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A Family of Theta-Function Identities Involving R_α and R_m-Functions Related to Jacobi's Triple-Product Identity

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Abstract The authors establish a set of four new theta-function identities involving R_α and R_m-Functions which are based upon a number of q-product identities and Jacobi's celebrated triple-product identity. These theta-function identities depict the inter-relationships that exist among theta-function identities and combinatorial partition-theoretic identities. Here, in this paper, we consider and relate the R_α and R_m-Functions to several interesting q-identities such as (for example) a number of q-product identities and Jacobi's celebrated triple-product identity. Several recent developments on the subject-matter of this article as well as some of its potential application areas are also briefly indicated.

1 Introduction

Throughout this article, we denote by $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{C}$ the set of positive integers, the set of integers and the set of complex numbers, respectively. We also let

$$\mathbb{N}_0 := \mathbb{N} \cup \{0\} = \{0, 1, 2, \cdots\}.$$

In what follows, we shall make use of the following q-notations for the details of which we refer the reader to a recent monograph on q-calculus by Ernst [25] (see also the earlier works [35, Chapter 3, Section 3.2.1], [44, Chapter 6] and [45, pp. 346 *et seq.*]).

The q-shifted factorial $(a;q)_n$ is defined (for $|q| < 1$) by

$$(a;q)_n := \begin{cases} 1 & (n=0) \\ \\ \prod_{k=0}^{n-1}(1-aq^k) & (n \in \mathbb{N}), \end{cases} \tag{1}$$

where $a, q \in \mathbb{C}$ and it is assumed *tacitly* that $a \neq q^{-m}$ $(m \in \mathbb{N}_0)$. We also write

$$(a;q)_\infty := \prod_{k=0}^{\infty}(1-aq^k) = \prod_{k=1}^{\infty}(1-aq^{k-1}) \qquad (a, q \in \mathbb{C};\ |q| < 1). \tag{2}$$

It should be noted that, when $a \neq 0$ and $|q| \geqq 1$, the infinite product in the equation (2) diverges. So, whenever $(a;q)_\infty$ is involved in a given formula, the constraint $|q| < 1$ will be *tacitly* assumed to be satisfied.

The following notations are also frequently used in our investigation:

$$(a_1, a_2, \cdots, a_m; q)_n := (a_1;q)_n\ (a_2;q)_n\ \cdots\ (a_m;q)_n \tag{3}$$

and

$$(a_1, a_2, \cdots, a_m; q)_\infty := (a_1; q)_\infty \ (a_2; q)_\infty \ \cdots \ (a_m; q)_\infty. \tag{4}$$

Ramanujan (see [32] and [33]) defined the general theta function $\mathfrak{f}(a, b)$ as follows (see, for details, [9, p. 31, Eq. (18.1)] and [7]; see also [39]):

$$\begin{aligned} \mathfrak{f}(a,b) &= 1 + \sum_{n=1}^{\infty} (ab)^{\frac{n(n-1)}{2}} \ (a^n + b^n) \\ &= \sum_{n=-\infty}^{\infty} a^{\frac{n(n+1)}{2}} \ b^{\frac{n(n-1)}{2}} = \mathfrak{f}(b,a) \qquad (|ab| < 1). \end{aligned} \tag{5}$$

We find from this last equation (5) that

$$\mathfrak{f}(a,b) = a^{\frac{n(n+1)}{2}} \ b^{\frac{n(n-1)}{2}} \ \mathfrak{f}\big(a(ab)^n, b(ab)^{-n}\big) = \mathfrak{f}(b,a) \qquad (n \in \mathbb{Z}). \tag{6}$$

In fact, Ramanujan (see [32] and [33]) also rediscovered Jacobi's famous triple-product identity which, in Ramanujan's notation, is given by (see [9, p. 35, Entry 19]):

$$\mathfrak{f}(a,b) = (-a; ab)_\infty \ (-b; ab)_\infty \ (ab; ab)_\infty \tag{7}$$

or, equivalently, by (see [28])

$$\begin{aligned} \sum_{n=-\infty}^{\infty} q^{n^2} \ z^n &= \prod_{n=1}^{\infty} \left(1 - q^{2n}\right) \left(1 + z q^{2n-1}\right) \left(1 + \frac{1}{z} \ q^{2n-1}\right) \\ &= \left(q^2; q^2\right)_\infty \ \left(-zq; q^2\right)_\infty \ \left(-\frac{q}{z}; q^2\right)_\infty \qquad (|q| < 1; \ z \neq 0). \end{aligned}$$

Remark 1.1. Equation (6) holds true as stated only if n is any integer. In case n is not an integer, this result (6) is only approximately true (see, for details, [32, Vol. **2**, Chapter XVI, p. 193, Entry 18 (iv)]). Moreover, historically speaking, the q-series identity (7) or its above-mentioned equivalent form was first proved by Carl Friedrich Gauss $(1777 - 1855)$.

Several q-series identities, which emerge naturally from Jacobi's triple-product identity (7), are worthy of note here (see, for details, [9, pp. 36–37, Entry 22]):

$$\begin{aligned} \varphi(q) &:= \sum_{n=-\infty}^{\infty} q^{n^2} = 1 + 2 \sum_{n=1}^{\infty} q^{n^2} \\ &= \left\{(-q; q^2)_\infty\right\}^2 (q^2; q^2)_\infty = \frac{(-q; q^2)_\infty \ (q^2; q^2)_\infty}{(q; q^2)_\infty \ (-q^2; q^2)_\infty}; \end{aligned} \tag{8}$$

$$\psi(q) := \mathfrak{f}(q, q^3) = \sum_{n=0}^{\infty} q^{\frac{n(n+1)}{2}} = \frac{(q^2; q^2)_\infty}{(q; q^2)_\infty}; \tag{9}$$

$$\begin{aligned} f(-q) &:= \mathfrak{f}(-q, -q^2) = \sum_{n=-\infty}^{\infty} (-1)^n \ q^{\frac{n(3n-1)}{2}} \\ &= \sum_{n=0}^{\infty} (-1)^n \ q^{\frac{n(3n-1)}{2}} + \sum_{n=1}^{\infty} (-1)^n \ q^{\frac{n(3n+1)}{2}} = (q; q)_\infty. \end{aligned} \tag{10}$$

Equation (10) is known as Euler's *Pentagonal Number Theorem*. Remarkably, the following q-series identity:

$$(-q; q)_\infty = \frac{1}{(q; q^2)_\infty} = \frac{1}{\chi(-q)} \tag{11}$$

provides the analytic equivalent form of Euler's famous theorem (see, for details, [6] and [27]).

Theorem 1.2. (Euler's Pentagonal Number Theorem) *The number of partitions of a given positive integer n into distinct parts is equal to the number of partitions of n into odd parts.*

We also recall the Rogers-Ramanujan continued fraction $R(q)$ given by

$$\begin{aligned} R(q) &:= q^{\frac{1}{5}} \frac{H(q)}{G(q)} = q^{\frac{1}{5}} \frac{\mathfrak{f}(-q,-q^4)}{\mathfrak{f}(-q^2,-q^3)} = q^{\frac{1}{5}} \frac{(q;q^5)_\infty\,(q^4;q^5)_\infty}{(q^2;q^5)_\infty\,(q^3;q^5)_\infty} \\ &= \frac{q^{\frac{1}{5}}}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+} \qquad (|q|<1). \end{aligned} \tag{12}$$

Here $G(q)$ and $H(q)$, which are associated with the widely-investigated Roger-Ramanujan identities, are defined as follows:

$$\begin{aligned} G(q) &:= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(q;q)_n} = \frac{f(-q^5)}{\mathfrak{f}(-q,-q^4)} \\ &= \frac{1}{(q;q^5)_\infty\,(q4;q^5)_\infty} = \frac{(q^2;q^5)_\infty\,(q^3;q^5)_\infty\,(q^5;q^5)_\infty}{(q;q)_\infty} \end{aligned} \tag{13}$$

and

$$\begin{aligned} H(q) &:= \sum_{n=0}^{\infty} \frac{q^{n(n+1)}}{(q;q)_n} = \frac{f(-q^5)}{\mathfrak{f}(-q^2,-q^3)} = \frac{1}{(q^2;q^5)_\infty\,(q^3;q^5)_\infty} \\ &= \frac{(q;q^5)_\infty\,(q^4;q^5)_\infty\,(q^5;q^5)_\infty}{(q;q)_\infty}, \end{aligned} \tag{14}$$

and the functions $\mathfrak{f}(a,b)$ and $f(-q)$ are given by the equations (5) and (10), respectively.

For a detailed historical account of (and for various related developments stemming from) the Rogers-Ramanujan continued fraction (12) as well as the Rogers-Ramanujan identities(13) and (14), the interested reader may refer to the monumental work [9, p. 77 *et seq.*] (see also [39] and [44]).

The following continued-fraction results may be recalled now (see, for example, [11, p. 5, Eq. (2.8)]).

Theorem 1.3. *Suppose that $|q|<1$. Then*

$$\begin{aligned} A(q) &:= (q^2;q^2)_\infty(-q;q)_\infty \\ &= \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty} = \frac{1}{1-}\,\frac{q}{1+}\,\frac{q(1-q)}{1-}\,\frac{q^3}{1+}\,\frac{q^2(1-q^2)}{1-}\,\frac{q^5}{1+}\,\frac{q^3(1-q^3)}{1-\cdots} \\ &= \cfrac{1}{1-\cfrac{q}{1+\cfrac{q(1-q)}{1-\cfrac{q^3}{1+\cfrac{q^2(1-q^2)}{1-\cfrac{q^5}{1+\cfrac{q^3(1-q^3)}{1-\cdots}}}}}}}, \end{aligned} \tag{15}$$

$$B(q) := \frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty} = \frac{1}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^4}{1+}\,\frac{q^5}{1+}\,\frac{q^6}{1+}\cdots$$
$$= \cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cfrac{q^4}{1+\cfrac{q^5}{1+\cfrac{q^6}{1+\cdots}}}}}}} \tag{16}$$

and

$$C(q) := \frac{(q^2;q^5)_\infty(q^3;q^5)_\infty}{(q;q^5)_\infty(q^4;q^5)_\infty} = 1+\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+}\,\frac{q^4}{1+}\,\frac{q^5}{1+}\,\frac{q^6}{1+}\cdots$$
$$= 1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cfrac{q^4}{1+\cfrac{q^5}{1+\cfrac{q^6}{1+\cdots}}}}}}\,. \tag{17}$$

By introducing the general family $R(s,t,l,u,v,w)$, Andrews *et al.* [5] investigated a number of interesting double-summation hypergeometric q-series representations for several families of partitions and further explored the rôle of double series in combinatorial-partition identities:

$$R(s,t,l,u,v,w) := \sum_{n=0}^{\infty} q^{s\binom{n}{2}+tn}\, r(l,u,v,w;n), \tag{18}$$

where

$$r(l,u,v,w:n) := \sum_{j=0}^{[\frac{n}{u}]} (-1)^j \frac{q^{uv\binom{j}{2}+(w-ul)j}}{(q;q)_{n-uj}\,(q^{uv};q^{uv})_j}. \tag{19}$$

We also recall the following interesting special cases of (18) (see, for details, [5, p. 106, Theorem 3]; see also [39]):

$$R(2,1,1,1,2,2) = (-q;q^2)_\infty, \tag{20}$$

$$R(2,2,1,1,2,2) = (-q^2;q^2)_\infty \tag{21}$$

and

$$R(m,m,1,1,1,2) = \frac{(q^{2m};q^{2m})_\infty}{(q^m;q^{2m})_\infty}. \tag{22}$$

Recently, Srivastva *et al* [43] has introduced following three notations:

$$R_\alpha = R(2,1,1,1,2,2);\quad R_\beta = R(2,2,1,1,2,2);\quad R_m = R(m,m,1,1,1,2)\ \ (m\in\mathbb{N}). \tag{23}$$

for multivaraite R-functions, which we shall use for computation of our main results in section 2.

Ever since the year 2015, several new advancements and generalizations of the existing results were made in regard to combinatorial partition-theoretic identities (see, for example, [12] to [24] and [39] to [41]). In particular, Chaudhary *et al.* generalized several known results on character formulas (see [24]), Roger-Ramanujan type identities (see [19]), Eisenstein series, the Ramanujan-Gollnitz-Gordon continued fraction (see [20]), the 3-dissection property (see [18]), Ramanujan's modular equations of degrees 3, 7 and 9 (see [13] and [17]), and so on, by using combinatorial partition-theoretic identities. An interesting recent investigation on the subject of combinatorial partition-theoretic identities by Hahn *et al.* [26] is also worth mentioning in this connection.

Here, in this paper, our main objective is to establish a set of four new theta-function identities which depict the inter-relationships that exist between the R_α and R_m-functions, q-product identities and partition-theoretic identities.

Each of the following preliminary results will be needed for the demonstration of our main results in this paper (see [8, pp. 1755–1756]):

I. If

$$P = \frac{\psi(q)}{q^{\frac{1}{2}}\psi(q^5)} \qquad \text{and} \qquad Q = \frac{\psi(q^2)}{q\psi(q^{10})}$$

then

$$\left(\frac{P}{Q}\right)^2 - \frac{5}{P^2} - P^2 + \left(\frac{Q}{P}\right)^2 + 4 = 0. \tag{24}$$

II. If

$$P = \frac{\psi(-q)}{q^{\frac{1}{2}}\psi(-q^5)} \qquad \text{and} \qquad Q = \frac{\psi(q^2)}{q\psi(q^{10})},$$

then

$$\left(\frac{P}{Q}\right)^2 - \frac{5}{P^2} - P^2 + \left(\frac{Q}{P}\right)^2 - 4 = 0. \tag{25}$$

III. If

$$P = \frac{\psi(-q)}{q^{\frac{1}{2}}\psi(-q^5)} \qquad \text{and} \qquad Q = \frac{\psi(q)}{q^{\frac{1}{2}}\phi(q^5)},$$

then

$$\left(\frac{P}{Q}\right)^2 + \left(\frac{Q}{P}\right)^2 + \left(\frac{P}{Q} - \frac{Q}{P}\right)\left(\frac{5}{PQ} - PQ\right) - 6 = 0. \tag{26}$$

IV. If

$$P = \frac{\psi(q)}{q^{\frac{1}{8}}\psi(q^2)} \qquad \text{and} \qquad Q = \frac{\psi(q^2)}{q^{\frac{1}{4}}\psi(q^4)},$$

$$P^2 - \left(\frac{2}{PQ}\right)^2 - \left(\frac{Q}{P}\right)^2 = 0. \tag{27}$$

2 A Set of Main Results

In this section, we state and prove a set of four new theta-function identities which depict inter-relationships among q-product identities; and the R_α and R_m-functions.

Theorem 2.1. *Each of the following relationships holds true*:

$$5q\left\{\frac{R_5}{R_1}\right\}^2 + \frac{1}{q}\left\{\frac{R_1}{R_5}\right\}^2 = q\left\{\frac{R_1R_{10}}{R_2R_5}\right\}^2 + + \frac{1}{q}\left\{\frac{R_2R_5}{R_1R_{10}}\right\}^2 + 4. \tag{28}$$

Equation (28) gives inter-relationships between R_1, R_2, R_5 and R_{10}.

$$\frac{1}{q}\left\{\frac{R_{10}(q^2;q^2)_\infty(-q^5;q^{10})_\infty}{R_\alpha R_2(q^{10};q^{10})_\infty}\right\}^2 + q\left\{\frac{R_\alpha R_2(q^{10};q^{10})_\infty}{R_{10}(q^2;q^2)_\infty(-q^5;q^{10})_\infty}\right\}^2$$

$$= 5q^3\left\{\frac{R_\alpha(q^{10};q^{10})_\infty}{(q^2;q^2)_\infty(-q^5;q^{10})_\infty}\right\}^2 + \frac{1}{q^3}\left\{\frac{(q^2;q^2)_\infty(-q^5;q^{10})_\infty}{R_\alpha(q^{10};q^{10})_\infty}\right\}^2 + 4. \tag{29}$$

Equation (29) gives inter-relationships between R_2, R_{10} and R_α.

$$\frac{1}{q}\left\{\frac{(q^2;q^2)_\infty(-q^5;q^{10})_\infty}{R_\alpha(q^{10};q^{10})_\infty}\right\}^2+5q\left\{\frac{R_\alpha(q^{10};q^{10})_\infty}{(q^2;q^2)_\infty(-q^5;q^{10})_\infty}\right\}^2+6$$

$$=\left\{\frac{R_5(q^2;q^2)_\infty(-q^5;q^{10})_\infty}{R_1R_\alpha(q^{10};q^{10})_\infty}\right\}^2+\frac{1}{q}\left\{\frac{R_1}{R_5}\right\}^2+\left\{\frac{R_1R_\alpha(q^{10};q^{10})_\infty}{R_5(q^2;q^2)_\infty(-q^5;q^{10})_\infty}\right\}^2+5q\left\{\frac{R_5}{R_1}\right\}^2 \quad (30)$$

Equation (30) gives inter-relationships between R_1, R_5 and R_α.

$$\frac{1}{q^{\frac{1}{4}}}\left\{\frac{R_1}{R_2}\right\}^2-4q^{\frac{3}{4}}\left\{\frac{R_4}{R_1}\right\}^2=\frac{1}{q^{\frac{1}{4}}}\frac{\left\{R_2\right\}^4}{\left\{R_1R_4\right\}^2}. \quad (31)$$

Equation (31) gives inter-relationships between R_1, R_2 and R_4.

It is assumed that each member of the assertions (28) *to* (31) *exists.*

Proof. First of all, in order to prove the assertion (28) of Theorem 2.1, we apply the identity (9) under the given precondition of the result (24), further using (23); and after some simplifications, we get the values for P and Q as follows:

$$P=\frac{\psi(q)}{q^{\frac{1}{2}}\psi(q^5)}=\frac{R_1}{q^{\frac{1}{2}}R_5}, \quad (32)$$

and

$$Q=\frac{\psi(q^2)}{q\psi(q^{10})}=\frac{R_2}{q(R_{10}}. \quad (33)$$

Now, upon substituting from these last results (32) and (33) into (24), if we rearrange the terms and use some algebraic manipulations, we are led to the first assertion (28) of Theorem 3.

Secondly, we prove the second relationship (29) of Theorem 2.1. Indeed, if we first apply the identity (9) under the given precondition of the assertion (25), and then make use of (23), after some simplifications the following values for P and Q would follow:

$$P=\frac{\psi(-q)}{q^{\frac{1}{2}}\psi(-q^5)}=\frac{(q^2;q^2)_\infty(-q^5;q^{10})_\infty}{q^{\frac{3}{2}}R_\alpha(q^{10};q^{10})_\infty}, \quad (34)$$

and

$$Q=\frac{\psi(q^2)}{q\psi(q^{10})}=\frac{R_2}{qR_{10}}. \quad (35)$$

Now, upon substituting from these last results (34) and (35) into (25), if we rearrange the terms and use some algebraic manipulations, we obtain the second assertion (29) of Theorem 3.

Thirdly, we prove the third relationship (30) of Theorem 3. For this purpose, we first apply the identity (9) under the given precondition of (26), and then use (23). We thus find for the values of P and Q that

$$P=\frac{\psi(-q)}{q^{\frac{1}{2}}\psi(-q^5)}=\frac{(q^2;q^2)_\infty(-q^5;q^{10})_\infty}{q^{\frac{1}{2}}R_\alpha(q^{10};q^{10})_\infty}, \quad (36)$$

and

$$Q=\frac{\psi(q)}{q^{\frac{1}{2}}\psi(q^5)}=\frac{R_1}{q^{\frac{1}{2}}R_5}. \quad (37)$$

Now, upon substituting from these last results (36) and (37) into (26), if we rearrange the terms and use some algebraic manipulations, we obtain the second assertion (30) of Theorem 3.

Finally, we proceed to prove the last identity (31) asserted by Theorem 3. We make use of the identity (9) under the given precondition of (27), and further using (23), we obtain the values

$$P = \frac{\psi(q)}{q^{\frac{1}{8}}\psi(q^2)} = \frac{R_1}{q^{\frac{1}{8}}R_2}, \tag{38}$$

and

$$Q = \frac{\psi(q^2)}{q^{\frac{1}{8}}\psi(q^4)} = \frac{R_2}{q^{\frac{1}{4}}R_4}. \tag{39}$$

Thus, upon using (38) and (39) in (27), we rearrange the terms and apply some algebraic simplifications. This leads us to the required result (31), thereby completing the proof of Theorem 3. □

3 Applications Based Upon Ramanujan's Continued-Fraction Identities

In this section, we first suggest some possible applications of our findings in Theorem 2.1 within the context of continued fraction identities. We begin by recalling that Naika *et al.* [30] studied the following continued fraction:

$$U(q) := \frac{q(1-q)}{(1-q^3)+}\ \frac{q^3(1-q^2)(1-q^4)}{(1-q^3)(1+q^6)+\cdots+}\ \frac{q^3(1-q^{6n-4})(1-q^{6n-2})}{(1-q^3)(1+q^{6n})+\cdots}\ , \tag{40}$$

which is a special case of a fascinating continued fraction recorded by Ramanujan in his second notebook [32] (see also [2]). On the other hand, Chaudhary *et al.* (see [p. 861, Eqs. (3.1) to (3.5)]) developed following identities for the continued fraction $U(q)$ in (40) by using such R-functions as (for example) $R(1,1,1,1,1,2)$, $R(2,2,1,1,2,2)$, $R(2,1,1,1,2,2)$, $R(3,3,1,1,1,2)$ and $R(6,6,1,1,1,2)$:

$$\frac{1}{U(q)} + U(q) = \frac{R(1,1,1,1,1,2)R(2,2,1,1,2,2)}{\{R(2,1,1,1,2,2)\}^2} \cdot \{(q^3;q^6)_\infty (q^6;q^{12})_\infty\}^3, \tag{41}$$

$$\frac{1}{\sqrt{U(q)}} + \sqrt{U(q)} = \frac{R(2,1,1,1,2,2)}{R(2,2,1,1,2,2)} \left\{ \frac{R(1,1,1,1,1,2)R(2,2,1,1,1,2)}{q\ R(3,3,1,1,1,2)R(6,6,1,1,1,2)} \right\}^{\frac{1}{2}}, \tag{42}$$

$$\begin{aligned} \frac{1}{\sqrt{U(q)}} - \sqrt{U(q)} &= f(-q,q^3) \\ &\quad \cdot \left\{ \frac{R(1,1,1,1,1,2)\{R(2,2,1,1,2,2)\}^2}{q\ R(6,6,1,1,1,2)R(3,3,1,1,1,2)R(2,2,1,1,1,2)} \right\}^{\frac{1}{2}}, \end{aligned} \tag{43}$$

$$\frac{1}{\sqrt{U(q)}} + \sqrt{U(q)} + 2 = \frac{R(2,1,1,1,2,2)\{R(1,1,1,1,1,2)\}^2}{q\ R(6,6,1,1,1,2)R(3,3,1,1,1,2)R(2,2,1,1,2,2)} \tag{44}$$

and

$$\frac{1}{\sqrt{U(q)}} + \sqrt{U(q)} - 2 = \frac{R(2,2,1,1,1,2)\{R(3,3,1,1,1,2)\}^3}{q\ R(1,1,1,1,1,2)\{R(6,6,1,1,1,2)\}^3}. \tag{45}$$

By using the above formulas (41) to (46), we can express our results (28) to (31) in Theorem 2.1 in term of Ramanujan's continued fraction $U(q)$ given here by (40).

Remark 3.1. Even though the results of Theorem 2.1 are apparently considerably-involved, each of the asserted theta-function identities does have the potential for other applications in analytic number theory and partition theory (see, for example, [29] and [48]) as well as in real and complex analysis, especially in connection with a significant number of wide-spread problems dealing with various basic (or q-) series and basic (or q-) operators (see, for example, [38]).

Each of the theta-function identities (28) to (31), which are asserted by Theorem 2.1, obviously depict the inter-relationships that exist between q-product identities and the multivariate R-functions. Some corollaries and consequences of Theorem 2.1 may be worth pursuing for further researches in the direction of the developments which we have presented in this article.

4 Concluding Remarks and Observations

The present investigation was motivated by several recent developments dealing essentially with theta-function identities and combinatorial partition-theoretic identities. Here, in this article, we have established a family of six presumably new theta-function identities which depict the inter-relationships that exist among q-product identities and combinatorial partition-theoretic identities. We have also considered several closely-related identities such as (for example) q-product identities and Jacobi's triple-product identities. And, with a view to further motivating researches involving theta-function identities and combinatorial partition-theoretic identities, we have chosen to indicate rather briefly a number of recent developments on the subject-matter of this article.

A view to further motivating researches involving theta-function identities and combinatorial partition-theoretic identities,we have chosen to indicate rather briefly a number of recent developments on the subject-matter of this article. In a recently-published review-cum-expository review article, in addition to applying the q-analysis to Geometric Function Theory of Complex Analysis, Srivastava [38] pointed out the fact that the results for the q-analogues can easily (and possibly trivially) be translated into the corresponding results for the (p, q)-analogues (with $0 < |q| < p \leqq 1$) by applying some obvious parametric and argument variations, the additional parameter p being *redundant*. Of course, this exposition and observation of Srivastava [38, p. 340] would apply also to the results which we have considered in our present investigation for $|q| < 1$.

The list of citations, which we have included in this article,is believed to be potentially useful for indicating some of the directions for further researches and related developments on the subject-matter which we have dealt with here. In particular, the recent works by Adiga *et al.* (see [1] and [2]), Cao *et al.* [10], Chaudhary *et al.* (see [11], [14], [21] and [24]), Hahn *et al.* [26], and Srivastava *et al.* (see [40], [42], [46], [47] and [48]) are worth mentioning here.

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