without a drift component by Madan and Seneta [13] as a model for uncertainty underlying security prices. Madan et al. [12] subsequently generalized the process as a three parameter process by evaluating a Brownian motion with constant drift and volatility at a random time change given by a gamma process. The VG process is a pure jump process that models high activity, as it has infinitely many jumps in any finite time interval, thus, making a diffusion component irrelevant. Lang et al. [11] studied how different choices of interest rate models by banks affect the overall financial stability and observed that interest rate models which are empirically best fitting do not entail aggregate financial stability. Filipovic and Willems [9] highlighted a structure based on jump-diffusions to price the term structure of dividends and interest rates while Mehrdoust et al. [14] studied the pricing of Bermuda option on an interest rate derivative driven by mixed fractional Vasicek model. Orlando et al. [15] observed that Cox-Ingersoll-Ross interest rate model is not suitable for modelling current market environment with negative short interest rates. Hence, we adopt and modify Vasicek interest rate model since it allows negative interest rates.

Bavouzet-Morel & Messaoud [2] discussed a Malliavin calculus for jump processes by working on a space of simple functionals of a finite set of random variables while Petrou [16] added some tools for the computation of sensitivities. Bayazit & Nolder [3] extended Bavouzet-Morel & Messaoud [2] and Petrou [16] to sensitivity analysis of an option driven by an exponential Lévy process built from subordinated Lévy processes.

This paper extends the work of Bayazit & Nolder [3] to an interest rate derivative Lévy market. Our main results concern the derivation of zero-coupon bond price using a modified Vasicek interest rate model driven by a Lévy process whose jump part is a VG process; and the derivation of the greeks: *delta* and *gamma* for the derived price.

This work, therefore, focuses on an interest rate derivative market with jumps, excess kurtosis and skewness, by extending the well-known *Vasicek model* to a market driven by a VG process and derives the price of the zero-coupon bond. We obtain the associated greeks to determine the sensitivity of an interest rate derivative in such markets.

The rest of this paper is structured as follows. Section 2 is a collection of some basic notions and results such as the Lévy process, VG process and Malliavin calculus which are employed in subsequent sections. In Section 3, we derive the modified Vasicek model driven by the VG process and use the result to derive an expression for the price of the zero-coupon bond. In Section 4, we compute the greeks. The final part is devoted to a sensitivity analysis of interest rate derivatives.

2 Fundamental notions

In this section, we highlight some important mathematical tools employed in the work.

2.1 Lévy process

The key stochastic process which features in the following discussion is a Lévy process.

Definition 2.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, where Ω is a set, $\mathcal{F}$ is a σ-algebra of subsets of Ω and $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$. We equip $(\Omega, \mathcal{F}, \mathbb{P})$ with a complete, right continuous filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}_+}$. Then, $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is a filtered probability space. Let $X = \{(X_t)_{0\leq t\leq T}\}$ with $X_0 = 0$ a.s., be a càdlàg, adapted, real-valued stochastic process on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.

A stochastic process X is called a *Lévy process* if: (i) $X_0 = 0$, (ii) X has independent increments, (iii) X has stationary increments, (iv) X has left limits, and (v) X is stochastically continuous from the right.

Proposition 2.2. ***(Lévy-Khintchine formula)**. If $X = (X_t)_{t\geq 0}$ is a Lévy process, then the characteristic function ϕ_t of X_t is given by*

$$\phi_t(u) = \mathbb{E}[e^{iuX_t}] = e^{t\psi(u)}, \qquad t \geq 0, \qquad u \in \mathbb{R}$$

where $\psi(u)$, called the characteristic exponent, is given by

$$\psi(u) = ibu - \frac{\sigma^2}{2}u^2 + \int_{|x|\geq 1}(e^{iux} - 1)\nu(dx) + \int_{|x|<1}(e^{iux} - 1 - iux)\nu(dx),$$

$\sigma^2 \geq 0$, $b \in \mathbb{R}$, ν is a measure on $\mathbb{R}\setminus\{0\}$, which counts the expected number of jumps, $\int_{\mathbb{R}\setminus\{0\}} \min(1, |x|^2)\nu(dx) < \infty$ and $\mathbb{E}$ denotes the mathematical expectation operator.

The Lévy process is said to have a characteristic triplet (b, σ^2, ν) and X_t may be written as

$$X_t = (b - \int_{\mathbb{R}} x\nu(dx))t + \sigma W_t + \sum_{0\le s\le t} \Delta X_t$$

where W_t is a standard Brownian motion and $\Delta X_t = X_t - X_{t^-}$.

Remark 2.3. In the next subsection, we introduce the subordinated Lévy process called *variance gamma (VG) process* needed in the derivation of our modified interest rate model.

2.2 The variance gamma process

A gamma process is a random process with independent gamma distributed increments and an infinite number of jumps [12]. The density function of a gamma process $G = G(t; \mu, \kappa)$ with mean μ and variance κ is given by

$$f_G(x) = \frac{(\frac{\mu}{\kappa})^{\frac{\mu^2 t}{\kappa}} x^{\frac{\mu^2 t}{\kappa}-1}}{\Gamma(\frac{\mu^2 t}{\kappa})} e^{-\frac{\mu}{\kappa}x}, \qquad x > 0,\ t \in \mathbb{R}^+$$

where $\Gamma(\cdot)$ is the gamma function. This density function has a semi-heavy right tail [16]. It is a pure jump increasing Lévy process with intensity measure $\nu(x) = \dfrac{\mu^2 e^{-\frac{\mu}{\kappa}x}}{\kappa x}, x > 0$, and characteristic function given by

$$\phi_t(u) = \left(1 - iu\frac{\kappa}{\mu}\right)^{-\frac{\mu^2 t}{\kappa}}, \ t \in \mathbb{R}^+,\ u \in \mathbb{R}.$$

Remark 2.4. 1. The VG process is a pure jump process with inter-arrival time that is a gamma process. It may be approximated by a compound Poisson process with high jump frequency and low jump magnitude ([12], [13]). Its benefits include good empirical fit, long-tailedness and finite moment of all orders [12].

2. Let $X_t(\theta, \widetilde{\sigma})$ be an arithmetic Brownian motion with drift θ and volatility $\widetilde{\sigma}$ given by

$$X_t(\theta, \widetilde{\sigma}) = \theta t + \widetilde{\sigma} W_t$$

where $W_t = W(t)$ is a standard Brownian motion. A VG process is obtained by evaluating an arithmetic Brownian motion with drift θ and volatility $\widetilde{\sigma}$ at a random time given by a gamma process having a mean μ per unit time and a variance κ. The resulting VG process $X_t(\widetilde{\sigma}, \theta, \kappa)$ has parameters θ and κ, for the control of skewness and kurtosis respectively. The VG process has the density function given by

$$f_{VG}(x) = \int_0^\infty \frac{1}{\sqrt{2\pi\widetilde{\sigma}^2 y}} \exp\left(- \frac{(x-\theta y)^2}{2\widetilde{\sigma}^2 y}\right) \frac{y^{(\frac{t}{\kappa}-1)}\kappa^{(-\frac{t}{\kappa})} e^{-\frac{y}{\kappa}}}{\Gamma(\frac{t}{\kappa})} dy$$

and the characteristic function of $X_t(\widetilde{\sigma}, \theta, \kappa)$ is

$$\phi_t(u) = (1 - i\theta\kappa u + \frac{1}{2}\widetilde{\sigma}^2 \kappa u^2)^{-t/\kappa}, t \ge 0.$$

The Lévy measure for this VG process [12] is

$$\nu(x)dx = \frac{\exp(\frac{\theta x}{\widetilde{\sigma}^2})}{\kappa \mid x \mid} \exp\left(- \widetilde{\sigma}^{-1}\sqrt{2(\frac{1}{\kappa}) + \frac{\theta^2}{\widetilde{\sigma}^2}} \mid x \mid \right) dx.$$

3. Let Z denote the standard Gaussian random variable and $G(t)$ denote a gamma process. A VG process that is derived from an arithmetic Brownian motion, with drift θ and volatility $\widetilde{\sigma}$, time-changed by a gamma process is given by ([12], [3])

$$X_t = \theta G(t) + \widetilde{\sigma} W(G(t)) = \theta G(t) + \widetilde{\sigma}\sqrt{G(t)} Z.$$

4. The next subsection highlights some aspects of the Malliavin calculus that we employ in our discussion of sensitivity analysis in the sequel.

2.3 Malliavin calculus for Lévy processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $X_i, i = 1, ..., n$ be a sequence of random variables with probability density functions which are piecewise differentiable. For $p \geq 1$ and $n \geq 1$, $C^p(\mathbb{R}^n)$ is the space of functions that are p times continuously differentiable on $\mathbb{R}^n$. The following basic definitions will be employed in the sequel.

Definition 2.5. Let $L^0(\Omega, \mathbb{R})$ be the linear space of all $\mathbb{R}$-valued random variables on $(\Omega, \mathcal{B}, \mathbb{P})$. A map $F : (L^0(\Omega, \mathbb{R}))^n \to L^0(\Omega, \mathbb{R}),\ n \in \mathbb{N}$, is called an (n, p)-*simple functional* of the n random variables if there exists an $\mathbb{R}$-valued function $\widetilde{F} \in C^p(\mathbb{R}^n)$ such that

$$F(X_1, ..., X_n)(\omega) = \widetilde{F}(X_1(\omega), ..., X_n(\omega)),\ \omega \in \Omega,\ X_1, ..., X_n \in L^0(\Omega, \mathbb{R}).$$

We write $S_{(n,p)}$ for the space of all (n, p)-simple functionals. An (n, p)-*simple process* of length n is a sequence of random variables $U = (U_i)_{i \leq n}$ such that $U_i(\omega) = u_i(X_1(\omega), ..., X_n(\omega))$ where $u_i \in C^p(\mathbb{R}^n),\ X_1, ..., X_n \in L^0(\Omega, \mathbb{R})$ and $\omega \in \Omega$.

We write $P_{(n,p)}$ for the space of all (n, p)-simple processes.

Definition 2.6. Let $F \in S_{(n,1)}$, with $F(X_1, ..., X_n)(\omega) = \widetilde{F}(X_1(\omega), ..., X_n(\omega))$, $\omega \in \Omega$, $\widetilde{F} \in C^1(\mathbb{R}^n)$, and $X_1, ..., X_n \in L^0(\Omega, \mathbb{R})$. Define the operator $D : S_{(n,1)} \to (P_{(n,0)})^n$, called the *Malliavin derivative operator*, by $DF = (D_iF)_{i \leq n}$ where

$$D_iF(X_1, ..., X_n)(\omega) = (\partial_i\widetilde{F})(X_1(\omega), ..., X_n(\omega)) = \left(\frac{\partial \widetilde{F}}{\partial x_i}\right)(X_1(\omega), ..., X_n(\omega)),\ \omega \in \Omega.$$

$$\text{For } n = 1,\ DF(X)(\omega) = \frac{\partial \widetilde{F}}{\partial x}(X(\omega)). \tag{2.1}$$

Definition 2.7. Let $F = (F_1, ..., F_d)$ be a d-dimensional vector of simple functionals such that $F_i \in S_{(n,1)}$. The matrix $\mathcal{M} = (\mathcal{M}(F)_{i,j})$ defined by

$$\mathcal{M}(F)_{i,j} = \langle DF_i, DF_j \rangle_n = \sum_{m=1}^{n} D_mF_iD_mF_j$$

is called the *Malliavin covariance matrix* of F [16].

$$\mathcal{M}(F)_{i,j} = DF_iDF_j,\ \text{if } n = 1. \tag{2.2}$$

Definition 2.8. The *Skorohod integral operator* $\delta : P_{(n,1)} \to S_{(n,0)}$ is defined for a simple process $U \in P_{(n,1)}$ by

$$\delta(U) = \sum_{i=1}^{n} \delta_i(U),$$

where

$$\delta_i(U)(X_1, ..., X_n) = -[D_iu_i(X_1, ..., X_n) + u_i(X_1, ..., X_n)\varphi_i(\mathbf{x})],\ U = (U_i)_{i=1,...,n},$$

$U_i(\omega) = u_i(X_1(\omega), ..., X_n(\omega)), \omega \in \Omega, \mathbf{x} = x_1, ..., x_n$;

$$\varphi_i(\mathbf{x}) = \frac{\partial \ln f_X(\mathbf{x})}{\partial x_i} = \left\{ \frac{f'_{X_i}(\mathbf{x})}{f_X(\mathbf{x})}, \quad \text{where } f_X(\mathbf{x}) \neq 0,\ 1 \leq i \leq n; \right.$$

and $f_X(x)$ denotes the density function of the random variable X.

Definition 2.9. The *Ornstein-Uhlenbeck (O-U) operator* $L : S_{(n,2)} \to S_{(n,0)}$ is defined as

$$(LF)(X_1, ..., X_n) = -\sum_{i=1}^{n}[(D^2_{ii}\widetilde{F})(X_1, ..., X_n) + \varphi_i(\mathbf{x})D_i\widetilde{F}(X_1, ..., X_n)],$$

where $X_1, ..., X_n \in L^0(\Omega, \mathbb{R})$, $F \in S_{(n,2)}$ and $\mathbf{x} = x_1, ..., x_n$.
This implies that

$$(LF)(X) = -[DD\widetilde{F}(X) + \varphi(x)D\widetilde{F}(X)],\ \text{if } n = 1 \tag{2.3}$$

where

$$\varphi(x) = \frac{\partial \ln f_X(x)}{\partial x} = \frac{f'_X(x)}{f_X(x)},\ f_X(x) \neq 0. \tag{2.4}$$

Malliavin Integration by Parts theorem

In the sensitivity analysis of interest rate derivatives, the following integration by parts theorem ([16], [3]), of Malliavin calculus will be employed.

Proposition 2.10. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $X_1, ..., X_n$ a sequence of real-valued random variables on $(\Omega, \mathcal{F}, \mathbb{P})$. Let $P = (P_1, ..., P_d) \in (S_{(n,2)})^d$, $Q \in S_{(n,1)}$, and $\mathcal{M} = (\mathcal{M}(P)_{ij})_{1\le i\le n, 1\le j\le n}$ be an invertible Malliavin covariance matrix with inverse denoted by $\mathcal{M}(P)^{-1} = (\mathcal{M}(P)_{ij})^{-1}_{1\le i\le n, 1\le j\le n}$. Assume that $\mathbb{E}[\det \mathcal{M}(P)^{-1}]^p < \infty$, $p \ge 1$, and $\Phi : \mathbb{R}^d \to \mathbb{R}$ is a smooth bounded function with bounded derivative. Then,*

$$\mathbb{E}[\partial_i \Phi(P) Q] = \mathbb{E}[\Phi(P) H_i(P,Q)] \qquad \text{where } \mathbb{E}[H_i(P,Q)] < \infty,$$

and

$$H_i(P,Q) = \sum_{j=1}^{n} Q\mathcal{M}(P)^{-1}_{ij} LP_j - \mathcal{M}(P)^{-1}_{ij}\langle DP_j, DQ\rangle - Q\langle DP_j, D\mathcal{M}(P)^{-1}_{ij}\rangle,\ i = 1,2,...,n$$

is called the Malliavin weight.

Remark 2.11. If $d = n = 1$, then the Malliavin weight is given by

$$H(P,Q) = Q\mathcal{M}(P)^{-1}LP - \mathcal{M}(P)^{-1}\langle DP, DQ\rangle - Q\langle DP, D\mathcal{M}(P)^{-1}\rangle. \tag{2.5}$$

Definition 2.12. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let X be a càdlàg process such that (i) X_t is $\mathcal{F}_t$-measurable (i.e. adapted to the filtration) (ii) $\mathbb{E}[\mid X_t \mid] < \infty$ for any $t \ge 0$. Then, X is called a *martingale* if for all $s < t$, $E[X_t \mid \mathcal{F}_s] = X_s$ where E is conditional expectation. Moreover, X is called a *semi-martingale* if for any $t \ge 0$, $X_t = X_0 + M_t + N_t$ where M_t and N_t denote a local martingale and an adapted process of finite variation, respectively [1].

Definition 2.13. Let Y be a semi-martingale. The quadratic variation of Y is the non-anticipating càdlàg process given by $[Y,Y]_t = \mid Y_t \mid^2 -2\int_0^t Y_{s_-} dY_s$.

In the next subsection, the Itô formula stated below will be employed.

Theorem 2.14. ***Itô formula for semi-martingale** [4]. Let $Y = (Y_t)_{0\le t\le T}$ be a semi-martingale. If $f : [0,T] \times \mathbb{R} \to \mathbb{R}$ is a $C^{1,2}$ function, then*

$$\begin{aligned} f(t,Y_t) = f(0,Y_0) &+ \int_0^t \frac{\partial f}{\partial s}(s,Y_s)ds + \int_0^t \frac{\partial f}{\partial y}(s,Y_{s_-})dY(s) \\ &+ \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial y^2}(s,Y_s)d[Y,Y]^c_s + \sum_{0\le s\le t, \Delta Y_s \ne 0} [f(s,Y_s) - f(s,Y_{s_-}) - \Delta Y_s \frac{\partial f}{\partial y}(s,Y_{s_-})] \end{aligned}$$

where $[Y,Y]^c_s$ denotes the continuous part of the quadratic variation of Y.

Remark 2.15. In the next section, we derive expression for an extended Vasicek model driven by a VG process. The result is used to derive an expression for an interest rate derivative called *zero-coupon bond* driven by the VG process. Then, we apply Malliavin calculus in the sensitivity analysis of the interest rate derivative.

3 The short rate model under the VG process

Let $P(t,T)$ be the value at time t of one currency unit received for sure at time T by an investor. This is called a *zero-coupon bond price* maturing with value 1 at time T. It can also be seen as a *discounting factor* for cash-flows occurring at time T.

We consider a zero-coupon bond whose underlying is a modified Vasicek (1977) interest rate model. The Vasicek model is a mean-reversion model given by the following stochastic differential equation for the interest rate r,

$$dr_t = a(b - r_t)dt + \sigma dX_t \tag{3.1}$$

where $X_t = X(t)$ is a Lévy process, b is the long-term mean rate, a is the speed of mean reversion and σ is the volatility of the interest rate.
Integrating equation (3.1), we obtain

$$r_t = r_0 e^{-at} + b(1 - e^{-at}) + \sigma \int_0^t e^{-a(t-s)} dX_s. \tag{3.2}$$

We adopt the VG model given by $X_t = \mathbf{w}t + \theta G_t + \widetilde{\sigma} W(G_t)$ [10] where $\mathbf{w}$ is the cumulant generating function given by $\mathbf{w} = -\frac{1}{t}\ln(\phi_t(-i)) = \frac{1}{\kappa}\ln\left(1 - \theta\kappa - \frac{1}{2}\widetilde{\sigma}^2\kappa\right)$ evaluated such that $\phi_t = e^{-\mathbf{w}t}$ [12] and $\phi_t(u) = (1 - i\theta\kappa u + \frac{1}{2}\widetilde{\sigma}^2\kappa u^2)^{-t/\kappa}$ is the characteristic function of the time-changed arithmetic Brownian motion $\theta G_t + \widetilde{\sigma} W(G_t)$. The parameter κ controls kurtosis which denotes the variance of the subordinator, and $G_t = G(t)$ denotes the gamma process. θ is the drift of the arithmetic Brownian motion which controls skewness and $\widetilde{\sigma}$ is the volatility of the arithmetic Brownian motion. We represent the standard Brownian motion as the process W such that $W(t) - W(s) = \sqrt{|\, t - s\,|}Z,\ r, s \geq 0$ where Z is a $N(0,1)$ Gaussian random variable. Then, $W(t) = \sqrt{t}Z$ and $\mathbb{E}(W(t)W(s)) = \min(t,s),\ t, s \geq 0$. Hence, we have

$$dX_t = \mathbf{w}dt + \theta\Delta G(t) + \widetilde{\sigma}\Delta(\sqrt{G(t)})Z$$

where $\Delta G(t) = G(t) - G(t_-)$ and $\Delta(\sqrt{G(t)}) = \sqrt{G(t)} - \sqrt{G(t_-)}$.
Then,

$$\int_0^t e^{-a(t-s)} dX_s = \frac{\mathbf{w}}{a}(1 - e^{-at}) + \theta \sum_{0 \leq s \leq t} \Delta G(s) e^{-a(t-s)} + \widetilde{\sigma} \sum_{0 \leq s \leq t} \Delta\sqrt{G(s)} e^{-a(t-s)} Z. \tag{3.3}$$

Substituting equation (3.3) into equation (3.2), we have

$$r_t = r_0 e^{-at} + b(1 - e^{-at}) + \sigma\left(\frac{\mathbf{w}}{a}(1 - e^{-at}) + \theta \sum_{0 \leq s \leq t} \Delta G(s) e^{-a(t-s)} + \widetilde{\sigma} \sum_{0 \leq s \leq t} \Delta\sqrt{G(s)} e^{-a(t-s)} Z\right). \tag{3.4}$$

Using the above information, we proceed to obtain the expression for the zero-coupon bond.

3.1 Expression for a zero-coupon bond with a Vasicek short rate model

Let $P = P(t,T)$ be the price of a zero-coupon bond. In a risk neutral world,

$$dP(t,T) = r_t P dt + \sigma P dX_t. \tag{3.5}$$

Applying Itô's formula, we have

$$d\ln P = (r_t dt + \sigma dX_t) - \frac{1}{2}\sigma^2 (dX_t)^2.$$

With

$$X_t = \mathbf{w}t + \theta G(t) + \widetilde{\sigma}\sqrt{G(t)}Z \Rightarrow dX_t = \mathbf{w}dt + \theta\Delta G(t) + \widetilde{\sigma}\Delta\sqrt{G(t)}Z,$$

we have

$$(dX)^2 = (\theta\Delta G(t) + \widetilde{\sigma}\Delta\sqrt{G(t)}Z)^2.$$

Hence,

$$d\ln P = r_t dt + \sigma(\mathbf{w}dt + \theta\Delta G(t) + \widetilde{\sigma}\Delta\sqrt{G(t)}Z) - \frac{1}{2}\sigma^2(\theta\Delta G(t) + \widetilde{\sigma}\Delta\sqrt{G(t)}Z)^2. \tag{3.6}$$

Integrating equation (3.6), noting that $P(T,T) = 1$, we have

$$\begin{aligned}
\ln P(t,T) = -\Bigg(&\int_t^T r_u du + \int_t^T \sigma\mathbf{w} du + \sum_{0 \leq u \leq T} \sigma(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z) \\
&- \sum_{0 \leq u \leq t} \sigma(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z) - \frac{1}{2}\sigma^2\Bigg(\sum_{0 \leq u \leq T} \sigma^2(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)^2 \\
&- \sum_{0 \leq u \leq t} (\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)^2\Bigg)\Bigg),
\end{aligned}$$

whence

$$P(t,T) = \exp\Big(-\Big(\int_t^T r_u du + \mathbf{w}\sigma[T-t] + \sigma\sum_{0\le u\le T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z) - \sum_{0\le u\le t}\sigma(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z) - \frac{\sigma^2}{2}\Big(\sum_{0\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)^2 - \sum_{0\le u\le t}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)^2\Big)\Big)\Big). \tag{3.7}$$

Since

$$r_t = r_0e^{-at} + b(1-e^{-at}) + \sigma\Big(\frac{\mathbf{w}}{a}(1-e^{-at}) + \theta\sum_{0\le s\le t}\Delta G(s)e^{-a(t-s)} + \widetilde{\sigma}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(t-s)}Z\Big),$$

we have

$$\int_t^T r_u du = -\frac{r_0}{a}\big(e^{-aT}-e^{-at}\big) + b\big(T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big) + \frac{\sigma\mathbf{w}}{a}\Big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\Big] + \sigma\Big(\theta\sum_{0\le u\le T}\sum_{0\le s\le t}\Delta G(s)e^{-a(u-s)} + \widetilde{\sigma}\sum_{0\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}Z\Big) - \sigma\Big(\theta\sum_{0\le u\le t}\sum_{0\le s\le t}\Delta G(s)e^{-a(u-s)} + \widetilde{\sigma}\sum_{0\le u\le t}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}Z\Big).$$

Hence, in a Lévy market driven by a VG process, the value of a zero-coupon bond is

$$P(t,T) = \exp\Big(-\Big(\Big[-\frac{r_0}{a}\big(e^{-aT}-e^{-at}\big) + b\big(T-t+\frac{1}{a}(e^{-aT}-e^{-at})\big) + \frac{\sigma\mathbf{w}}{a}\Big[T-t+\frac{1}{a}(e^{-aT}-e^{-at})\Big] + \sigma\sum_{0\le u\le T}\sum_{0\le s\le t}\big(\theta\Delta G(s)e^{-a(u-s)} + \widetilde{\sigma}\Delta\sqrt{G(s)}e^{-a(u-s)}Z\big) - \sigma\sum_{0\le u\le t}\sum_{0\le s\le t}\big(\theta\Delta G(s)e^{-a(u-s)} + \widetilde{\sigma}\Delta\sqrt{G(s)}e^{-a(u-s)}Z\big)\Big] + \mathbf{w}\sigma[T-t] + \sigma\sum_{0\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z) - \sigma\sum_{0\le u\le t}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z) - \frac{\sigma^2}{2}\Big(\sum_{0\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)^2 - \sum_{0\le u\le t}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)^2\Big)\Big)\Big). \tag{3.8}$$

Besides being a function of t and T, the expression on the right hand side of (3.8) also depends on r_0, σ, $\widetilde{\sigma}$, $\mathbf{w}$ and Z. Consequently, in the sequel, we shall regard P as a function of t, T, r_0, σ, $\widetilde{\sigma}$, $\mathbf{w}$ and Z.

Remark 3.1. We shall use the following information in the sequel.

$$\sum_{t\le u\le T} f(u)\Delta(u) = \sum_{0\le u\le T} f(u)\Delta(u) - \sum_{0\le u\le t} f(u)\Delta(u)$$

implies that when $t = T$, we have

$$\sum_{t\le u\le T} f(u)\Delta(u) = 0.$$

By Remark 3.1, the expression given by equation (3.8) becomes

$$
\begin{aligned}
P(t,T) = \exp\Bigg(- \Bigg(\Bigg[&- \frac{r_0}{a}\big(e^{-aT} - e^{-at}\big) + b\big(T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big) \\
&+ \frac{\sigma \mathbf{w}}{a}\Big[T - t + \frac{1}{a}(e^{-aT} - e^{-at})\Big] \\
&+ \sigma \sum_{t \le u \le T} \sum_{0 \le s \le t} \big(\theta \Delta G(s) e^{-a(u-s)} + \widetilde{\sigma}\Delta\sqrt{G(s)} e^{-a(u-s)} Z\big)\Bigg] + \mathbf{w}\sigma[T - t] \\
&+ \sigma \sum_{t \le u \le T} (\theta \Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)} Z) - \frac{\sigma^2}{2}\Big(\sum_{t \le u \le T} (\theta \Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)} Z)^2 \Big)\Bigg)\Bigg)
\end{aligned}
\tag{3.9}
$$

where $\mathbf{w} = \frac{1}{\kappa}\ln(1 - \theta\kappa - \frac{1}{2}\widetilde{\sigma}^2\kappa)$.

Remark 3.2. The above expression for the price of a zero-coupon bond will be used in the subsequent subsections.

Let P be the price of a zero-coupon bond. Suppose Q is of the form $Q = \frac{\partial P}{\partial \eta}$ for some parameter η of the zero-coupon bond. By Proposition 2.2, equations (2.1) and (2.2) with $i = j = 1$, we write $\mathcal{M}(P) = \langle DP, DP \rangle = DP \cdot DP$ for the Malliavin covariance matrix, with inverse $\mathcal{M}(P)^{-1} = \frac{1}{\mathcal{M}(P)}$, where DP is given by equation (2.1). For a smooth function $\Phi : \mathbb{R} \to \mathbb{R}$, the following equation holds:

$$E[\partial\Phi(P)Q] = \mathbb{E}[\Phi(P)H(P,Q)]$$

where $H(P,Q)$ is the Malliavin weight given by

$$H(P,Q) = Q\mathcal{M}(P)^{-1}LP - \mathcal{M}(P)^{-1}\langle DP, DQ\rangle - Q\langle DP, D\mathcal{M}(P)^{-1}\rangle.$$

Let P be the price of a zero-coupon bond given by equation (3.9). Then, from Definition 2.6 with equations (2.3) and (2.4), the Ornstein-Uhlenbeck (O-U) operator on P is given by

$$LP(t,T,Z) = -[DDP(t,T,Z) + \varphi(z)DP(t,T,Z)] \tag{3.10}$$

where DP denotes the Malliavin derivative of $P(t,T,Z)$; $Z \sim \mathcal{N}(0,1)$; $\varphi(z) = \frac{\partial \ln f_{\mathcal{N}}(z)}{\partial z} = -z$, $f_{\mathcal{N}}(z) \neq 0$; and $f_{\mathcal{N}}(z)$ is the density function of the Gaussian random variable Z.

We state Lemmas 3.1-3.4 which are necessary for Theorems 4.7 and 4.11.

Lemma 3.3. *Let P be the price of a zero-coupon bond under the VG process. Then, the Malliavin derivative of P is given by*

$$
\begin{aligned}
DP = -\Bigg[&\sigma\widetilde{\sigma}\Big(\sum_{t \le u \le T} \sum_{0 \le s \le t} \Delta\sqrt{G(s)} e^{-a(u-s)} + \sum_{t \le u \le T} (\Delta\sqrt{G(u)}) \Big) \\
&- \sigma^2\widetilde{\sigma}\Big(\sum_{t \le u \le T} (\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)} Z)\Delta\sqrt{G(u)} \Big)\Bigg] P.
\end{aligned}
$$

Proof. Let $P = P(t,T)$ be as given in equation (3.9). Then, from Definition 2.3 and equation (2.1), we obtain the Malliavin derivative

$$
\begin{aligned}
DP = -\Bigg[&\sigma \sum_{t \le u \le T} \sum_{0 \le s \le t} \widetilde{\sigma}\Delta\sqrt{G(s)} e^{-a(u-s)} + \sigma \sum_{t \le u \le T} (\widetilde{\sigma}\Delta\sqrt{G(u)}) \\
&- \frac{\sigma^2\widetilde{\sigma}}{2}\big(2 \sum_{t \le u \le T} (\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)} Z)\Delta\sqrt{G(u)}\big)\Bigg] P.
\end{aligned}
$$

Thus, the result follows. □

Remark 3.4. Lemma 3.3 shows that $DP \neq 0$ for all t. Thus, its inverse exists for all t.

Lemma 3.5. *Let P be the price of a zero-coupon bond under the VG process and L be the Ornstein-Uhlenbeck operator as in equation (3.10). Then*

$$\begin{aligned} LP = -\Bigg[&\sigma^2\widetilde{\sigma}^2 \sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2 + \Bigg(\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)} \\ &+ \sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)}) - \sigma^2\widetilde{\sigma}(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg)^2 \\ &- \varphi(z)\Bigg(\sigma\widetilde{\sigma}(\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)} + \sum_{t\le u\le T}\Delta\sqrt{G(u)}) \\ &- \sigma^2\widetilde{\sigma}\Bigg(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg)\Bigg)\Bigg]P, \quad \textit{where } \varphi(z) = -z. \end{aligned} \tag{3.11}$$

Proof. Taking the Malliavin derivative of the price of the zero-coupon bond P, it follows from Lemma 3.3 that:

$$\begin{aligned} D(DP) = \sigma^2\Bigg(\sum_{t\le u\le T}(\widetilde{\sigma}\Delta\sqrt{G(u)})^2\Bigg)P + \Bigg(\Bigg[&\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)} \\ &+ \sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)}) - \sigma^2\widetilde{\sigma}(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg]^2\Bigg)P. \end{aligned}$$

From Definition 2.6 and equation (2.4), we have

$$\varphi(z) = \frac{\partial \ln f_{\mathcal{N}}(z)}{\partial z} = \frac{\partial}{\partial z}\ln\left(\frac{1}{\sqrt{2}}e^{-\frac{1}{2}z^2}\right) = -z,$$

where $f_{\mathcal{N}}(z)$ denotes the density function of the random variable Z.
Thus,

$$LP(t,T) = -[DDP(t,T) + \varphi(z)DP(t,T)].$$

Whence, with $\varphi(z) = -z$, we get

$$\begin{aligned} LP = -\Bigg[\Bigg(&\sigma^2\sum_{t\le u\le T}(\widetilde{\sigma}\Delta\sqrt{G(u)})^2 + \Bigg[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)} \\ &+ \sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)}) - \sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg]^2\Bigg)P \\ &+ \varphi(z)\Bigg(-\Bigg[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)}) \\ &- \sigma^2\widetilde{\sigma}\Bigg(\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg)\Bigg]\Bigg)P\Bigg]. \end{aligned}$$

□

Lemma 3.6. *Let P be the price of a zero-coupon bond under the VG process. Then, its Malliavin covariance matrix is given by*

$$\begin{aligned} \mathcal{M}(P) = \Bigg[&\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)}) \\ &- \sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg]^2 P^2. \end{aligned}$$

Also,

$$\mathcal{M}(P)^{-1} = \bigg(\bigg[\sigma\widetilde{\sigma}\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma}\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)}) - \sigma^2\widetilde{\sigma}\sum_{t\leq u\leq T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\bigg]P\bigg)^{-2}$$

for all t *since by Remark 3.4,* $DP(t,T) \neq 0$ *for any* t*.*

Proof. By Lemma 3.3,

$$DP = -\bigg[\sigma\widetilde{\sigma}\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma}\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)}) - \sigma^2\widetilde{\sigma}\sum_{t\leq u\leq T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\bigg]P.$$

Then, the Malliavin covariance matrix $\mathcal{M}(P)$ is given by

$$\mathcal{M}(P) = \langle DP, DP\rangle = \bigg(-\bigg[\sigma\widetilde{\sigma}\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma}\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)}) - \sigma^2\widetilde{\sigma}\sum_{t\leq u\leq T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\bigg]\bigg)^2 P^2.$$

The expression for $\mathcal{M}(P)^{-1}$ follows. □

Lemma 3.7. *Let* $P = P(t,T)$ *be the price of a zero-coupon bond under the VG process. If* $\mathcal{M}(P)^{-1}$ *is the inverse Malliavin covariance matrix of* $P(t,T)$*, then,*

$$D\mathcal{M}(P)^{-1} = 2\bigg[\bigg(\sigma\widetilde{\sigma}\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma}\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)}) - \sigma^2\widetilde{\sigma}\sum_{t\leq u\leq T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\bigg)^{-3}\bigg]P^{-2}$$
$$\times\bigg[\sigma^2\widetilde{\sigma}^2\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)})^2 + \bigg[\sigma\widetilde{\sigma}\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma}\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)}) - \sigma^2\widetilde{\sigma}\sum_{t\leq u\leq T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\bigg]^2\bigg].$$

Proof. From Lemma 3.6,

$$\mathcal{M}(P)^{-1} = \bigg(-\bigg[\sigma\widetilde{\sigma}\sum_{t\leq u\leq T}\sum_{0\leq s\leq t}\Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma}\sum_{t\leq u\leq T}(\Delta\sqrt{G(u)}) -\sigma^2\widetilde{\sigma}\sum_{t\leq u\leq T}(\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\bigg]P\bigg)^{-2}.$$

Hence,

$$\begin{aligned}D\mathcal{M}(P)^{-1} = -2\Bigg[\Bigg(&\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)})\\&-\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg)P\Bigg]^{-3}\\&\times\Bigg[-\sigma^2\sum_{t\le u\le T}(\widetilde{\sigma}\Delta\sqrt{G(u)})^2P - \Bigg[\sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\Delta\sqrt{G(s)}e^{-a(u-s)}\\&+\sigma\widetilde{\sigma}\sum_{t\le u\le T}(\Delta\sqrt{G(u)}) - \sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta\Delta G(u)+\widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Bigg]^2P\Bigg]\end{aligned}$$

which gives the result. □

4 Computation of some greeks

Greeks are used to describe different aspects of risks embroiled in taking a bond options' position. Investors use values of greeks such as delta, gamma, vega and Theta to predict bond options' risks in order to manage them. All the greeks are important but for now, we shall concentrate on the computation of the 'delta' that measures the sensitivity of the bond option price to changes in the initial interest rate and 'gamma' that measures the bond option price sensitivity to changes in delta.

Let $\Phi(P) = \max(P(0,T,r_0) - K, 0)$ be the payoff of a call option on a zero-coupon bond price P with strike price K. Then, the price of the call option at time T is given by

$$\mathbb{V} = e^{-r_0T}\mathbb{E}[\Phi(P)].$$

The greeks are computed using

$$\frac{\partial\mathbb{V}}{\partial\eta} = \frac{\partial}{\partial\eta}\big(e^{-r_0T}\mathbb{E}[\Phi(P)]\big)$$

where η represents any of the parameters of the bond price whose sensitivity with respect to changes in η is to be determined.

4.1 Computation of Delta for VG-driven Interest Rate Derivatives

The greek 'delta' measures the sensitivity of the zero-coupon bond option price to changes in the initial interest rate. Bond option investors study delta since movements in the initial interest rate can alter the worth of their investments [5]. Hence, delta assists players in the interest rate derivative market to predict what happens if the initial interest rate of the bond option price changes. This helps in decision making and risk management.

Let $\Phi(P)$ be the payoff of the zero-coupon bond price P given by equation (3.9). Then

$$\begin{aligned}\triangle_{VG} := \frac{\partial}{\partial r_0}[e^{-r_0T}\mathbb{E}(\Phi(P))] &= -Te^{-r_0T}\mathbb{E}(\Phi(P)) + e^{-r_0T}\mathbb{E}\left[\Phi'(P)\frac{\partial P}{\partial r_0}\right]\\&= -Te^{-r_0T}\mathbb{E}(\Phi(P)) + e^{-r_0T}\mathbb{E}\left[\Phi(P)H\left(P, \frac{\partial P}{\partial r_0}\right)\right].\end{aligned}$$

Lemma 4.1. *Let P denote the price of the zero-coupon bond and $Q = \dfrac{\partial P}{\partial r_0}$. Then,*

$$Q = \frac{1}{a}(e^{-aT} - e^{-at})P$$

and

$$DQ = -\frac{1}{a}(e^{-aT} - e^{-at})\Big(\sigma\widetilde{\sigma} \sum_{t\le u\le T} \sum_{0\le s\le t} \Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma} \sum_{t\le u\le T} (\Delta\sqrt{G(u)}) - \sigma^2\widetilde{\sigma} \sum_{t\le u\le T} (\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}\Big) P.$$

Proof. From the definition of P and since

$$Q = \frac{\partial P}{\partial r_0} = \frac{1}{a}(e^{-aT} - e^{-at})P \text{ and } DQ = \frac{1}{a}(e^{-aT} - e^{-at})DP,$$

the result follows by Lemma 3.3. □

Remark 4.2. Using Lemmas 3.3-3.7 and Lemma 4.1, we establish Lemmas 4.3 and 4.4 in order to obtain each term of the Malliavin weight for $\triangle_{VG}$ in Theorem 4.7.

Lemma 4.3. *Given that P is the price of a zero-coupon bond under the VG process, then the following expression holds:*

$$Q\mathcal{M}(P)^{-1}LP = -\frac{\sigma^2}{a}(e^{-aT} - e^{-at})\big(\sum_{t\le u\le T} (\widetilde{\sigma}\Delta\sqrt{G(u)})^2\big)\cdot\mathcal{K}^{-2} - \frac{1}{a}(e^{-aT} - e^{-at}) + \frac{\frac{1}{a}(e^{-aT} - e^{-at})\varphi(z)}{\mathcal{K}} \tag{4.1}$$

where $\varphi(z) = -z$ and

$$\mathcal{K} = \sigma\widetilde{\sigma} \sum_{t\le u\le T} \sum_{0\le s\le t} \Delta\sqrt{G(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma} \sum_{t\le u\le T} (\Delta\sqrt{G(u)}) - \sigma^2\widetilde{\sigma} \sum_{t\le u\le T} (\theta\Delta G(u) + \widetilde{\sigma}\Delta\sqrt{G(u)}Z)\Delta\sqrt{G(u)}. \tag{4.2}$$

Proof. The result follows from Lemmas 3.5, 3.6 and 4.1 by substituting Q from Lemma 4.1, $\mathcal{M}(P)^{-1}$ from Lemma 3.6 and LP from equation (3.11) of Lemma 3.5 into $Q\mathcal{M}(P)^{-1}LP$. □

Lemma 4.4. *Let P be the price of a zero-coupon bond under the VG process and let $\mathcal{M}(P)^{-1}$ be the inverse Malliavin covariance matrix of P. Then, the following holds:*

(i) $\mathcal{M}(P)^{-1}\langle DP, DQ\rangle = \frac{1}{a}(e^{-aT} - e^{-at})$.

(ii) $Q\langle DP, D\mathcal{M}(P)^{-1}\rangle = -\frac{2}{a}(e^{-aT} - e^{-at}) - \frac{2\sigma^2}{a}(e^{-aT} - e^{-at})(\widetilde{\sigma}^2 \sum_{t\le u\le T} (\Delta\sqrt{G(u)})^2)\mathcal{K}^{-2}$

where $\mathcal{K}$ is given by equation (4.2).

Proof. (i) The result follows from Lemmas 3.3, 3.6 and 4.1 by substituting $\mathcal{M}(P)^{-1}$ from Lemma 3.6, DP from Lemma 3.3 and DQ from Lemma 4.1 into $\mathcal{M}(P)^{-1}\langle DP, DQ\rangle$.

(ii) The result follows from Lemmas 3.3, 3.7 and 4.1 by substituting Q from Lemma 4.1, DP from Lemma 3.3 and $D\mathcal{M}(P)^{-1}$ from Lemma 3.7 into $Q\langle DP, D\mathcal{M}(P)^{-1}\rangle$.

□

Lemma 4.5. *Let P be the price of a zero-coupon bond under the VG process. Then,*

$$\mathbb{E}[\Phi(P)] = \int_K^\infty \int_K^\infty (p(t, T, g, z) - K)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\left(\frac{\kappa^{-\frac{t}{\kappa}}}{\Gamma(\frac{t}{\kappa})}g^{\frac{t}{\kappa}-1}e^{-\frac{1}{\kappa}g}\right)dzdg$$

where the strike price K is a constant and by equation (3.9), we have

$$\begin{aligned} p(t,T) = p(t,T,z,g) = \exp\Bigg(- \Bigg(\Bigg[- \frac{r_0}{a}\big(e^{-aT} - e^{-at}\big) + b\big(T - t + \frac{1}{a}(e^{-aT} - e^{-at})\big) \\ + \frac{\sigma \mathbf{w}}{a}\Big[T - t + \frac{1}{a}(e^{-aT} - e^{-at})\Big] \\ + \sigma \sum_{t\le u\le T}\sum_{0\le s\le t}\big(\theta g(s)e^{-a(u-s)} + \widetilde{\sigma}\sqrt{g(s)}e^{-a(u-s)}z\big)\Bigg] + \mathbf{w}\sigma[T-t] \\ + \sigma \sum_{t\le u\le T}\big(\theta g(u) + \widetilde{\sigma}\sqrt{g(u)}z\big) - \frac{\sigma^2}{2}\Big(\sum_{t\le u\le T}\big(\theta g(u) + \widetilde{\sigma}\sqrt{g(u)}z\big)^2\Big)\Bigg)\Bigg). \end{aligned} \tag{4.3}$$

Proof. Suppose that the gamma process is independent. Let $f_{\mathcal{N}}(z;0,1)$ and $f_G(g;\frac{t}{\kappa})$ denote the probability density functions for a Gaussian random variable and a gamma random variable, respectively. Then,

$$\begin{aligned} \mathbb{E}[\Phi(P)] &= \int_{\mathbb{R}}\int_{\mathbb{R}} \Phi(p(t,T,g,z)f_{\mathcal{N}}(z;0,1)f_G(g;t/\kappa,1/\kappa)dzdg \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} \max(p(t,T,g,z)-K,0)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\left(\frac{\kappa^{-\frac{t}{\kappa}}}{\Gamma(\frac{t}{\kappa})}g^{\frac{t}{\kappa}-1}e^{-\frac{1}{\kappa}g}\right)dzdg \\ &= \int_K^\infty\int_K^\infty (p(t,T,g,z)-K)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\left(\frac{\kappa^{-\frac{t}{\kappa}}}{\Gamma(\frac{t}{\kappa})}g^{\frac{t}{\kappa}-1}e^{-\frac{1}{\kappa}g}\right)dzdg. \end{aligned}$$

□

Lemma 4.6. *Let P be the price of a zero-coupon bond driven by the VG process and $H(P,Q) = H\left(P, \frac{\partial P}{\partial r_0}\right)$, the Malliavin weight for the greek 'delta'. Then, the following expression holds:*

$$H\left(p, \frac{\partial p}{\partial r_0}\right) = \frac{\sigma^2}{a}(e^{-aT} - e^{-at})\big(\widetilde{\sigma}^2 \sum_{t\le u\le T}(\sqrt{g(u)})^2\big)\mathcal{K}^{*^{-2}} + \frac{\frac{1}{a}(e^{-aT}-e^{-at})\varphi(z)}{\mathcal{K}^*} \tag{4.4}$$

where

$$\begin{aligned} \mathcal{K}^* = \sigma\widetilde{\sigma}\sum_{t\le u\le T}\sum_{0\le s\le t}\sqrt{g(s)}e^{-a(u-s)} + \sigma\widetilde{\sigma}\sum_{t\le u\le T}(\sqrt{g(u)}) \\ -\sigma^2\widetilde{\sigma}\sum_{t\le u\le T}(\theta g(u) + \widetilde{\sigma}\sqrt{g(u)}z)\sqrt{g(u)}. \end{aligned} \tag{4.5}$$

Proof. The Malliavin weight $H(P,Q)$ is given by

$$H(P,Q) = Q\mathcal{M}(P)^{-1}LP - \mathcal{M}(P)^{-1}\langle DP, DQ\rangle - Q\langle DP, D\mathcal{M}(P)^{-1}\rangle.$$

The expression for $H\left(p, \frac{\partial p}{\partial r_0}\right)$ in equation (4.4) is obtained from Lemmas 4.3 and 4.4. By substituting $Q\mathcal{M}(P)^{-1}LP$ from equation (4.1) of Lemma 4.3, and $\mathcal{M}(P)^{-1}\langle DP, DQ\rangle$ and $\mathcal{M}(P)^{-1}\langle DP, DQ\rangle$ from Lemma 4.4, we obtain the expression

$$H(P,Q) = \frac{\sigma^2}{a}(e^{-aT}-e^{-at})\big(\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big)\mathcal{K}^{-2} + \frac{\frac{1}{a}(e^{-aT}-e^{-at})\varphi(z)}{\mathcal{K}}, \ \varphi(z) = -z \tag{4.6}$$

where $\mathcal{K}$ is given by equation (4.2). Since $H(P,Q) = H\left(P, \frac{\partial P}{\partial r_0}\right)$, we obtain the expression in equation (4.4) where $\mathcal{K}^*$ follows from equation (4.2). Since $Q = \frac{\partial P}{\partial r_0}$, the result follows. □

Lemma 4.7. *Let P be the price of a zero-coupon bond driven by the VG process and $H(P,Q)$ be its Malliavin weight. Then,*

$$\mathbb{E}\left[\Phi(P)H\left(P,\frac{\partial P}{\partial r_0}\right)\right]=\int_K^\infty\int_K^\infty (p(t,T,g,z)-K)H\left(p,\frac{\partial p}{\partial r_0}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\left(\frac{\kappa^{-\frac{t}{\kappa}}}{\Gamma\left(\frac{t}{\kappa}\right)}g(u)^{\frac{t}{\kappa}-1}e^{-\frac{1}{\kappa}g(u)}\right)dzdg$$

where $p(t,T,g,z)$ is given by equation (4.3) and $H\left(p,\dfrac{\partial p}{\partial r_0}\right)$ is given by Lemma 4.6.

Proof. From the density functions of the Gaussian and gamma random variables,

$$\begin{aligned}\mathbb{E}[\Phi(P)H(P,Q)]&=\int_{\mathbb{R}}\int_{\mathbb{R}}\Phi(p)H\left(p,\frac{\partial p}{\partial r_0}\right)f_{\mathcal{N}}(z;0,1)f_G(g;t/\kappa,1/\kappa)dzdg\\&=\int_{\mathbb{R}}\int_{\mathbb{R}}\max(p(t,T)-K,0)H\left(p,\frac{\partial p}{\partial r_0}\right)f_{\mathcal{N}}(z;0,1)f_G(g;t/\kappa,1/\kappa)dzdg\\&=\int_{\mathbb{R}}\int_{\mathbb{R}}\max(p(t,T)-K,0)H\left(p,\frac{\partial p}{\partial r_0}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\left(\frac{\kappa^{-\frac{t}{\kappa}}}{\Gamma\left(\frac{t}{\kappa}\right)}g^{\frac{t}{\kappa}-1}e^{-\frac{1}{\kappa}g}\right)dzdg.\end{aligned}$$

Hence, the result follows. □

Theorem 4.8. *Let P be the price of a zero-coupon bond driven by the VG process. Assume that $\Phi(P)=\max(P-K,0)$ is the payoff with strike price K on the bond. Given that the price of the call option is $e^{-r_0T}\mathbb{E}[\Phi(P)]$, then, its sensitivity with respect to its initial interest rate denoted as $\triangle_{VG}$ is given by*

$$\begin{aligned}\triangle_{VG}=e^{-r_0T}\Bigg(&-T\left(\int_K^\infty\int_K^\infty (p(t,T,g,z)-K)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\left(\frac{\kappa^{-\frac{t}{\kappa}}}{\Gamma\left(\frac{t}{\kappa}\right)}g(u)^{\frac{t}{\kappa}-1}e^{-\frac{1}{\kappa}g(u)}\right)dzdg\right)\\&+\int_K^\infty\int_K^\infty (p(t,T,g,z)-K)H\left(p,\frac{\partial p}{\partial r_0}\right)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\left(\frac{\kappa^{-\frac{t}{\kappa}}}{\Gamma\left(\frac{t}{\kappa}\right)}g(u)^{\frac{t}{\kappa}-1}e^{-\frac{1}{\kappa}g(u)}\right)dzdg\Bigg)\end{aligned}$$

where $H\left(p,\dfrac{\partial p}{\partial r_0}\right)$ is given by equation (4.4).

Proof. The greek 'delta' is given by

$$\triangle_{VG}=e^{-r_0T}(-T\mathbb{E}(\Phi(P))+\mathbb{E}[\Phi(P)H(P,Q)]).$$

If we substitute $E[\Phi(P)]$ from Lemma 4.5 and $\mathbb{E}[\Phi(P)H(P,Q)]$ from Lemma 4.7 into the above expression, we have

$$\begin{aligned}\triangle_{VG}=e^{-r_0T}\Bigg(&-T\left(\int_{\mathbb{R}}\int_{\mathbb{R}}\max(p(t,T,g,z)-K,0)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\left(\frac{\kappa^{-\frac{t}{\kappa}}}{\Gamma\left(\frac{t}{\kappa}\right)}g(u)^{\frac{t}{\kappa}-1}e^{-\frac{1}{\kappa}g(u)}\right)dzdg\right)\\&+\int_{\mathbb{R}}\int_{\mathbb{R}}\max(p(t,T,g,z)-K,0)H(p,Q)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\left(\frac{\kappa^{-\frac{t}{\kappa}}}{\Gamma\left(\frac{t}{\kappa}\right)}g(u)^{\frac{t}{\kappa}-1}e^{-\frac{1}{\kappa}g(u)}\right)dzdg\Bigg).\end{aligned}$$

This leads to the result. □

4.2 Computation of Gamma under VG-driven Interest Rate Derivatives

The greek '*gamma* Γ_{VG}' measures the sensitivity of the zero-coupon bond option price to changes in the greek 'delta'. It helps investors and risk managers to determine the extent to which the risk in the bond option's position changes with respect to changes in delta. Thus, it is obtained from the second partial derivative with respect to the initial interest rate and is given by

$$\begin{aligned}\Gamma_{VG}&=\frac{\partial^2}{\partial r_0^2}(e^{-r_0T}\mathbb{E}[\Phi(P)])=\frac{\partial}{\partial r_0}(-Te^{-r_0T}\mathbb{E}(\Phi(P))+e^{-r_0T}\mathbb{E}[\Phi'(P)Q])\\&=T^2e^{-r_0T}\mathbb{E}[\Phi(P)]-2Te^{-r_0T}\mathbb{E}[\Phi(P)H(P,Q)]+e^{-r_0T}\mathbb{E}[\Phi(P)H(P,QH(P,Q))]\end{aligned}$$

where $Q = \dfrac{\partial P}{\partial r_0}$; $H(P,Q)$ is the Malliavin weight for delta given by equation (4.6) under Lemma 4.6 and $H(P, QH(P,Q)) = H(P, Q_{\Gamma_{VG}})$ is the Maliavin weight for gamma.
Using Lemmas 3.3-3.7 and Lemma 4.1, we establish Lemmas 4.8-4.10 which are required for Theorem 4.11.

Lemma 4.9. *Let P be the price of a zero-coupon bond under the VG process and define*

$$Q_{\Gamma_{VG}} = \frac{\partial P}{\partial r_0} H\left(P, \frac{\partial P}{\partial r_0}\right) = QH(P,Q) \tag{4.7}$$

where $H(P,Q)$ is the Malliavin weight obtained by Lemma 4.6.
Then, for $\mathcal{K}$ given by equation (4.2), we obtain

$$Q_{\Gamma_{VG}} = \frac{1}{a}(e^{-aT} - e^{-at})P\left[\frac{\sigma^2}{a}(e^{-aT} - e^{-at})\big(\widetilde{\sigma}^2 \sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big)\mathcal{K}^{-2} + \frac{1}{a}(e^{-aT} - e^{-at})\varphi(z)\mathcal{K}^{-1}\right] \tag{4.8}$$

$$\begin{aligned} DQ_{\Gamma_{VG}} = \Bigg[&-\big(\frac{1}{a}(e^{-aT} - e^{-at})\big)\mathcal{K}H(P,Q) - \big(\frac{1}{a}(e^{-aT} - e^{-at})\big)^2\mathcal{K}^{-1} + 2\big(\frac{1}{a}(e^{-aT} - e^{-at})\big)^2\mathcal{K}^{-3} \\ &\cdot \big[\sigma^2\widetilde{\sigma}^2 \sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big]^2 + \big(\frac{1}{a}(e^{-aT} - e^{-at})\big)^2\varphi(z)\mathcal{K}^{-2}\big[-\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big]\Bigg]P. \end{aligned} \tag{4.9}$$

Proof. From Lemma 4.1 and equation (4.7), it follows that

$$Q_{\Gamma_{VG}} = QH(P,Q) = \frac{1}{a}(e^{-aT} - e^{-at})PH(P,Q).$$

Substituting $H(P,Q)$ from equation (4.6), we obtain equation (4.8).
The Malliavin derivative of $Q_{\Gamma_{VG}}$ is given by

$$DQ_{\Gamma_{VG}} = \frac{1}{a}(e^{-aT} - e^{-at})[-\mathcal{K}H(P,Q) + DH(P,Q)]P. \tag{4.10}$$

By equation (4.6), we have

$$\begin{aligned} DH(P,Q) = &\frac{2}{a}(e^{-aT} - e^{-at})\mathcal{K}^{-3}\big(\sigma^2\widetilde{\sigma}^2 \sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big)^2 - \frac{1}{a}(e^{-aT} - e^{-at})\mathcal{K}^{-1} \\ &- \big(\frac{1}{a}(e^{-aT} - e^{-at})\big)\varphi(z)\mathcal{K}^{-2}\big(-\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big),\ \varphi(z) = -z. \end{aligned} \tag{4.11}$$

Substituting equation (4.11) into equation (4.10), we obtain equation (4.9). Hence, the result follows. □

Lemma 4.10. *Let P be the price of a zero-coupon bond, and $Q_{\Gamma_{VG}}$ and $\mathcal{K}$ be given by equations (4.7) and (4.2), respectively. We obtain*

$$Q_{\Gamma_{VG}}\mathcal{M}(P)^{-1}LP = -\frac{1}{a}(e^{-aT} - e^{-at})H(P,Q)\left[\big(\sigma^2\widetilde{\sigma}^2 \sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big)\mathcal{K}^{-2} + 1 - \varphi(z)\mathcal{K}^{-1}\right] \tag{4.12}$$

where $H(P,Q)$ is given by equation (4.6).

Proof. The expression for $Q_{\Gamma_{VG}}\mathcal{M}(P)^{-1}LP$ given by equation (4.12) is obtained from Lemmas 3.5, 3.6 and 4.9 by substituting $Q_{\Gamma_{VG}}$ from equation (4.8) of Lemma 4.9, $\mathcal{M}(P)^{-1}$ from Lemma 3.6 and LP from equation (3.11) of Lemma 3.5 into $Q_{\Gamma_{VG}}\mathcal{M}(P)^{-1}LP$. □

Lemma 4.11. *Let P be a zero-coupon bond price and $Q_{\Gamma_{VG}}$ be given by equation (4.7). Then, the following hold:*

(i) $$\mathcal{M}(P)^{-1}\langle DP, DQ_{\Gamma_{VG}}\rangle = \frac{1}{a}(e^{-aT}-e^{-at})H(P,Q) + \big(\frac{1}{a}(e^{-aT}-e^{-at})\big)^2\Big[\mathcal{K}^{-2} -2\big(\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big)^2\mathcal{K}^{-4} + \varphi(z)\mathcal{K}^{-3}\big(\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big)\Big]. \tag{4.13}$$

(ii) For $H(P,Q)$ given by Lemma (4.6),

$$Q_{\Gamma_{VG}}\langle DP, D\mathcal{M}(P)^{-1}\rangle = -2\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)H(P,Q)\Big[\mathcal{K}^{-2}\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2)+1\Big]. \tag{4.14}$$

Proof. (i) The expression given by equation (4.13) is obtained from Lemmas 3.3, 3.6 and 4.9 by substituting $\mathcal{M}(P)^{-1}$ from Lemma 3.6, DP from Lemma 3.3 and $DQ_{\Gamma_{VG}}$ from equation (4.9) of Lemma 4.9 into $\mathcal{M}(P)^{-1}\langle DP, DQ_{\Gamma_{VG}}\rangle$.

(ii) The expression given by equation (4.14) is obtained from Lemmas 3.3, 3.7 and 4.9 by substituting $Q_{\Gamma_{VG}}$ from Lemma 4.9, DP from Lemma 3.3 and $D\mathcal{M}(P)^{-1}$ from Lemma 3.7 into $Q_{\Gamma_{VG}}\langle DP, D\mathcal{M}(P)^{-1}\rangle$.

□

Theorem 4.12. *Let P be the price of a zero-coupon bond driven by the VG process, $\mathbb{E}[\Phi(P)]$ and $\mathbb{E}[\Phi(P)H(P,Q)]$ be given by Lemmas 4.5 and 4.7, respectively. Then, the greek "Γ_{VG}" is given by*

$$\Gamma_{VG} = T^2e^{-r_0T}\mathbb{E}[\Phi(P)] - 2Te^{-r_0T}\mathbb{E}[\Phi(P)H(P,Q)] + e^{-r_0T}\mathbb{E}[\Phi(P)H(P,Q_{\Gamma_{VG}})]$$

where $Q = \dfrac{\partial P}{\partial r_0}$ and $\mathbb{E}[\Phi(P)H(P,Q)]$ is given by Lemma 4.7 while $\mathbb{E}[\Phi(P)H(P,Q_{\Gamma_{VG}})] = \mathbb{E}[\Phi(P)H(P,QH(P,Q))]$ is given by

$$\mathbb{E}\Big[\Phi(P)H\Big(P,\frac{\partial P}{\partial r_0}H\Big(P,\frac{\partial P}{\partial r_0}\Big)\Big)\Big] = \int_K^\infty\int_K^\infty (p(t,T,g,z)-K)H\Big(p,\frac{\partial p}{\partial r_0}H\Big(p,\frac{\partial p}{\partial r_0}\Big)\Big) \cdot \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\Big(\frac{\kappa^{-\frac{t}{\kappa}}}{\Gamma(\frac{t}{\kappa})}g(u)^{\frac{t}{\kappa}-1}e^{-\frac{1}{\kappa}g(u)}\Big)dzdg$$

and $H\Big(p,\dfrac{\partial p}{\partial r_0}H\Big(p,\dfrac{\partial p}{\partial r_0}\Big)\Big)$

$$= \frac{1}{a}(e^{-aT}-e^{-at})H\Big(p,\frac{\partial p}{\partial r_0}\Big)\mathcal{K}^{*-2} - z\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)H\Big(p,\frac{\partial p}{\partial r_0}\Big)\mathcal{K}^{*-1}\big(\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\sqrt{g(u)})^2\big) + 2\big(\frac{1}{a}(e^{-aT}-e^{-at})\big)^2\mathcal{K}^{*-4}\big(\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\sqrt{g(u)})^2\big)^2 - \big(\frac{1}{a}(e^{-aT}-e^{-at})\big)^2\mathcal{K}^{*-2} + \big(\frac{1}{a}(e^{-aT}-e^{-at})\big)^2 z\mathcal{K}^{*-3}\big(-\sigma^2\widetilde{\sigma}^2\sum_{t\le u\le T}(\sqrt{g(u)})^2\big). \tag{4.15}$$

$\mathcal{K}^*$ is given by equation (4.5).

Proof. The expression given by equation (4.15) follows from Lemmas 4.10 and 4.11 by substituting the expression for $Q_{\Gamma_{VG}}\mathcal{M}(P)^{-1}LP$ from Lemma 4.10, and the expressions for $\mathcal{M}(P)^{-1}\langle DP, DQ_{\Gamma_{VG}}\rangle$ and $Q_{\Gamma_{VG}}\langle DP, D\mathcal{M}(P)^{-1}\rangle$ from Lemma 4.11 into

$$H(P,Q_{\Gamma_{VG}}) = Q_{\Gamma_{VG}}\mathcal{M}(P)^{-1}LP - \mathcal{M}(P)^{-1}\langle DP, DQ_{\Gamma_{VG}}\rangle - Q_{\Gamma_{VG}}\langle DP, D\mathcal{M}(P)^{-1}\rangle = H(P,QH(P,Q)).$$

Thus,

$$H(P,Q_{\Gamma_{VG}}) = \frac{1}{a}(e^{-aT} - e^{-at})H(P,Q)\mathcal{K}^{-2}\big(\sigma^2\widetilde{\sigma}^2 \sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big)$$

$$+\varphi(z)\big(\frac{1}{a}(e^{-aT} - e^{-at})\big)H(P,Q)\mathcal{K}^{-1} + 2\big(\frac{1}{a}(e^{-aT} - e^{-at})\big)^2\mathcal{K}^{-4}\big(\sigma^2\widetilde{\sigma}^2 \sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big)^2$$

$$-\big(\frac{1}{a}(e^{-aT} - e^{-at})\big)^2\mathcal{K}^{-2} + \big(\frac{1}{a}(e^{-aT} - e^{-at})\big)^2 Z\mathcal{K}^{-3}\big(-\sigma^2\widetilde{\sigma}^2 \sum_{t\le u\le T}(\Delta\sqrt{G(u)})^2\big).$$

Moreover,

$$\mathbb{E}[\Phi(P)H(P,Q_{\Gamma_{VG}})] = \mathbb{E}\Big[\Phi(P)H\Big(P,\frac{\partial P}{\partial r_0}H\Big(P,\frac{\partial P}{\partial r_0}\Big)\Big)\Big]$$

$$= \int_{\mathbb{R}}\int_{\mathbb{R}} \max(p(t,T,g,z) - K, 0)H\Big(p,\frac{\partial p}{\partial r_0}H\Big(p,\frac{\partial p}{\partial r_0}\Big)\Big)\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}\Big(\frac{\kappa^{-\frac{t}{\kappa}}}{\Gamma(\frac{t}{\kappa})}g(u)^{\frac{t}{\kappa}-1}e^{-\frac{1}{\kappa}g(u)}\Big)dzdg.$$

Hence, the result follows. □

Remark 4.13. We have extended the work of Bavouzet-Morel & Messaoud [2] and Bayazit & Nolder [3] to the sensitivity analysis of interest rate derivatives in a Lévy market. We extended the Vasicek interest rate model by considering a subordinated Lévy process called the *variance gamma process*, whose properties capture jumps, excess kurtosis and skewness in the interest rate derivative market. The jumps generated by the gamma process using the variance parameter 'κ' takes care of sudden or rare events in the bond market. The extended Vasicek model was used to derive the price of the zero-coupon bond. The greeks, namely: delta $\triangle_{VG}$ and gamma Γ_{VG} were obtained using Malliavin integration by parts formula. For a call option, $\triangle_{VG}$ lies between 0 and 1. Γ_{VG} measures the rate of change in the bond option's price as a result of changes in $\triangle_{VG}$. The greeks derived in this paper will help bond market investors in a variance gamma Lévy market to predict possible outcomes of investing in a given bond and the effect of taking certain decisions on an investment in the future, so as to minimize risks. The work has bridged the gap of not having adequate model that captures jumps of different magnitudes which occur in variance gamma interest rate derivative Lévy markets. For example, the presence of the pandemic 'corona virus' has contributed to jumps of different magnitudes in the market and can only be adequately modelled using the Lévy processes such as the variance gamma process.

References

[1] D. Applebaum, *Lévy processes and stochastic calculus*, Cambridge University press (2004).

[2] M.-P. Bavouzet-Morel and M. Messaoud, Computation of greeks using Malliavin's calculus in jump type market models, *Electronic Journal of Probability* **11**, no. 10, 276–300 (2006).

[3] D. Bayazit and C. A. Nolder, Malliavin calculus for Lévy markets and new sensitivities, *Quantitative Finance* **13**, no. 8, 1257–1287 (2013).

[4] R. Cont and P. Tankov, *Financial modelling with jump processes*, Chapman & Hall/CRC Financial Mathematics Series, Chapman & Hall/CRC, BocaRaton, FL (2004).

[5] H. Corb, *Interest rate swaps and other derivatives*, Columbia University Press (2012).

[6] E. A. Daal and D. B. Madan, An empirical examination of the variance-gamma model for foreign currency options, *Journal of Business* **78**, no. 6, 2121–2152 (2005).

[7] G. O. S. Ekhaguere, *Lecture notes on Financial Mathematics* (Unpublished), University of Ibadan, Ibadan, Nigeria, (2010).

[8] F. A. Fard and N. Rong, Pricing and managing risks of ruin contingent life annuities under regime-switching variance gamma process, *Ann Finance* **10**, 315–332 (2014).

[9] D. Filipovic and S. Williams: A term structure model for dividends and interest rates, arXiv:1803.02249v2 (2019).

[10] N. Gabrielli, What is an affine process, https://nicolettagabrielli.weebly .com, (2011).

[11] S. Lang, R. Signer and K. Spremann, The choice of interest rate models and its effect on bank capital requirements regulation and financial stability, *International Journal of Economics and Finance* **10**, no. 1, 74–92 (2018).

[12] D. B. Madan, P. Carr and E. C. Chang, The variance gamma process and option pricing, *European Finance Review* **2**, 79–105 (1998).

[13] D. B. Madan and E. Seneta, The variance gamma (VG) model for share market returns, *The Journal of Business* **63**, no. 4, 511–524 (1990).

[14] F. Mehrdoust, A. R. Najafi and H. Samimi, A mixed fractional Vasicek model and pricing Bermuda option on zero-coupon bonds, *Sādhanā* **45**, no. 58, 1–12 (2020).

[15] G. Orlando, R. M. Mininni and M. Bufalo, On the calibration of short-term Interest Rates Through a CIR Model. www.arXiv:1806.03683v1, (2018).

[16] E. Petrou, Malliavin Calculus in Lévy spaces and applications to finance, *Electronic Journal of Probability* **13**, no. 27, 852–879 (2008).

[17] A. Rathgeber, J. Stadler and S. Stöck, Modelling share returns - an empirical study on the variance gamma model, *Journal of Economics and Finance* **40**, no. 4, 653–682 (2013).

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