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Some results on Lorentzian para-Kenmotsu manifolds admitting η-Ricci solitons

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Abstract The object of the present paper is to characterize 3-dimensional Lorentzian para-Kenmotsu manifolds admitting η-Ricci solitons. Finally, the existence of $\eta-$Ricci soliton on 3-dimensional Lorentzian para-Kenmotsu manifolds has been proved by a concrete example.

1 Introduction

In 1982, Hamilton [14] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian (or a semi Riemannian) manifold defined as follows:

$$\frac{\partial}{\partial t}g_{ij}(t) = -2R_{ij}.$$

A Ricci soliton is a generalization of an Einstein metric. On the manifold M, a Ricci soliton is a triple (g, V, λ) with g a Riemannian (or semi Riemannian) metric, V a vector field, called potential vector field and λ a real scalar such that

$$\pounds_V g + 2S + 2\lambda g = 0, \tag{1.1}$$

where $\pounds_V$ is the Lie derivative operator along the vector field V on M. The Ricci soliton is said to be shrinking, steady and expanding according to λ being negative, zero and positive, respectively. Ricci solitons have been studied by several authors such as [12, 13, 15] and many others.

As a generalization of Ricci soliton, the problem of studying η-Ricci solitons in the context of contact geometry was initiated by Cho and Kimura [10]. η-Ricci solitons has also been studied for Hopf hypersurfaces in complex space forms by Calin and Crasmareanu [9]. An $\eta-$Ricci soliton is a tuple (g, V, λ, μ), where V is a vector field on M, λ and μ are constants, and g is a Riemannian (or a semi Riemannian) metric satisfying the equation

$$\pounds_V g + 2S + 2\lambda g + 2\mu\eta \otimes \eta = 0, \tag{1.2}$$

where λ and μ are real numbers. In particular, if $\mu = 0$, then the notion of η-Ricci soliton (g, V, λ, μ) reduces to the notion of Ricci soliton (g, V, λ). Recenty, $\eta-$Ricci solitons have been studied by various authors such as $[3, 5-8, 11]$ and many others.

Motivated by the above studies, in this paper we study $\eta-$Ricci soliton on 3-dimensional Lorentzian para-Kenmotsu manifolds satisfying certain curvature conditions. The paper is organized as follows: In Section 2, we give a brief introduction of Lorentzian para-Kenmotsu manifolds. Section 3 deals with the study of $\eta-$Ricci solitons on 3-dimensional Lorentzian para-Kenmotsu manifolds. In Section 4, we study $\eta-$Ricci solitons on 3-dimensional Lorentzian para-Kenmotsu manifolds admitting Codazzi type of Ricci tensor and cyclic $\eta-$recurrent Ricci tensor. $\eta-$Ricci solitons on 3-dimensional Lorentzian para-Kenmotsu manifolds satisfying the curvature conditions $P(\xi, X)\cdot S = 0$, $Q\cdot P = 0$ and $S\cdot R = 0$ have been studied in sections 5, 6 and 7, respectively. Finally, we construct a 3-dimensional example of Lorentzian para-Kenmotsu manifolds which admits an $\eta-$Ricci soliton.

2 Preliminaries

An n-dimensional differentiable manifold M with a structure (ϕ, ξ, η, g) is said to be a Lorentzian almost paracontact metric manifold, if it admits a $(1,1)$-tensor field ϕ, a contravariant vector field ξ, a 1-form η and a Lorentzian metric g satisfying

$$\eta(\xi) = -1, \tag{2.1}$$

$$\phi^2 X = X + \eta(X)\xi, \tag{2.2}$$

$$\phi\xi = 0, \quad \eta(\phi X) = 0, \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X, Y) + \eta(X)\eta(Y), \tag{2.4}$$

$$g(X, \xi) = \eta(X), \tag{2.5}$$

$$\Phi(X, Y) = \Phi(Y, X) = g(X, \phi Y) \tag{2.6}$$

for any vector fields $X, Y \in \chi(M)$; where $\chi(M)$ is the Lie algebra of vector fields on the manifold M.
If ξ is a killing vector field, the (para) contact structure is called a K-(para) contact. In such a case, we have

$$\nabla_X \xi = \phi X. \tag{2.7}$$

A Lorentzian almost paracontact manifold M is called a Lorentzian para-Sasakian manifold if

$$(\nabla_X \phi)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi \tag{2.8}$$

for any vector fields X, Y on M.

Definition 2.1. A Lorentzian almost paracontact manifold M is called Lorentzian para-Kenmostu manifold if $[2, 4]$

$$(\nabla_X \phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X \tag{2.9}$$

for any vector fields X, Y on M.

In a Lorentzian para-Kenmostu manifold, we have

$$\nabla_X \xi = -X - \eta(X)\xi, \tag{2.10}$$

$$(\nabla_X \eta)Y = -g(X, Y) - \eta(X)\eta(Y), \tag{2.11}$$

where ∇ is the Levi-Civita connection with respect to the Lorentzian metric g.
Furthermore, in a Lorentzian para-Kenmotsu manifold with respect to the Levi-Civita connection, the following relations hold [2, 4]:

$$g(R(X, Y)Z, \xi) = \eta(R(X, Y)Z) = g(Y, Z)\eta(X) - g(X, Z)\eta(Y), \tag{2.12}$$

$$R(\xi, X)Y = -R(X, \xi)Y = g(X, Y)\xi - \eta(Y)X, \tag{2.13}$$

$$R(X, Y)\xi = \eta(Y)X - \eta(X)Y, \tag{2.14}$$

$$R(\xi, X)\xi = X + \eta(X)\xi, \tag{2.15}$$

$$S(X, \xi) = (n-1)\eta(X), \quad S(\xi, \xi) = -(n-1), \tag{2.16}$$

$$Q\xi = (n-1)\xi, \tag{2.17}$$

$$S(\phi X, \phi Y) = S(X, Y) + (n-1)\eta(X)\eta(Y) \tag{2.18}$$

for any vector fields $X, Y, Z \in \chi(M)$; where R is the curvature tensor, S is the Ricci tensor and Q is the Ricci operator.

Definition 2.2. A Lorentzian para-Kenmotsu manifold M is said to be an η-Einstein manifold if its non-vanishing Ricci tensor S of the form

$$S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y), \tag{2.19}$$

where a and b are smooth functions on M. In particular, if $b = 0$, then the manifold is said to be an Einstein manifold.

Definition 2.3. The projective curvature tensor P in an n-dimensional Lorentzian para-Kenmotsu manifold is defined by [17]

$$P(X, Y)Z = R(X, Y)Z - \frac{1}{n-1}[S(Y, Z)X - S(X, Z)Y] \tag{2.20}$$

for any $X, Y, Z \in \chi(M)$.

It is known that every 3-dimensional Kenmotsu manifold is an η-Einstein manifold and its Ricci tensor S is given by [16]

$$S(X, Y) = (\frac{r}{2} + 1)g(X, Y) - (3 + \frac{r}{2})\eta(X)\eta(Y),$$

where r is the scalar curvature of the manifold. In the same way we can easily prove the following:

Proposition 2.4. *Let M be a 3-dimensional Lorentzian para-Kenmotsu manifold. Then, we have*

$$R(X, Y)Z = (\frac{r}{2} - 2)[g(Y, Z)X - g(X, Z)Y] + (\frac{r}{2} - 3)[\eta(Y)X - \eta(X)Y]\eta(Z) \tag{2.21}$$

$$+(\frac{r}{2} - 3)[g(Y, Z)\eta(X) - g(X, Z)\eta(Y)]\xi,$$

$$S(X, Y) = (\frac{r}{2} - 1)g(X, Y) + (\frac{r}{2} - 3)\eta(X)\eta(Y) \tag{2.22}$$

for any vectore fields $X, Y, Z \in \chi(M)$.

3 η-Ricci solitions on 3-dimensional Lorentzian para-Kenmotsu manifolds

Suppose that a 3-dimensional Lorentzian para-Kenmotsu manifold admits an η-Ricci soliton (g, ξ, λ, μ). Then (1.2) implies

$$(\pounds_\xi g)(Y, Z) + 2S(Y, Z) + 2\lambda g(Y, Z) + 2\mu\eta(Y)\eta(Z) = 0. \tag{3.1}$$

In a 3-dimensional Lorentzian para-Kenmotsu manifold, we have

$$(\pounds_\xi g)(Y, Z) = g(\nabla_Y \xi, Z) + g(Y, \nabla_Z \xi) = -2[g(Y, Z) + \eta(Y)\eta(Z)]. \tag{3.2}$$

By using (3.2) in (3.1), we get

$$S(Y,Z) = (1-\lambda)g(Y,Z) + (1-\mu)\eta(Y)\eta(Z). \tag{3.3}$$

From the last equation, it follows that

$$S(Y,\xi) = (\mu-\lambda)\eta(Y), \tag{3.4}$$

$$QY = (1-\lambda)Y + (1-\mu)\eta(Y)\xi, \tag{3.5}$$

$$Q\xi = (\mu-\lambda)\xi. \tag{3.6}$$

Comparing (3.3) with (2.21) we find $\frac{r}{2} - 1 = 1 - \lambda$ and $\frac{r}{2} - 3 = 1 - \mu$, therefore from these two relations we obtain $\mu - \lambda = 2$. Thus we have the following:

Theorem 3.1. *Let M be a 3-dimensional Lorentzian para-Kenmotsu manifold. If M admits an $\eta-$Ricci soliton (g,ξ,λ,μ), then $\mu - \lambda = 2$.*

4 $\eta-$Ricci solitons on 3-dimensional Lorentzian para-Kenmotsu manifold with Codazzi type of Ricci tensor and cyclic $\eta-$recurrent Ricci tensor

Definition 4.1. A 3-dimensional Lorentzian para-Kenmotsu manifold is said to have Codazzi type of Ricci tensor if its Ricci tensor S of type $(0,2)$ is non zero and satisfies following condition [1]

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z)$$

for all $X, Y, Z \in \chi(M)$.

Taking covariant derivative of (3.3) and making use of (2.11), we find

$$(\nabla_X S)(Y,Z) = (\mu-1)[g(X,Y)\eta(Z) + g(X,Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z)]. \tag{4.1}$$

If the Ricci tensor S is of Codazzi type, then

$$(\nabla_X S)(Y,Z) = (\nabla_Y S)(X,Z). \tag{4.2}$$

In view of (4.1), (4.2) takes the form

$$(\mu-1)[g(X,Z)\eta(Y) - g(Y,Z)\eta(X)] = 0$$

from which it follows that $\mu = 1$ and hence from Theorem 3.1, we find $\lambda = -1$. Thus we have the following:

Theorem 4.2. *Let (g,ξ,λ,μ) be an $\eta-$Ricci soliton in a 3-dimensional Lorentzian para-Kenmotsu manifold. If M has Codazzi type of Ricci tensor, then $\lambda = -1$ and $\mu = 1$.*

Definition 4.3. A 3-dimensional Lorentzian para-Kenmotsu manifold is said to have cyclic $\eta-$recurrent Ricci tensor, if

$$\begin{aligned}(\nabla_X S)(Y,Z) + (\nabla_Y S)(Z,X) + (\nabla_Z S)(X,Y)\\ = \eta(X)S(Y,Z) + \eta(Y)S(Z,X) + \eta(Z)S(X,Y)\end{aligned} \tag{4.3}$$

for all $X, Y, Z \in \chi(M)$.

By the cyclic permutations of X, Y and Z in (4.1), we write two more following equations:

$$(\nabla_Y S)(Z,X) = (\mu-1)[g(Y,Z)\eta(X) + g(Y,X)\eta(Z) + 2\eta(X)\eta(Y)\eta(Z)], \tag{4.4}$$

and

$$(\nabla_Z S)(X,Y) = (\mu-1)[g(Z,X)\eta(Y) + g(Z,Y)\eta(X) + 2\eta(X)\eta(Y)\eta(Z)]. \tag{4.5}$$

By using (4.1), (4.4) and (4.5) in (4.3), we obtain

$$(2\mu+\lambda-3)[g(X,Y)\eta(Z) + g(Y,Z)\eta(X) + g(Z,X)\eta(Y)] + 9(\mu-1)\eta(X)\eta(Y)\eta(Z) = 0$$

which by putting $Y = Z = \xi$ and making use of (2.1) and (2.5) gives $\mu - \lambda = 0$. Thus we can state:

Theorem 4.4. *Let (g,ξ,λ,μ) be an $\eta-$Ricci soliton on a 3-dimensional Lorentzian para-Kenmotsu manifold. If M has cyclic $\eta-$recurrent Ricci tensor, then $\mu - \lambda = 0$.*

5 $\eta-$Ricci solitons on 3-dimensional Lorentzian para-Kenmotsu manifolds satisfying $P(\xi, X) \cdot S = 0$

Suppose that a 3-dimensional Lorentzian para-Kenmotsu manifold admitting an $\eta-$Ricci soliton (g, ξ, λ, μ) satisfies $P(\xi, X) \cdot S = 0$. Then we have

$$S(P(\xi, X)Y, Z) + S(Y, P(\xi, X)Z) = 0 \tag{5.1}$$

which in view of (3.3) becomes

$$(1-\lambda)[g(P(\xi, X)Y, Z) + g(Y, P(\xi, X)Z)] + (1-\mu)[\eta(P(\xi, X)Y)\eta(Z) + \eta(Y)\eta(P(\xi, X)Z)] = 0. \tag{5.2}$$

From (2.1), (2.5), 2.13), (2.20), (3.3), and (3.4), we find

$$P(\xi, X)Y = (1 + \frac{\lambda - 1}{2})g(X, Y)\xi - (1 + \frac{\lambda - \mu}{2})\eta(Y)X + \frac{\mu - 1}{2}\eta(X)\eta(Y)\xi, \tag{5.3}$$

$$\eta(P(\xi, X)Y) = -(1 + \frac{\lambda - 1}{2})g(X, Y) - (1 + \frac{\lambda - \mu}{2})\eta(X)\eta(Y). \tag{5.4}$$

In view of (5.3) and (5.4), (5.2) takes the form

$$(1-\lambda)[(\frac{\mu - 1}{2})(g(X, Y)\eta(Z) + g(X, Z)\eta(Y)) + (\mu - 1)\eta(X)\eta(Y)\eta(Z)]$$

$$-\frac{(1-\mu)(1+\lambda)}{2}[g(X, Y)\eta(Z) + g(X, Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z)] = 0$$

from which it follows that

$$(\mu - 1)[g(X, Y)\eta(Z) + g(X, Z)\eta(Y) + 2\eta(X)\eta(Y)\eta(Z)] = 0. \tag{5.5}$$

Taking $Z = \xi$ and using (2.1), (2.4) and (2.5), (5.5) reduces to

$$(\mu - 1)g(\phi X, \phi Y) = 0 \tag{5.6}$$

from which we obtain $\mu = 1$ and hence from Theorem 3.1, we find $\lambda = -1$. By using these values of μ and λ in (3.3) we get $S(Y, Z) = 2g(Y, Z)$, from which we obtain $r = 6$. In a 3-dimensional semi-Riemannian manifold, we have

$$R(X, Y)Z = g(Y, Z)QX - g(X, Z)QY + S(Y, Z)X - S(X, Z)Y - \frac{r}{2}(g(Y, Z)X - g(X, Z)Y). \tag{5.7}$$

By using above values of S, Q and r in (5.7), we get

$$R(X, Y)Z = g(Y, Z)X - g(X, Z)Y. \tag{5.8}$$

Thus we have the following:

Theorem 5.1. *Let M be a 3-dimensional Lorentzian para-Kenmotsu manifold. If M satisfies the curvature condition $P(\xi, X) \cdot S = 0$, then M admits an $\eta-$Ricci soliton of type $(g, V, -1, 1)$ and is locally isometric to the unit sphere $S^3(1)$.*

6 $\eta-$Ricci solitons on 3-dimensional Lorentzian para-Kenmotsu manifolds satisfying $Q \cdot P = 0$

In this section we suppose that a 3-dimensional Lorentzian para-Kenmotsu manifold admitting an $\eta-$Ricci soliton satisfies $Q \cdot P = 0$. Then we have

$$Q(P(X, Y)Z) - P(QX, Y)Z - P(X, QY)Z - P(X, Y)QZ = 0 \tag{6.1}$$

for all $X, Y, Z \in \chi(M)$. In view of (2.20), (6.1) takes the form

$$Q(R(X,Y)Z) - R(QX,Y)Z - R(X,QY)Z - R(X,Y)QZ$$
$$+(1-\lambda)(S(Y,Z)X - S(X,Z)Y) + (1-\mu)(\mu-\lambda)(\eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y) = 0.$$

The inner product of last equation with ξ leads to

$$\eta(Q(R(X,Y)Z)) - \eta(R(QX,Y)Z) - \eta(R(X,QY)Z) - \eta(R(X,Y)QZ) \tag{6.2}$$
$$+(1-\lambda)(S(Y,Z)\eta(X) - S(X,Z)\eta(Y)) = 0.$$

Putting $Y = \xi$ in (6.2) and using (3.4), we have

$$\eta(Q(R(X,\xi)Z)) - \eta(R(QX,\xi)Z) - \eta(R(X,Q\xi)Z) - \eta(R(X,\xi)QZ) \tag{6.3}$$
$$+(1-\lambda)(S(X,Z) + (\mu-\lambda)\eta(X)\eta(Z)) = 0.$$

From (2.1), (2.5), (2.13), (3.5) and (3.6), we find

$$\eta(Q(R(X,\xi)Z)) = \eta(R(X,Q\xi)Z) = (\mu-\lambda)[g(X,Z) + \eta(X)\eta(Z)], \tag{6.4}$$
$$\eta(R(QX,\xi)Z) = \eta(R(X,\xi)QZ) = S(X,Z) + (\mu-\lambda)\eta(X)\eta(Z).$$

From (6.3) and (6.4), we obtain

$$(1+\lambda)(S(X,Z) + (\mu-\lambda)\eta(X)\eta(Z)) = 0.$$

It follows that $\lambda = -1$. Thus, likewise in the Section 5, we have the following:

Theorem 6.1. *Let M be a 3-dimensional Lorentzian para-Kenmotsu manifold. If M satisfies the curvature condition $Q \cdot P = 0$, then M admits an $\eta-$Ricci soliton of type $(g, V, -1, 1)$ and is locally isometric to the unit sphere $S^3(1)$.*

7 $\eta-$Ricci solitons on 3-dimensional Lorentzian para-Kenmotsu manifolds satisfying the curvature condition $S \cdot R = 0$

In this section we consider a 3-dimensional Lorentzian para-Kenmotsu manifold satisfying the curvature condition

$$(S(X,Y)\cdot R)(U,V)W = 0$$

for any vector fields $X, Y, U, V, W \in \chi(M)$. This implies that

$$(X\wedge_S Y)R(U,V)W + R((X\wedge_S Y)U,V)W \tag{7.1}$$
$$+R(U,(X\wedge_S Y)V)W + R(U,V)(X\wedge_S Y)W = 0.$$

We define endomorphisms $X \wedge_A Y$ by

$$(X \wedge_A Y)U = A(Y,U)X - A(X,U)Y, \tag{7.2}$$

where A is a symmetric $(0,2)$-tensor. By virtue of (7.2), (7.1) takes the form

$$S(Y,R(U,V)W)X - S(X,R(U,V)W)Y + S(Y,U)R(X,V)W \tag{7.3}$$
$$-S(X,U)R(Y,V)W + S(Y,V)R(U,X)W - S(X,V)R(U,Y)W$$
$$+S(Y,W)R(U,V)X - S(X,W)R(U,V)Y = 0.$$

Taking $U = W = \xi$ in (7.3), then using (2.14)-(2.16), we find

$$2S(Y,V)X - 2S(X,V)Y + 4\eta(Y)\eta(V)X$$
$$-4\eta(X)\eta(V)Y + \eta(X)S(Y,V)\xi - \eta(Y)S(X,V)\xi$$
$$+2g(V,X)\eta(Y)\xi - 2g(V,Y)\eta(X)\xi = 0.$$

Replacing X by ξ in the last equation and using (2.1), (2.5) and (2.16), we find

$$S(Y,V)\xi + 2g(V,Y)\xi + 4\eta(Y)\eta(V)\xi = 0$$

which by taking inner product with ξ yields

$$S(Y,V) = -2g(V,Y) - 4\eta(Y)\eta(V). \tag{7.4}$$

By making use of (7.4) in (3.1), we have

$$g(\nabla_Y \xi, V) + g(Y, \nabla_V \xi) + 2(\lambda - 1)g(Y,V) + 2(\mu - 4)\eta(Y)\eta(V) = 0. \tag{7.5}$$

Putting $Y = V = \xi$ in (7.5) and using (2.1) and (2.5), we find

$$g(\nabla_\xi \xi, V) + \mu - \lambda - 2 = 0.$$

Since $g(\nabla_\xi \xi, V) = 0$, so we get $\mu - \lambda = 2$. Now by using $\mu = 2 + \lambda$ in (7.5), we have

$$g(\nabla_Y \xi, V) + g(Y, \nabla_V \xi) + 2(\lambda - 2)(g(Y,V) + \eta(Y)\eta(V)) = 0. \tag{7.6}$$

Taking $Y = \xi$ in (7.6) and using (2.1), (2.5) and (2.10), we get $g(\nabla_\xi \xi, V) = 0$ for any vector field V on M and hence we have $\nabla_\xi \xi = 0$, that is, ξ is a geodesic vector field. Thus we have the following:

Theorem 7.1. *Let M be a 3-dimensional Lorentzian para-Kenmotsu manifold admiting an $\eta-$Ricci soliton (g, ξ, λ, μ). If M satisfies the curvature condition $S \cdot R = 0$, then*
1. $\mu - \lambda = 2$,
2. *ξ is a geodesic vector field.*

Example. We consider the 3-dimensional manifold $M = \{(x,y,z) \in R^3 : z > 0\}$, where (x,y,z) are the standard coordinates of R^3. Let e_1, e_2 and e_3 be the vector fields on M given by

$$e_1 = z\frac{\partial}{\partial x}, \quad e_2 = z\frac{\partial}{\partial y}, \quad e_3 = z\frac{\partial}{\partial z} = \xi,$$

which are linearly independent at each point of M. Let g be the Lorentzian metric defined by

$$g(e_1,e_1) = 1, \;\; g(e_2,e_2) = 1, \;\; g(e_3,e_3) = -1, \;\; g(e_1,e_2) = g(e_1,e_3) = g(e_2,e_3) = 0.$$

Let η be the 1-form defined by $\eta(X) = g(X, e_3) = g(X, \xi)$ for all $X \in \chi(M)$, and let ϕ be the $(1,1)$-tensor field defined by

$$\phi e_1 = -e_1, \;\; \phi e_2 = -e_2, \;\; \phi e_3 = 0.$$

By using linearity of ϕ and g, we have

$$\eta(\xi) = g(\xi,\xi) = -1, \;\; \phi^2 X = X + \eta(X)\xi \;\text{ and }\; g(\phi X, \phi Y) = g(X,Y) + \eta(X)\eta(Y)$$

for all $X, Y \in \chi(M)$. Thus for $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines a Lorentzian almost paracontact metric structure on M.
Also we have

$$[e_1,e_2] = 0, \;\; [e_2,e_1] = 0, \;\; [e_1,e_3] = -e_1, \;\; [e_3,e_1] = e_1, \;\; [e_2,e_3] = -e_2, \;\; [e_3,e_2] = e_2.$$

The Levi-Civita connection ∇ of the Lorentzian metric g is given by

$$2g(\nabla_X Y, Z) = Xg(Y,Z) + Yg(Z,X) - Zg(X,Y) - g(X,[Y,Z]) + g(Y,[Z,X]) + g(Z,[X,Y]),$$

which is known as Koszul's formula. Using Koszul's formula, we can easily calculate

$$\nabla_{e_1} e_1 = -e_3, \;\; \nabla_{e_1} e_2 = 0, \;\; \nabla_{e_1} e_3 = -e_1, \;\; \nabla_{e_2} e_1 = 0, \tag{7.7}$$

$$\nabla_{e_2} e_2 = -e_3, \;\; \nabla_{e_2} e_3 = -e_2, \;\; \nabla_{e_3} e_1 = 0, \;\;\; \nabla_{e_3} e_2 = 0, \;\;\; \nabla_{e_3} e_3 = 0.$$

Also, one can easily verify that

$$\nabla_X \xi = -X - \eta(X)\xi \quad \text{and} \quad (\nabla_X \phi)Y = -g(\phi X, Y)\xi - \eta(Y)\phi X.$$

Therefore, the manifold is a Lorentzian para-Kenmotsu manifold. It is known that

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z. \tag{7.8}$$

With the help of the above results we find the components of the Ricci tensor as follows:

$$R(e_1,e_2)e_1 = -e_2, \quad R(e_1,e_3)e_1 = -e_3, \quad R(e_2,e_3)e_1 = 0, \tag{7.9}$$

$$R(e_1,e_2)e_2 = e_1, \qquad R(e_1,e_3)e_2 = 0, \qquad R(e_2,e_3)e_2 = -e_3,$$

$$R(e_1,e_2)e_3 = 0, \qquad R(e_1,e_3)e_3 = -e_1, \quad R(e_2,e_3)e_3 = -e_2$$

from which it is clear that

$$R(X,Y)Z = g(Y,Z)X - g(X,Z)Y.$$

Thus the manifold (M, ϕ, ξ, η, g) is a Lorentzian para-Kenmotsu manifold of constant curvature 1 and hence is locally isometric to the unit sphere $S^3(1)$.
From (7.9), we calculate the Ricci tensors as follows:

$$S(e_1,e_1) = S(e_2,e_2) = 2, \quad S(e_3,e_3) = -2. \tag{7.10}$$

Therefore, $r = \sum_{i=1}^{3} \epsilon_i S(e_i, e_i) = 6$, where $\epsilon_i = g(e_i, e_i)$. Now from (3.3) and (7.10), we obtain $\lambda = -1$ and $\mu = 1$. Therefore the data (g, ξ, λ, μ) for $\lambda = -1$ and $\mu = 1$ defines an $\eta-$Ricci soliton on (M, ϕ, ξ, η, g).

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