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CERTAIN SUBCLASSES OF MULTIVALENT FUNCTIONS ASSOCIATED WITH $q-$ANALOGUE OF MITTAG LEFFLER FUNCTIONS

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Abstract The main object of this paper is to investigate convolution properties and coefficient estimates for some subclasses of multivalent functions defined by $q-$derivative operator in the open unit disc. The results presented here would provide extensions of those given in earlier works.

1 Introduction

Quantum calculus or $q-$calculus is an ordinary calculus without limit. In recent years, the study of $q-$theory attracted the researches due to its applications in various branches of mathematics and physics, for example, in the areas of special functions, ordinary fractional calculus, $q-$difference, $q-$integral equations and in $q-$transform analysis (see, for instance, [1, 2, 3, 9, 11, 14, 15, 20]).

For $p \in \mathbb{N} = \{1, 2, 3, ...\}$, let $\mathcal{A}_p$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \tag{1.1}$$

which are analytic and multivalent in open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$. In particular, we write $\mathcal{A}(1) = \mathcal{A}$. Let $\mathcal{S}_p^*(\eta)$ and $\mathcal{C}_p(\eta)$ denote the subclasses of multivalent starlike and convex functions of order η $(0 \leq \eta < p)$ (see Owa [18], Aouf [5] and Aouf et al. [6] and Srivastava et al. [25]). If $f(z)$ and $g(z)$ are analytic in $\mathbb{U}$, we say that $f(z)$ is subordinate to $g(z)$, written $f(z) \prec g(z)$ if there exists a Schwarz function ω, which (by definition) is analytic in $\mathbb{U}$ with $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in \mathbb{U}$, such that $f(z) = g(\omega(z)),\ z \in \mathbb{U}$. Furthermore, if the function g is univalent in $\mathbb{U}$, then we have the following equivalence, (see [12, 16]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \text{ and } f(\mathbb{U}) \subset g(\mathbb{U}).$$

For functions f given by (1.1) and g given by

$$g(z) = z^p + \sum_{k=p+1}^{\infty} b_k z^k$$

the Hadamard product or convolution of f and g is defined by

$$(f * g)(z) = z^p + \sum_{k=p+1}^{\infty} a_k b_k z^k = (g * f)(z).$$

Also, for $f \in \mathcal{A}_p$ given by (1.1) and $0 < q < 1$, the $q-$derivative of f is defined by (see Gasper and Rahman [13] and Srivastava et al. [26])

$$D_{p,q}f(z) := \begin{cases} f'(0) & \text{if } z = 0, \\ \dfrac{f(qz) - f(z)}{(q-1)z} & \text{if } z \neq 0, \end{cases} \tag{1.2}$$

provided that $f'(0)$ exists. From (1.2), we deduce that

$$D_{p,q}f(z) = [p]_q\, z^{p-1} + \sum_{k=p+1}^{\infty} [k]_q\, a_k z^{k-1} \ (z \neq 0), \tag{1.3}$$

where

$$[i]_q := \frac{1-q^i}{1-q} = 1 + q + q^2 + ... + q^{i-1}, \tag{1.4}$$

and

$$\lim_{q \to 1^-} D_{p,q}f(z) = \lim_{q \to 1^-} \frac{f(qz) - f(z)}{(q-1)z} = f'(z),$$

for a function f which is differentiable in a given subset of $\mathbb{C}$. Further, for $p = 1$, we have $D_{q,1}f(z) = D_q f(z)$ (see Seoudy and Aouf [22]).

Making use of the $q-$derivative operator $D_{p,q}$ $(0 < q < 1,\ p \in \mathbb{N})$ given by (1.2), we introduce the subclass $\mathcal{S}^*_{p,q}(\eta)$ of $p-$valently $q-$starlike functions of order η in $\mathbb{U}$ and the subclass $\mathcal{C}_{p,q}(\eta)$ of $p-$valently $q-$convex functions of order η in $\mathbb{U}$, $0 \leq \eta < [p]_q$, as follows:

$$\mathcal{S}^*_{p,q}(\eta) = \left\{ f \in \mathcal{A}_p : \Re\left\{ \frac{zD_{p,q}f(z)}{f(z)} \right\} > \eta \right\},$$

and

$$\mathcal{C}_{p,q}(\eta) = \left\{ f \in \mathcal{A}_p : \Re\left\{ \frac{D_{p,q}(zD_{p,q}f(z))}{D_{p,q}f(z)} \right\} > \eta \right\},$$

respectively. It is easy to check that

$$f \in \mathcal{C}_{p,q}(\eta) \iff \frac{zD_{p,q}f}{[p]_q} \in \mathcal{S}^*_q(\eta).$$

We note also that $\lim_{q \to 1^-} \mathcal{S}^*_{p,q}(\eta) = \mathcal{S}^*_p(\eta)$ and $\lim_{q \to 1^-} \mathcal{C}_{p,q}(\eta) = \mathcal{C}_p(\eta)$.

We next introduce the subclasses $\mathcal{S}^*_{p,q}[\alpha; A, B]$ and $\mathcal{C}_{p,q}[\alpha; A, B]$ as follows.

Definition 1.1. For $0 < q < 1$, $0 \leq \eta < [p]_q$, $-1 \leqq B < A \leqq 1$ and $p \in \mathbb{N}$, let $\mathcal{S}^*_{p,q}[\eta; A, B]$ and $\mathcal{C}_{p,q}[\eta, A, B]$ be the subclasses of $\mathcal{A}_p$ consisting of functions f of the form (1.1) and satisfy the analytic criterion:

$$\frac{1}{[p]_q - \eta}\left(\frac{zD_{p,q}f(z)}{f(z)} - \eta \right) \prec \frac{1+Az}{1+Bz}, \tag{1.5}$$

and

$$\frac{1}{[p]_q - \eta}\left(\frac{D_{p,q}(zD_{p,q}f(z))}{D_{p,q}f(z)} - \eta \right) \prec \frac{1+Az}{1+Bz}, \tag{1.6}$$

From (1.5) and (1.6), it follows that

$$f \in \mathcal{C}_{p,q}[\eta; A, B] \Leftrightarrow \frac{zD_{p,q}f}{[p]_q} \in \mathcal{S}^*_{p,q}[\eta; A, B]. \tag{1.7}$$

We remark the following special cases:

(i) $\mathcal{S}^*_{p,q}[\eta; 1, -1] =: \mathcal{S}^*_{p,q}(\eta)$ and $\mathcal{C}_{p,q}[\eta; 1, -1] =: \mathcal{C}_{p,q}(\eta)$ $(0 \leq \eta < [p]_q)$;
(ii) $\mathcal{S}^*_{1,q}[0; A, B] =: \mathcal{S}^*_q[A, B]$ and $\mathcal{C}_{1,q}[0; A, B] =: \mathcal{C}_q[A, B]$ (see [22, 23]);
(iii) $\lim_{q \to 1^-} \mathcal{S}^*_{p,q}[\eta; 1, -1] =: \mathcal{S}^*_p(\eta)$ and $\lim_{q \to 1^-} \mathcal{C}_{p,q}[\eta; 1, -1] =: \mathcal{C}_p(\eta)$ $(0 \leqq \eta < p)$ (see [18] and [5]);

(iv) $\lim_{q\to 1^-} \mathcal{S}^*_{p,q}[0;A,B] = \mathcal{S}_p[A,B]$ and $\lim_{q\to 1^-} \mathcal{C}_{p,q}[0;A,B] = \mathcal{C}_p[A,B]$ (see [21], with $\lambda = 0$ and $\phi(z) = \frac{1+Az}{1+Bz}$] and [4]);
(v) $\lim_{q\to 1^-} \mathcal{S}^*_{1,q}[0;A,B] = \mathcal{S}[A,B]$ and $\lim_{q\to 1^-} \mathcal{C}_{1,q}[0;A,B] = \mathcal{C}[A,B]$ (see [7]).

The q-*shifted factorials*, for any complex number α, are defined by

$$(\alpha;q)_0 := 1; \quad (\alpha;q)_n := \prod_{k=0}^{n-1}\left(1-\alpha q^k\right), \ n\in\mathbb{N}. \tag{1.8}$$

The definition (1.8) remains meaningful for $n=\infty$ as a convergent infinite product

$$(\alpha;q)_\infty = \prod_{j=0}^{\infty}\left(1-\alpha q^j\right) \ \text{for } |q|<1.$$

Furthermore, in terms of the basic (or $q-$) Gamma function $\Gamma_q(z)$ defined by

$$\Gamma_q(z) := \frac{(q;q)_\infty(1-q)^{1-z}}{(q^z;q)_\infty} \qquad (0<q<1;\ z\in\mathbb{C}), \tag{1.9}$$

so that

$$\lim_{q\to 1^-}\{\Gamma_q(z)\} = \Gamma(z)$$

for the familiar Gamma function $\Gamma(z)$, we find from (1.8) that

$$(q^\alpha;q)_n = \frac{\Gamma_q(\alpha+n)}{\Gamma_q(\alpha)}(1-q)^n \qquad (n\in\mathbb{N};\ \alpha\in\mathbb{C}).$$

We note that

$$\lim_{q\to 1^-}\frac{(q^\alpha;q)_n}{(1-q)^n} = (\alpha)_n,$$

where

$$(\alpha)_n = \begin{cases} 1, & \text{if} \quad n=0, \\ \alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1), & \text{if} \quad n\in\mathbb{N}. \end{cases}$$

For $0<q<1$, α, β, $\gamma\in\mathbb{C}$, $\Re(\alpha)>0$, $\Re(\beta)>0$, $\Re(\gamma)>0$, consider the q-*analogue of Mittag Leffler* defined by (see [24], [8] and [10])

$$E^\gamma_{\alpha,\beta}(z;q) = \sum_{k=0}^{\infty}\frac{(q^\gamma;q)_k}{(q;q)_k}\frac{z^k}{\Gamma_q(\alpha k+\beta)}.$$

As $q\to 1^-$, the operator $E^\gamma_{\alpha,\beta}(z;q)$ reduces to $E^\gamma_{\alpha,\beta}(z)$ introduced by Prabhakar [19]. Now, let us define

$$\mathbb{E}^{\gamma,p}_{\alpha,\beta}(z;q) := z^p\,\Gamma_q(\beta)\,E^\gamma_{\alpha,\beta}(z;q) = z^p + \sum_{k=p+1}^{\infty}\frac{(q^\gamma;q)_{k-p}}{(q;q)_{k-p}}\frac{\Gamma_q(\beta)}{\Gamma_q(\alpha(k-p)+\beta)}z^k.$$

We remark that:
(i) $\mathbb{E}^{1,p}_{1,1}(z;q) = z^p e_q(z)$;
(ii) $\mathbb{E}^{1,p}_{1,2}(z;q) = z^{p-1}(e_q(z)-1)$;
where $e_q(z)$ is one of the $q-$analogues of the exponential function e^z given by

$$e_q(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma_q(k+1)} = \sum_{k=0}^{\infty}\frac{(1-q)^k z^k}{(q;q)_k} = \frac{1}{(z;q)_\infty}.$$

Using the Hadamard product (or convolution), we define the linear operator $\mathbb{E}^{\gamma,p}_{\alpha,\beta}:\mathcal{A}_p\to\mathcal{A}_p$ by (see [8] and [10])

$$\mathbb{H}^{\gamma,p}_{\alpha,\beta}f(z) = \mathbb{E}^{\gamma,p}_{\alpha,\beta}(z;q)*f(z) = z^p + \sum_{k=p+1}^{\infty}\frac{(q^\gamma;q)_{k-p}}{(q;q)_{k-p}}\frac{\Gamma_q(\beta)}{\Gamma_q(\alpha(k-p)+\beta)}a_k z^k,\ z\in\mathbb{U}. \tag{1.10}$$

Definition 1.2. For $0 < q < 1$, $\alpha, \beta, \gamma \in \mathbb{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $-1 \le B < A \le 1$ and $0 \le \eta < [p]_q$, let

$$\mathcal{S}^{\gamma}_{p,q,\alpha,\beta}[\eta; A, B] := \left\{ f \in \mathcal{A}_p : \mathbb{H}^{\gamma,p}_{\alpha,\beta} f \in \mathcal{S}^{*}_{p,q}[\eta; A, B] \right\},$$

and

$$\mathcal{C}^{\gamma}_{p,q,\alpha,\beta}[\eta; A, B] := \left\{ f \in \mathcal{A}_p : \mathbb{H}^{\gamma,p}_{\alpha,\beta} f \in \mathcal{C}_{p,q}[\eta; A, B] \right\}.$$

It is easy to check that

$$f \in \mathcal{C}^{\gamma}_{p,q,\alpha,\beta}[\eta; A, B] \Leftrightarrow \frac{zD_{p,q}f}{[p]_q} \in \mathcal{S}^{\gamma}_{p,q,\alpha,\beta}[\eta; A, B].$$

Seoudy and Aouf in [22] (and Mostafa et al. in [17]) introduced some subclasses of $q-$starlike (meromorphic) and $q-$convex (meromorphic) functions involving $q-$derivative operator, they obtained convolution properties and coefficient estimates for functions belonging to these classes. In this paper, we investigate convolution properties two subclasses $\mathcal{S}^{*}_{p,q}[\eta; A, B]$ and $\mathcal{K}_{p,q}[\eta; A, B]$. Also, we obtain coefficient estimates for the subclasses $\mathcal{S}^{\gamma}_{p,q,\alpha,\beta}[\eta; A, B]$ and $\mathcal{C}^{\gamma}_{p,q,\alpha,\beta}[\eta; A, B]$.

2 Main Results

Unless otherwise mentioned, we assume throughout this paper that $0 < q < 1$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $\Re(\gamma) > 0$, $-1 \leqq B < A \leqq 1$, $\mathbb{U}^* = \mathbb{U} \setminus \{0\}$ and $0 \le \eta < [p]_q$.

Theorem 2.1. *If $f \in \mathcal{A}_p$, then $f \in \mathcal{S}^{*}_{p,q}[\eta; A, B]$ if and only if*

$$\frac{1}{z^p}\left[f(z) * \frac{z^p - Cz^{p+1}}{(1-z)(1-qz)} \right] \neq 0 \quad (z \in \mathbb{U}^*; \theta \in [0, 2\pi)), \tag{2.1}$$

where

$$C = q + \frac{q^p\left(e^{-i\theta} + B\right)}{(A-B)\left([p]_q - \eta\right)}. \tag{2.2}$$

Proof. For any function $f \in \mathcal{A}_p$, we can verify that

$$f(z) = f(z) * \frac{z^p}{1-z} \tag{2.3}$$

and

$$zD_{p,q}f(z) = f(z) * \frac{[p]_q z^p + \left(1 - [p]_q\right) z^{p+1}}{(1-z)(1-qz)}. \tag{2.4}$$

First, in order to prove that (2.1) holds, we will write (1.5) by using the principle of subordination between analytic functions, that is,

$$\frac{zD_{p,q}f(z)}{f(z)} = \frac{[p]_q + \left\{ [p]_q B + (A-B)\left([p]_q - \eta\right)\right\} w(z)}{1 + Bw(z)},$$

where w is a Schwarz function, hence

$$\frac{1}{z^p}\left[zD_{p,q}f(z)\left(1 + Be^{i\theta}\right) - \left([p]_q + \left\{ [p]_q B + (A-B)\left([p]_q - \eta\right)\right\} e^{i\theta}\right) f(z)\right] \neq 0, \tag{2.5}$$

for all $z \in \mathbb{U}^*$ and $\theta \in [0, 2\pi)$. Since the convolution operator satisfy the distributivity $f * (g+h) = f*g + f*h$ for any functions $f,\ g,\ h \in \mathcal{A}_p$, and from (2.3) and (2.4), the relation (2.5) may be written as

$$\frac{1}{z^p}\left[\left(1 + Be^{i\theta}\right)\left(f(z) * \left[\frac{[p]_q z^p + \left(1 - [p]_q\right) z^{p+1}}{(1-z)(1-qz)} \right]\right)\right.$$

$$-\left([p]_q+\left\{[p]_q B+(A-B)\left([p]_q-\eta\right)\right\}e^{i\theta}\right)\left(f(z)*\frac{z^p}{1-z}\right)\Bigg]\neq 0,$$

which is equivalent to

$$\frac{1}{z^p}\left[f(z)*\frac{z^p-\left[q+\dfrac{q^p\left(e^{-i\theta}+B\right)}{(A-B)\left([p]_q-\eta\right)}\right]z^{p+1}}{(1-z)(1-qz)}\right]\neq 0,\ z\in\mathbb{U}^*,\ \theta\in[0,2\pi),$$

that is (2.1).

Reversely, suppose that $f\in\mathcal{A}_p$ satisfy the condition (2.1). Like it was previously shown, the assumption (2.1) is equivalent to (2.4), that is,

$$\frac{zD_{p,q}f(z)}{f(z)}\neq\frac{[p]_q+\left\{[p]_q B+(A-B)\left([p]_q-\eta\right)\right\}e^{i\theta}}{1+Be^{i\theta}}\quad\left(z\in\mathbb{U}^*;\theta\in[0,2\pi)\right). \tag{2.6}$$

Denoting

$$\varphi(z)=\frac{zD_{p,q}f(z)}{f(z)}\quad\text{and}\quad\psi(z)=\frac{[p]_q+\left\{[p]_q B+(A-B)\left([p]_q-\eta\right)\right\}z}{1+Bz},$$

the relation (2.6) could be written as $\varphi(\mathbb{U})\cap\psi(\partial\mathbb{U})=\emptyset$. Therefore, the simply connected domain $\varphi(\mathbb{U})$ is included in a connected component of $\mathbb{C}\setminus\psi(\partial\mathbb{U})$. From this fact, using that $\varphi(0)=\psi(0)=[p]_q$ together with the univalence of the function ψ, it follows that $\varphi(z)\prec\psi(z)$, that is $f\in\mathcal{S}^*_{p,q}[\eta;A,B]$. □

Remark 2.2. Putting $q\to 1^-,\eta=0$ and $e^{i\theta}=x$ in Theorem 2.1, we obtain the result of Sarkar et al. [21, Theorem 2.1 with $\lambda=0$ and $\phi(z)=\frac{1+Az}{1+Bz}$];

Remark 2.3. Putting $\eta=0$ and $p=1$ in Theorem 2.1, we obtain the result of Seoudy and Aouf [22, Theorem 1].

Theorem 2.4. *If $f\in\mathcal{A}_p$, then $f\in\mathcal{C}_{p,q}[\eta;A,B]$ if and only if*

$$\frac{1}{z^p}\left[f(z)*\frac{[p]_q z^p-\left\{[p]_q-(q+1)+C\left(1+q[p]_q\right)\right\}z^{p+1}+q\left([p]_q-1\right)C\ z^{p+2}}{(1-z)(1-qz)(1-q^2z)}\right]\neq 0, \tag{2.7}$$

for all $z\in\mathbb{U}^*$ and $\theta\in[0,2\pi)$, where C is given by (2.2).

Proof. From (1.7) it follows that $f\in\mathcal{C}_{p,q}[\eta;A,B]$ if and only if $\Phi(z):=\dfrac{zD_{p,q}f}{[p]_q}\in\mathcal{S}^*_{p,q}[\eta;A,B]$. Then, according to Theorem 2.1, the function $\Phi(z)$ belongs to $\mathcal{S}^*_{p,q}[\eta;A,B]$ if and only if

$$\frac{1}{z^p}\left[\Phi(z)*g(z)\right]\neq 0,\ \text{for all } z\in\mathbb{U}^*\text{ and }\theta\in[0,2\pi), \tag{2.8}$$

where

$$g(z)=\frac{z^p-Cz^{p+1}}{(1-z)(1-qz)}.$$

But (2.8) is equivalent to

$$\frac{1}{z^p}\left[\frac{1}{[p]_q}\left(f(z)*zD_{p,q}g(z)\right)\right]\neq 0,$$

that is,

$$\frac{1}{[p]_q z^p}\left[f(z)*zD_{p,q}\,g(z)\right]\neq 0\Leftrightarrow\frac{1}{z^p}\left[f(z)*zD_{p,q}\,g(z)\right]\neq 0,$$

for all $z \in \mathbb{U}^*$ and $\theta \in [0, 2\pi)$. Using the fact that

$$zD_{p,q}g(z) = \frac{[p]_q z^p - \left\{[p]_q - (q+1) + C\left(1 + q[p]_q\right)\right\} z^{p+1} + q\left([p]_q - 1\right) C\ z^{p+2}}{(1-z)(1-qz)(1-q^2 z)},$$

it is easy to check that (2.8) is equivalent to (2.7). □

Remark 2.5. For $q \to 1^-, \eta = 0$ and $e^{i\theta} = x$ in Theorem 2.4, we obtain the result of Sarkar et al. [21, Theorem 2.3 with $\lambda = 0$ and $\phi(z) = \frac{1+Az}{1+Bz}$];

Remark 2.6. For $\eta = 0$ and $p = 1$ in Theorem 2.4, we obtain the result of Seoudy and Aouf [22, Theorem 5].

Theorem 2.7. *If $f \in \mathcal{A}_p$, then $f \in \mathcal{S}^{\gamma}_{p,q,\alpha,\beta}[\eta; A, B]$ if and only if*

$$1 + \sum_{k=p+1}^{\infty} \frac{(q^\gamma; q)_{k-p}}{(q;q)_{k-p}} \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha(k-p)+\beta)} \tag{2.9}$$

$$\times \frac{(A-B)\left([p]_q - \eta\right)[k-p+1]_q - [k-p]_q \left\{q(A-B)\left([p]_q - \eta\right) + \left(1 + q[p]_q - [p]_q\right)\left(e^{-i\theta} + B\right)\right\}}{(A-B)\left([p]_q - \eta\right)} a_k z^{k-p} \neq 0,$$

for all $z \in \mathbb{U}^*$ and $\theta \in [0, 2\pi)$.

Proof. If $f \in \mathcal{A}_p$, then from Definition 1.2 and according to Theorem 2.1, we have $f \in \mathcal{S}^{\gamma}_{p,q,\alpha,\beta}[\eta; A, B]$ if and only if

$$\frac{1}{z^p}\left[\left(\mathbb{H}^{\gamma,p}_{\alpha,\beta} f\right)(z) * \frac{z^p - Cz^{p+1}}{(1-z)(1-qz)}\right] \neq 0, \text{for all } z \in \mathbb{U}^* \text{ and all } \theta \in [0, 2\pi), \tag{2.10}$$

where C is given by (2.2). Since

$$\frac{z^p}{(1-z)(1-qz)} = z^p + \sum_{k=p+1}^{\infty} [k-p+1]_q z^k, \quad \frac{z^{p+1}}{(1-z)(1-qz)} = \sum_{k=p+1}^{\infty} [k-p]_q z^k.$$

After some computations, we get

$$\frac{z^p - Cz^{p+1}}{(1-z)(1-qz)} = z^p + \sum_{k=p+1}^{\infty} \left([k-p+1]_q - [k-p]_q C\right) z^k,$$

then we may deduce that (2.10) is equivalent to (2.9), and the proof of Theorem 2.7 is completed. □

Theorem 2.8. *If $f \in \mathcal{A}_p$, then $f \in \mathcal{C}^{\gamma}_{p,q,\alpha,\beta}[\eta; A, B]$ if and only if*

$$1 + \sum_{k=p+1}^{\infty} \frac{[k]_q}{[p]_q} \frac{(q^\gamma; q)_{k-p}}{(q;q)_{k-p}} \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha(k-p)+\beta)} \tag{2.11}$$

$$\times \frac{(A-B)\left([p]_q - \eta\right)[k-p+1]_q - [k-p]_q \left\{q(A-B)\left([p]_q - \eta\right) + \left(1 + q[p]_q - [p]_q\right)\left(e^{-i\theta} + B\right)\right\}}{(A-B)\left([p]_q - \eta\right)} a_k z^{k-p} \neq 0,$$

for all $z \in \mathbb{U}^*$ and $\theta \in [0, 2\pi)$.

Proof. If $f \in \mathcal{A}_p$, then from Definition 1.2 and Theorem 2.4, we have that $f \in \mathcal{C}^{\gamma}_{p,q,\alpha,\beta}[\eta; A, B]$ if and only if

$$\frac{1}{z^p}\left[\left(\mathbb{H}^{\gamma,p}_{\alpha,\beta} f\right)(z) * \frac{[p]_q z^p - \left\{[p]_q - (q+1) + C\left(1 + q[p]_q\right)\right\} z^{p+1} + q\left([p]_q - 1\right) C\ z^{p+2}}{(1-z)(1-qz)(1-q^2 z)}\right] \neq 0, \tag{2.12}$$

for all $z \in \mathbb{U}^*$ and $\theta \in [0, 2\pi)$, where C is given by (2.2). Since

$$\frac{[p]_q z^p - \left\{[p]_q - (q+1) + C\left(1 + q[p]_q\right)\right\} z^{p+1} + q\left([p]_q - 1\right) Cz^{p+2}}{(1-z)(1-qz)(1-q^2 z)}$$

$$= [p]_q z^p + \sum_{k=p+1}^{\infty} [k]_q \left([k-p+1]_q - [k-p]_q C\right) z^k, \ z \in \mathbb{U}.$$

Now, we may check that (2.12) is equivalent to (2.11) which proves our result. □

Unless otherwise mentioned, we assume throughout the remainder part of this section that α, β and γ are real numbers.

Theorem 2.9. *If $f \in \mathcal{A}_p$ satisfies the inequality*

$$\sum_{k=p+1}^{\infty} \frac{(q^\gamma;q)_{k-p}}{(q;q)_{k-p}} \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha(k-p)+\beta)} \tag{2.13}$$
$$\times \frac{(A-B)\left([p]_q-\eta\right)[k-p+1]_q+[k-p]_q\left\{q(A-B)\left([p]_q-\eta\right)+\left(1+q[p]_q-[p]_q\right)(1+|B|)\right\}}{(A-B)\left([p]_q-\eta\right)}|a_k|<1$$

then $f \in \mathcal{S}^{\gamma}_{p,q,\alpha,\beta}[\eta;A,B]$.

Proof. If $f \in \mathcal{A}_p$ has the form (1.1) and assuming that (2.9) holds, we obtain

$$\Bigg|1+\sum_{k=p+1}^{\infty} \frac{(q^\gamma;q)_{k-p}}{(q;q)_{k-p}} \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha(k-p)+\beta)}$$
$$\times \frac{(A-B)\left([p]_q-\eta\right)[k-p+1]_q-[k-p]_q\left\{q(A-B)\left([p]_q-\eta\right)+\left(1+q[p]_q-[p]_q\right)\left(e^{-i\theta}+B\right)\right\}}{(A-B)\left([p]_q-\eta\right)}a_k z^{k-p}\Bigg|$$
$$\geq 1-\Bigg|\sum_{k=p+1}^{\infty} \frac{(q^\gamma;q)_{k-p}}{(q;q)_{k-p}} \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha(k-p)+\beta)}$$
$$\times \frac{(A-B)\left([p]_q-\eta\right)[k-p+1]_q-[k-p]_q\left\{q(A-B)\left([p]_q-\eta\right)+\left(1+q[p]_q-[p]_q\right)\left(e^{-i\theta}+B\right)\right\}}{(A-B)\left([p]_q-\eta\right)}a_k z^{k-p}\Bigg|$$
$$\geq 1-\sum_{k=p+1}^{\infty} \frac{(q^\gamma;q)_{k-p}}{(q;q)_{k-p}} \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha(k-p)+\beta)}$$
$$\times \frac{(A-B)\left([p]_q-\eta\right)[k-p+1]_q+[k-p]_q\left\{q(A-B)\left([p]_q-\eta\right)+\left(1+q[p]_q-[p]_q\right)(1+|B|)\right\}}{(A-B)\left([p]_q-\eta\right)}|a_k|>0,$$

for all $z \in \mathbb{U}^*$ and $\theta \in [0,2\pi)$. It follows that (2.9) holds, and from Theorem 2.7 we obtain our conclusion. □

Using similar arguments to those in the proof of Theorem 2.9, we obtain the following theorem:

Theorem 2.10. *If $f \in \mathcal{A}_p$ and satisfies the inequality*

$$\sum_{k=p+1}^{\infty} \frac{[k]_q}{[p]_q} \frac{(q^\gamma;q)_{k-p}}{(q;q)_{k-p}} \frac{\Gamma_q(\beta)}{\Gamma_q(\alpha(k-p)+\beta)}$$
$$\times \frac{(A-B)\left([p]_q-\eta\right)[k-p+1]_q+[k-p]_q\left\{q(A-B)\left([p]_q-\eta\right)+\left(1+q[p]_q-[p]_q\right)(1+|B|)\right\}}{(A-B)\left([p]_q-\eta\right)}|a_k|<1,$$

then $f \in \mathcal{C}^{\gamma}_{p,q,\alpha,\beta}[\eta;A,B]$.

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