Palestine Journal of Mathematics

Vol. 11(2)(2022) , 229–242

Mild and classical solutions for fractional evolution differential equation

J. Vanterler da C. Sousa [1], Thabet Abdeljawad * and D. S. Oliveira [2]

Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 26A33, 34A08, 34A12; Secondary 34G20, 47Dxx.

Keywords and phrases: Fractional evolution differential equation, existence and uniqueness, mild and classical solutions, Banach contraction theorem, Gronwall inequality.

Abstract Investigating the existence, uniqueness, stability, continuous dependence of data among other properties of solutions of fractional differential equations, has been the object of study by an important range of researchers in the scientific community, especially in fractional calculus. And over the years, these properties have been investigated more vehemently, as they enable more general and new results. In this paper, we investigate the existence and uniqueness of a class of mild and classical solutions of the fractional evolution differential equation in the Banach space Ω. To obtain such results, we use fundamental tools, namely: Banach contraction theorem, Gronwall inequality and the β-times integrated β-times integrated α-resolvent operator function of an (α, β)-resolvent operator function.

1 Introduction

It is common to hear from several researchers that better results were obtained when working with derivatives and fractional integrals [1, 3]. In fact, as time goes by the theory of fractional calculus has been increasingly consolidated, with well-founded results and consequently applicable results that allow us to better describe reality. Some papers are suggested for a brief read on some applications [2, 4] and references therein.

Differential equations have long appeared in many branches of engineering, physics, among others, and many concepts and methods have been developed to solve various differential equations, as we know they are fundamental tools for describing physical phenomena such as diffusion processes, world population growth, computational models, among others [4, 5, 6]. On the other hand, we also have an exponential growth in the interest in the existence, uniqueness, stability and continuous dependence of the data of mild, classical, strong solutions of fractional differential equations of the type: functional, impulsive, evolution, among others [15, 16, 22, 23]. Therefore, in recent years considerable attention has been given to investigating various types of fractional differential equations and certainly will continue to be studied for many long years.

In 2010 Zhang et. al. [24] investigated the existence and uniqueness of mild solutions for fractional neutral impulsive functional infinite delay integrodifferential systems with nonlocal initial conditions, through the fixed point theorem with the strongly continuous operator semigroup theory. In 2013, Hernández et al. [13] elaborated a work on some errors about abstract fractional differential equations, i.e., they proposed a different approach to this kind of problem involving the existence of mild solutions for a class of abstract fractional differential equations with nonlocal conditions. In 2014, Dubey and Sharma [10] decided to present a paper on mild and classical solutions for fractional functional differential equations with non-local conditions, highlighting the existence and uniqueness of global mild solutions in Banach space using the Banach fixed point theorem. In this sense, innumerable works on existence, uniqueness, attractivity and other properties of solutions of fractional differential equations, have been published, and consolidates this field. As there is a wide range of interesting and relevant work, we suggest some of these for further reading [9, 11, 12].

Discussing qualitative properties of solutions of fractional differential equations over these two decades has been the subject of study by numerous researchers, due to the importance and impact of the results obtained. The fact has been occurring for some reasons, in particular, for the possibilities that the order of the fractional operator that is directly linked to the solution of the problem, allows carrying out analyzes and obtain answers which the whole case does not allow. In addition, the information on the problem discussed in the fractional case, already contains as a particular case what will be discussed in the entire case, this a priory, is in fact advantageous, since the analysis of the results is not restricted to the entire case. There are many works involving fractional operators with α-resolvent, in most cases, they are in the sense of the Caputo and Riemann-Liouville fractional derivatives, however when it comes to the Hilfer fractional derivative, since this derivative is an interpolation between the Caputo and Riemann-Liouville fractional derivatives, there are few published works.

Since the general purpose of this paper is to discuss qualitative properties (existence and uniqueness) of solutions of fractional differential equations and to provide a wide class of possible particular cases from the choice of $\beta \to 1$, $\beta \to 0$ and the variation of $0 < \alpha \leq 1$, the results discussed here are more general and contribute significantly to area and also allowing the continuation of other works with the same type of fractional differential equations via Hilfer, as here addressed.

In this paper, we consider fractional functional differential evolution equations given by

$$^{H}\mathbb{D}^{\alpha,\beta}_{t_0^+}u(t) + \mathcal{A}u(t) = f\left(t, u(t), u(b_1(t)), \cdots, u(b_r(t))\right), \ t \in J/\{t_0\} \tag{1.1}$$

satisfying the condition

$$I^{1-\gamma}_{t_{0^+}}u(t_0^+) + \sum_{k=1}^{p}\lambda_k I^{1-\gamma}_{t_{0^+}}u(t_k) = u_0 \tag{1.2}$$

where $^{H}\mathbb{D}^{\alpha,\beta}_{t_0^+}(\cdot)$ is the Hilfer fractional derivative of order $0 < \alpha \leq 1$ and type $0 \leq \beta \leq 1$, $I^{1-\gamma}_{t_{0^+}}(\cdot)$ is the Riemann-Liouville fractional integral of order $1-\gamma(\gamma = \alpha + \beta(1-\alpha))$, $0 \leq \gamma \leq 1$ f and b_i $(i = 1, 2, \cdots, r)$ are given functions satisfying some assumptions $u_0 \in \mathbf{\Omega}$, $\lambda_k \neq 0\,(k = 1, 2, \cdots, p)$ and $p, r \in \mathbb{N}$.

The first step of the work, as highlighted above, was the formulation of the problem in the case Eq.(1.1)-Eq.(1.2) via the more general Hilfer fractional derivative (global) than the Caputo and Riemann-Liouville classics. In order to make clear what are the main contributions of this paper to the literature, we perform a rigorous analysis of Eq.(1.1) and our main strategy and results can be summarized as follows:

(i) We present a new class of mild and classic solutions has been introduced as we are free to choose $0 < \alpha \leq 1$ e $0 \leq \beta \leq 1$;

(ii) The mild and classic solutions are well defined as noted in Remark 1;

After discussing points 1 and 2 (above), we discuss the first main result of this paper, i.e., the following result:

Theorem 1.1. *If*

(i) $f : J \times \mathbf{\Omega}^{r+1} \to \mathbf{\Omega}$ *is continuous with respect to the first variable on* J, $b_i : J \to J$ $(i = 1, 2, \cdots, r)$ *are continuous on* J *and there is* $L > 0$ *such that*

$$\left\| f\left(s, z_0, z_1, \cdots, z_r\right) - f\left(s, \overline{z}_0, \overline{z}_1, \cdots, \overline{z}_r\right)\right\| \leq L\sum_{i=0}^{r}\left\|z_i - \overline{z}_i\right\|_{C_{1-\gamma}} \tag{1.3}$$

for $s \in J$, $z_i, \overline{z}_i \in \mathbf{\Omega}$, $i = 0, 1, \cdots, r$;

(ii) $(r+1)MLa\left(1 + M\left\|\mathbf{B}\right\|\sum_{k=1}^{p}|\lambda_k|\right) < 1$;

(iii) $u_0 \in \mathbf{\Omega}$;

(iv) $\kappa = \max\limits_{t \in J}\left\|f\left(s, 0, ..., 0\right)\right\|$;

(v) $M\left(\left\|\mathbf{B}\right\|\left\|u_0\right\| + aLrR + a\kappa\right) + M^2\left\|\mathbf{B}\right\|\sum\limits_{k=1}^{p}\lambda_k a\left(LrR + \kappa\right) \leq R$ *with* $R > 0$.

Then, the problem Eq.(1.1) *has a unique mild solution.*

The second main result of this paper, is to investigate the following:

Theorem 1.2. *Assume that* $f : J \times \Omega^{r+1} \to \Omega$ *is Lipschitz continuous on* $J \times \Omega^{r+1}$. *If* u *is a classical solution to the problem* Eq.(1.1) *then* u *is a mild solution of this problem.*

Theorem 1.3. *Suppose that*

(i) $f : J \times \Omega^{r+1} \to \Omega$, $b_i : J \to J (i = 1, 2, \cdots, r)$ *are continuous on* J *and there is* $\eta > 0$ *such that*

$$\|f(s, z_0, z_1, \cdots, z_r) - f(s, \overline{z}_0, \overline{z}_1, \cdots, \overline{z}_r\| \leq \eta \left(|s - \overline{s}| + \sum_{i=0}^{r} \|z_i - \overline{z}_i\|_{C_{1-\gamma}} \right); \quad (1.4)$$

for $s, \overline{s} \in J$, $z_i, \overline{z}_i \in \Omega$, $i = 0, 1, \cdots, r$;

(ii) $(r + 1)M\eta a \left(1 + M \|\mathbf{B}\| \sum_{k=1}^{p} |\lambda_k|\right) < 1$;

(iii) $u_0 \in \Omega$.

(iv) Then the fractional functional differential nonlocal evolution problem Eq.(1.1) *has a unique mild solution denoted by* u. *Moreover, if* $\mathbf{B}u_0 \in D(\mathcal{A})$ *and*

$$\mathbf{B} \int_{t_0^+}^{t_k} \mathbf{K}_\alpha(t_k - s) f(s, u(s), u(b_1(s)), \cdots, u(b_r(s)))\, \mathrm{d}s \in D(\mathcal{A}) \quad (1.5)$$

$k = 1, 2, \cdots, p$ *and if there is* $K > 0$ *such that*

$$\|u(b_i(s)) - u(b_i(\overline{s}))\|_{C_{1-\gamma}} \leq \|u(s) - u(\overline{s})\|, \text{ for } s, \overline{s} \in J \quad (1.6)$$

then u *is the unique classical solution to the problem* Eq.(1.1).

The information of the parameters for the Theorem 3.1, Theorem 4.2 and Theorem 4.3, will be presented in during the article.

Note that, from the choice of $0 < \alpha \leq 1$ and the limits of $\beta \to 1$ or $\beta \to 0$ in Eq.(1.1), we have a class of fractional functional differential evolution equation as particular cases. Especially, when we choose $\alpha = 1$ and one of the limits of $\beta \to 1$ or $\beta \to 0$, we have the integer case. As the solution is directly related to the fractional differential equation investigated, therefore, we also have their respective particular cases, i.e., the results investigated here, are also valid for their respective particular cases, preserving the investigated properties

This paper is organized as follows. In section 2, we present spaces with their respective norms, definitions of Riemann-Liouville fractional integral with respect to another function and ψ-Hilfer fractional derivative. In this sense, we present the concept of β-times integrated α-resolvent operator function of an (α, β)-resolvent operator function, Banach contraction theorem and Gronwall inequality, fundamental to achieve the proposed results. In section 3, we investigate our first and main result, namely, the existence and uniqueness of mild solutions for Eq.(1.1) towards the Hilfer fractional derivative using the Banach contraction theorem. To finalize the work, section 4, is intended for the second main result, to investigate the existence and uniqueness of classical solutions for Eq.(1.1) via the Gronwall inequality. Concluding remarks closing the paper.

2 Preliminaries

Let the interval $J' = [0, a]$. The weighted space of continuous functions is given by [20]

$$C_{1-\gamma}(J', \Omega) = \left\{\psi \in C(J', \Omega),\ t^{1-\gamma} u(t) \in C(J', \Omega)\right\}$$

where $0 \leq \gamma \leq 1$, with norm

$$\|u\|_{C_{1-\gamma}} = \sup_{t \in I} \left\|t^{1-\gamma} u(t)\right\|.$$

Note that Ω be a Banach space with norm $\|\cdot\|_{C_{1-\gamma}}$ and let $\mathcal{A}:\Omega\to\Omega$ be a closed densely-defined linear operator. For an operator $\mathcal{A}$, we denote by $D(\mathcal{A})$, $\rho(\mathcal{A})$ and $\mathcal{A}^*$, domain, resolvent set and adjoint, respectively [8].

For a Banach space Ω, $L(\Omega)$ denotes the set of closed linear operator from Ω into itself. We shall need the class $G(\widetilde{M},\beta)$ of operators $\mathcal{A}$ satisfying the conditions: There exists constants $\widetilde{M}>0$ and $\beta\in\mathbb{R}$ such that

(i) $\mathcal{A}\in L(\Omega)$, $\overline{D(\mathcal{A})}=\Omega$ and $(\beta,+\infty)\subset\rho(-\mathcal{A})$;

(ii) $\left\|(\mathcal{A}+\xi)^{-k}\right\|\leq\widetilde{M}(\xi-\beta)^{-k}$ for each $\xi>\beta$ and $k=1,2,\cdots$.

Assumption A: The adjoint operator $\mathcal{A}^*$ is densely defined in Ω^*, i.e., $\overline{D(\mathcal{A}^*)}=\Omega^*$.

Definition 2.1. [8] Let $\alpha>0$ and $\beta\geq 0$. A function $\mathbb{S}_{\alpha,\beta}:\mathbb{R}_+\to L(\Omega)$ is called a β-times integrated α-resolvent operator function of an (α,β)-resolvent operator function (ROF) if the following conditions are satisfied:

(A) $\mathbb{S}_{\alpha,\beta}(\cdot)$ is strongly continuous on $\mathbb{R}_+$and $\mathbb{S}_{\alpha,\beta}(0)=g_{\beta+1}(0)I$;

(B) $\mathbb{S}_{\alpha,\beta}(s)\mathbb{S}_{\alpha,\beta}(t)=\mathbb{S}_{\alpha,\beta}(t)\mathbb{S}_{\alpha,\beta}(s)$ for all $t,s\geq 0$;

(C) the function equation $\mathbb{S}_{\alpha,\beta}(s)I_t^{\alpha}\mathbb{S}_{\alpha,\beta}(t)-I_s^{\alpha}\mathbb{S}_{\alpha,\beta}(s)\mathbb{S}_{\alpha,\beta}(t)$

$=g_{\beta+1}(s)I_t^{\alpha}\mathbb{S}_{\alpha,\beta}(t)-g_{\beta+1}(t)I_s^{\alpha}\mathbb{S}_{\alpha,\beta}(s)$ for all $t,s\geq 0$.

The generator $\mathcal{A}$ of $\mathbb{S}_{\alpha,\beta}$ is defined by

$$D(\mathcal{A}):=\left\{x\in\Omega:\lim_{t\to 0^+}\frac{\mathbb{S}_{\alpha,\beta}(t)\,x-g_{\beta+1}(t)\,x}{g_{\alpha+\beta+1}(t)}\text{ exists}\right\}\tag{2.1}$$

and

$$\mathcal{A}\,x:=\lim_{t\to 0^+}\frac{\mathbb{S}_{\alpha,\beta}(t)\,x-g_{\beta+1}(t)\,x}{g_{\alpha+\beta+1}(t)},\quad x\in D(\mathcal{A}),\tag{2.2}$$

where $g_{\alpha+\beta+1}(t):=\dfrac{t^{\alpha+\beta}}{\Gamma(\alpha+\beta+1)}$ $(\alpha+\beta>0)$ for $t>0$.

It is known that for $\mathcal{A}\in G(\widetilde{M},\beta)$ there exists exactly one (α,β)-resolvent operator function $\mathbb{S}_{\alpha,\beta}(t):\Omega\to\Omega$ for $t\geq 0$ such that $-\mathcal{A}$ is the infinitesimal operator and

$$\|\mathbb{S}_{\alpha,\beta}(t)\|\leq\widetilde{M}\,e^{\beta t},\ \text{for } t\geq 0.$$

Throughout the paper we shall assume the conditions 1, 2 and Assumption A. Moreover, we shall use the notation $0\leq t_0<t_1<\cdots<t_p\leq t_0+a$, $a>0$; $J:=[t_0,t_0+a]$;

$$M:=\sup_{t\in[0,t_0+a]}\|\mathbb{S}_{\alpha,\beta}(t)\|$$

and $X:=C_{1-\gamma}(J,\Omega)$.

Also, we shall assume that there exists the operator $\mathbf{B}$ with $D(\mathbf{B})=\Omega$ given by the formula

$$\mathbf{B}:=\left(\mathrm{I}+\sum_{k=1}^{p}\lambda_k I_{0^+}^{1-\gamma}\mathbb{S}_{\alpha,\beta}(t_k-t_0)\right)^{-1}$$

where I is the identity operator on Ω.

Let (a,b) $(-\infty\leq a<b\leq\infty)$ be a finite interval (or infinite) of the real line $\mathbb{R}$ and let $\alpha>0$. Also let $\psi(t)$ be an increasing and positive monotone function on $(a,b]$, having a continuous derivative $\psi'(x)$ (we denote first derivative as $\dfrac{d}{dt}\psi(t)=\psi'(t)$) on (a,b). The left-sided fractional integral of a function f with respect to another function ψ on $[a,b]$ is defined by [19]

$$I_{a+}^{\alpha;\psi}f(t)=\frac{1}{\Gamma(\alpha)}\int_a^t\psi'(s)\,(\psi(t)-\psi(s))^{\alpha-1}f(s)\,ds.\tag{2.3}$$

On the other hand, let $n-1<\alpha<n$ with $n\in\mathbb{N}$, let $I'=[a,b]$ be an interval such that $-\infty\leq a<b\leq\infty$ and let $f,\psi\in C^n[a,b]$ be two functions such that ψ is increasing and $\psi'(t)\neq 0$, for all $t\in I'$. The left-sided $\psi-$Hilfer fractional derivative ${}^H\mathbb{D}_{a+}^{\alpha,\beta;\psi}(\cdot)$ of a function f of order α and type $0\leq\beta\leq 1$, is defined by [19]

$$^H\mathbb{D}_{a+}^{\alpha,\beta;\psi}f(t)=I_{a+}^{\beta(n-\alpha);\psi}\left(\frac{1}{\psi'(t)}\frac{d}{dt}\right)^n I_{a+}^{(1-\beta)(n-\alpha);\psi}f(t), \tag{2.4}$$

where $I_{a+}^{\varepsilon;\psi}(\cdot)$ is ψ-Riemann-Liouville fractional integral with $\varepsilon=\beta(n-\alpha)$ or $\varepsilon=(1-\beta)(n-\alpha)$.

Choosing $\psi(t)=t$ in Eq.(2.4), we have the Hilfer fractional derivative given by

$$^H\mathbb{D}_{a+}^{\alpha,\beta}f(t)=I_{a+}^{\beta(n-\alpha)}\left(\frac{d}{dt}\right)^n I_{a+}^{(1-\beta)(n-\alpha)}f(t),$$

where $I_{a+}^{\varepsilon}(\cdot)$ is Riemann-Liouville fractional integral with $\varepsilon=\beta(n-\alpha)$ or $\varepsilon=(1-\beta)(n-\alpha)$.

To study our problem Eq.(1.1), first we shall need the following linear problem,

$$\left\{\begin{array}{rcl} ^H\mathbb{D}_{t_0^+}^{\alpha,\beta}u(t)+\mathcal{A}u(t) & = & y(t),\ t\in J/\{t_0^+\} \\ I_{t_{0+}}^{1-\gamma}u(t_0^+) & = & u_0 \end{array}\right. \tag{2.5}$$

and the following definition.

We will introduce two fundamental results for the investigation of the main results of this paper, namely the Banach contraction theorem and the Gronwall Inequality.

Theorem 2.2. [20] (BANACH CONTRACTION PRINCIPLE). *Let (X,d) be a generalized complete metric space. Assume that $\Omega:X\to X$ is a strictly contractive operator with the Lipschitz constant $L<1$. If there exists a nonnegative integer k such that $d\left(\Omega^{k+1},\Omega^k\right)<\infty$ for some $x\in X$, then the following are true:*

(i) The sequence $\left\{\Omega^k x\right\}$ converges to a point x^* of Ω;

(ii) x^* is the unique fixed point of Ω in $\Omega^*=\left\{y\in X/d\left(\Omega^k x,y\right)<\infty\right\}$;

(iii) If $y\in X^*$, then $d(y,x^*)\leq\dfrac{1}{1-L}d(\Omega y,y)$.

Theorem 2.3. [21] (GRONWALL THEOREM). *Let u,v be two integrable functions and y a continuous function, it domain $[a,b]$. Let $\psi\in C^1(\Omega,\mathbb{R})$ an increasing function such that $\psi'(t)\neq 0$, $\forall t\in\Omega$. Assume that:*

(i) u and v are nonnegative;

(ii) y is nonnegative and nondecreasing.

If

$$u(t)\leq v(t)+y(t)\int_a^b \mathbf{Q}_\psi^\mu(t,s)\,u(s)\,ds,$$

then

$$u(t)\leq v(t)+\int_a^b\sum_{k=1}^\infty\frac{[y(t)\xi(\mu)]^k}{\Gamma(\mu k)}\mathbf{Q}_\psi^{k\mu}(t,s)\,v(s)\,ds,$$

$\forall t\in\Omega$ and $\mathbf{Q}_\psi^{k\mu}(t,s):=\psi'(s)(\psi(t)-\psi(s))^{k\mu-1}$.

Lemma 2.4. [21] (GRONWALL LEMMA) *Under the hypotheses of* Theorem 2.3*, let v be a nondecreasing function on Ω. Then, we have*

$$u(t)\leq v(t)\mathbb{E}_\mu\left(y(t)\Gamma(\mu)[(\psi(t)-\psi(a))^\mu]\right)$$

$t\in\Omega$, where $\mathbb{E}_\mu(\cdot)$ is the Mittag-Leffler function with one parameter given by $\mathbb{E}_\mu(t)=\sum_{k=0}^\infty\frac{t^k}{\Gamma(\mu k+1)}$ [17, 18].

Definition 2.5. A function $u : J \to \Omega$ is said to be a classical solution to the problem Eq.(1.1)

(i) u is continuous on J and continuously differentiable on $J/\{t_0^+\}$;

(ii) ${}^H\mathbb{D}^{\alpha,\beta}_{t_0^+} u(t) + \mathcal{A}u(t) = y(t)$, for $t \in J/\{t_0^+\}$;

(iii) $I^{1-\gamma}_{t_{0^+}} u(t_0^+) = u_0$.

To study problem Eq.(1.1), we shall need the following theorem.

Theorem 2.6. *Let $y : J \to \Omega$ be Lipschitz continuous on J and $u_0 \in D(\mathcal{A})$. Then, the Cauchy problem* Eq.(2.5) *has exactly one classical solution, denoted by u, given by the formula*

$$u(t) = \mathbb{S}_{\alpha,\beta}(t - t_0)u_0 + \int_{t_0^+}^{t} \mathbf{K}_\alpha(t - s)y(s)\, \mathrm{d}s, \quad t \in J.$$

On the other hand, a function $u \in \Lambda$ (Λ is a Banach space) satisfying the integral equation

$$\begin{aligned} u(t) &= \mathbb{S}_{\alpha,\beta}(t - t_0)\mathbf{B}u_0 + \int_{t_0^+}^{t} \mathbf{K}_\alpha(t - s) f(s, u(s), u(b_1(s)), \cdots, u(b_r(s)))ds \\ &- \mathbb{S}_{\alpha,\beta}(t - t_0)\mathbf{B} \sum_{k=1}^{p} \lambda_k I^{1-\gamma}_{t_0^+} \int_{t_0^+}^{t_k} \mathbf{K}_\alpha(t_k - s) f(s, u(s), u(b_1(s)), \cdots, u(b_r(s)))ds \end{aligned} \tag{2.6}$$

is said to be a mild solution of the fractional functional differential nonlocal evolution Cauchy problem Eq.(1.1) satisfying the condition Eq.(1.2), where $\mathbf{K}_\alpha(t) = t^{\alpha-1} G_\alpha(t)$, $G_\alpha(t) = \int_0^\infty \alpha\theta M_\alpha(\theta)\, \mathcal{A}(t^\alpha\theta)\, d\theta$, $\mathbb{S}_{\alpha,\beta}(t) = I_\theta^{\beta(1-\alpha)} \mathbf{K}_\alpha(t)$, $0 < \alpha \le 1$ and $0 \le \beta \le 1$. For more details see [22] and references therein.

Remark 2.7. A mild solution of the nonlocal fractional Cauchy problem Eq.(1.1) satisfying the condition Eq.(1.2). Indeed, observe that by Eq.(2.6)

$$u(t_0) = \mathbb{S}_{\alpha,\beta}(0)\mathbf{B}u_0 - \mathbb{S}_{\alpha,\beta}(0)\mathbf{B} \sum_{k=1}^{p} \lambda_k I^{1-\gamma}_{0^+} \int_{t_0}^{t_k} \mathbf{K}_\alpha(t_k - s) f(s, u(s), u(b_1(s)), \cdots, u(b_r(s)))ds \tag{2.7}$$

and

$$\begin{aligned} u(t_i) &= \mathbb{S}_{\alpha,\beta}(t_i - t_0)\mathbf{B}u_0 + \int_{t_0^+}^{t_i} K_\alpha(t_i - s) f(s, u(s), u(b_1(s)), \cdots, u(b_r(s)))ds \\ &\quad - \mathbb{S}_{\alpha,\beta}(t_i - t_0)\mathbf{B} \sum_{k=1}^{p} \lambda_k I^{1-\gamma}_{t_0^+} \int_{t_0^+}^{t_k} \mathbf{K}_\alpha(t_k - s) f(s, u(s), u(b_1(s)), \cdots, u(b_r(s)))\, ds. \end{aligned}$$

From Eq.(2.7) and Eq.(2.8) and the definition of operator $\mathbf{B}$ we obtain the formula

$$\begin{aligned} & I^{1-\gamma}_{t_0^+} u(t_0) + \sum_{i=1}^{p} \lambda_i I^{1-\gamma}_{t_0^+} u(t_i) \\ = & \left(I^{1-\gamma}_{t_0^+} \mathbb{S}_{\alpha,\beta}(0) + \sum_{i=1}^{p} \lambda_i I^{1-\gamma}_{t_0^+} \mathbb{S}_{\alpha,\beta}(t_i = t_0) \right) \mathbf{B}u_0 \\ & - \left(I^{1-\gamma}_{t_0^+} \mathbb{S}_{\alpha,\beta}(0) + \sum_{i=1}^{p} \lambda_i I^{1-\gamma}_{t_0^+} \mathbb{S}_{\alpha,\beta}(t_i = t_0) \right) \mathbf{B} \times \\ & \times \sum_{k=1}^{p} \lambda_k I^{1-\gamma}_{t_0^+} \int_{t_0^+}^{t_k} \mathbf{K}_\alpha(t_k - s) f(s, u(s), u(b_1(s)), \cdots, u(b_r(s)))ds \\ & + \sum_{i=1}^{p} \lambda_i I^{1-\gamma}_{t_0^+} \int_{t_0^+}^{t_i} \mathbf{K}_\alpha(t_i - s) f(s, u(s), u(b_1(s)), \cdots, u(b_r(s)))\, ds. \\ & = u_0. \end{aligned} \tag{2.8}$$

Thus we have that the mild solution given by Eq.(1.1) satisfies the condition given by Eq.(1.2), so it is well defined.

3 Existence and Uniqueness of mild solution

In this section, we investigate the existence and uniqueness of mild solutions for the fractional functional differential evolution equations introduced by the Hilfer fractional derivative and using the Banach contraction principle.

Theorem 3.1. *If*

(i) $f : J \times \Omega^{r+1} \to \Omega$ *is continuous with respect to the first variable on* J*,* $b_i : J \to J$ $(i = 1, 2, \cdots, r)$ *are continuous on* J *and there is* $L > 0$ *such that*

$$\|f(s, z_0, z_1, \cdots, z_r) - f(s, \overline{z}_0, \overline{z}_1, \cdots, \overline{z}_r)\| \leq L \sum_{i=0}^{r} \|z_i - \overline{z}_i\|_{C_{1-\gamma}} \tag{3.1}$$

for $s \in J$*,* $z_i, \overline{z}_i \in \Omega$*,* $i = 0, 1, \cdots, r$*;*

(ii) $(r+1)MLa\left(1 + M\|\mathbf{B}\| \sum_{k=1}^{p} |\lambda_k|\right) < 1$*;*

(iii) $u_0 \in \Omega$*;*

(iv) $\kappa = \max\limits_{t \in J} \|f(s, 0, ..., 0)\|$*;*

(v) $M\left(\|\mathbf{B}\|\|u_0\| + aLrR + a\kappa\right) + M^2\|\mathbf{B}\| \sum\limits_{k=1}^{p} \lambda_k a\left(LrR + \kappa\right) \leq R$ *with* $R > 0$*.*

Then, the problem Eq.(1.1) *has a unique mild solution.*

Proof. Let $B_R = \left\{w : \|w\|_{C_{1-\gamma}} \leq R\right\} \subset \Lambda$ where Λ is a Banach space and consider the following operator

$$\begin{aligned}(F\omega)(t) \quad : \quad &= \mathbb{S}_{\alpha,\beta}(t-t_0)\mathbf{B}u_0 - \mathbb{S}_{\alpha,\beta}(t-t_0)\mathbf{B} \times \\ &\times \sum_{k=1}^{p} \lambda_k I_{t_0^+}^{1-\gamma} \int_{t_0^+}^{t_k} \mathbf{K}_\alpha(t_k - s) f(s, \omega(s), \omega(b_1(s)), \cdots, \omega(b_r(s)))ds \\ &+ \int_{t_0^+}^{t} \mathbf{K}_\alpha(t-s) f(s, \omega(s), \omega(b_1(s)), \cdots, \omega(b_r(s)))ds \end{aligned} \tag{3.2}$$

with $t \in J$ and $\omega \in \Lambda$.

Let's prove that $FB_R \subset B_R$. To this end, from Eq.(3.2) and our hypotheses, we obtain

$$\begin{aligned} & \|(Fw)(t)\| \\ \leq\ & \|\mathbb{S}_{\alpha,\beta}(t-t_0)\| \|\mathbf{B}\| \|u_0\| \\ & + \|\mathbb{S}_{\alpha,\beta}(t-t_0)\| \|\mathbf{B}\| \sum_{k=1}^{p} \lambda_k \left\| I_{t_{0+}}^{1-\gamma} \right\| \int_{t_{0+}}^{t_k} \|\mathbf{K}_\alpha(t_k - s)\| \times \\ & \|f(s, w(s), w(b_1(s)), ..., w(b_r(s)))\| \, ds \\ & + \int_{t_{0+}}^{t} \|\mathbf{K}_\alpha(t-s)\| \|f(s, w(s), w(b_1(s)), ..., w(b_r(s)))\| \, ds \end{aligned}$$

$$
\begin{aligned}
&\le \left\|\mathbb{S}_{\alpha,\beta}\left(t-t_0\right)\right\|\left\|\mathbf{B}\right\|\left\|u_0\right\| \\
&\quad +\left\|\mathbb{S}_{\alpha,\beta}\left(t-t_0\right)\right\|\left\|\mathbf{B}\right\|\sum_{k=1}^{p}\lambda_k\left\|I_{t_{0+}}^{1-\gamma}\right\|\int_{t_{0+}}^{t_k}\left\|\mathbf{K}_\alpha\left(t_k-s\right)\right\|\times \\
&\quad \left(\left\|f\left(s,w\left(s\right),w\left(b_1\left(s\right)\right),...,w\left(b_r\left(s\right)\right)\right)-f\left(s,0,...,0\right)\right\|+\left\|f\left(s,0,...,0\right)\right\|\right)ds \\
&\quad +\int_{t_{0+}}^{t}\left\|\mathbf{K}_\alpha\left(t-s\right)\right\|\left(\begin{array}{c}\left\|f\left(s,w\left(s\right),w\left(b_1\left(s\right)\right),...,w\left(b_r\left(s\right)\right)\right)-f\left(s,0,...,0\right)\right\| \\ +\left\|f\left(s,0,...,0\right)\right\|\end{array}\right)ds \\
&\le M\left\|\mathbf{B}\right\|\left\|u_0\right\|+M^2\left\|\mathbf{B}\right\|\sum_{k=1}^{p}\lambda_k a\left(Lr\left\|w\right\|_{C_{1-\gamma}}+\kappa\right)+Ma\left(Lr\left\|w\right\|_{C_{1-\gamma}}+\kappa\right) \\
&\le M\left(\left\|\mathbf{B}\right\|\left\|u_0\right\|+aLrR+a\kappa\right)+M^2\left\|\mathbf{B}\right\|\sum_{k=1}^{p}\lambda_k a\left(LrR+\kappa\right) \\
&\le R
\end{aligned}
\tag{3.3}
$$

for $w\in B_R$ and $t\in J$. Hence $\left\|Fw\right\|_{C_{1-\gamma}}\le R$. Therefore, the Eq.(3.3) shows that the operator F maps B_R into itself.

Now, we go prove that in fact F is a contraction, and so use Banach contraction theorem. So, we have

$$
\begin{aligned}
&\left\|(F\omega)(t)-(F\widetilde{\omega})(t)\right\| \\
&= -\mathbb{S}_{\alpha,\beta}(t-t_0)\mathbf{B}\sum_{k=1}^{p}\lambda_k I_{t_0^+}^{1-\gamma}\int_{t_0}^{t_k}\left[\begin{array}{c} f\left(s,\omega\left(s\right),\omega\left(b_1\left(s\right)\right),\cdots,\omega\left(b_r\left(s\right)\right)\right) \\ -f(s,\widetilde{\omega}(s),\widetilde{\omega}(b_1(s)),\cdots,\overline{\omega}(b_r(s)))\end{array}\right]ds \\
&\quad +\int_{t_0}^{t}\mathbf{K}_\alpha(t-s)\left[\begin{array}{c} f(s,\omega(s),\omega(b_1(s)),\cdots,\omega(b_r(s))) \\ -f(s,\widetilde{\omega}(s),\widetilde{\omega}(b_1(s)),\cdots,\overline{\omega}(b_r(s)))\end{array}\right]ds \\
&\le \sum_{k=1}^{p}\left|\lambda_k\right|\left\|\mathbb{S}_{\alpha,\beta}(t-t_0)\right\|\left\|\mathbf{B}\right\|\left\|I_{t_0}^{1-\gamma}\right\|\times \\
&\quad \times\int_{t_0^+}^{t_k}\left\|\mathbf{K}_\alpha(t_k-s)\right\|\left\|f(s,\widetilde{\omega}(s),\widetilde{\omega}(b_1(s)),\cdots,\widetilde{\omega}(b_r(s)))\right\|ds \\
&\quad +\int_{t_0}^{t}\left\|\mathbf{K}_\alpha(t-s)\right\|\left\|\begin{array}{c} f(s,\omega(s),\omega(b_1(s)),\cdots,\omega(b_r(s))) \\ -f(s,\widetilde{\omega}(s),\widetilde{\omega}(b_1(s)),\cdots,\overline{\omega}(b_r(s)))\end{array}\right\|ds \\
&\le \sum_{k=1}^{p}\left|\lambda_k\right|M\left\|\mathbf{B}\right\|\int_{t_0^+}^{t_k}M\,L\sum_{i=0}^{r}\left\|\omega-\widetilde{\omega}\right\|_{C_{1-\gamma}}ds+\int_{t_0^+}^{t}M\,L\sum_{i=0}^{r}\left\|\omega-\widetilde{\omega}\right\|_{C_{1-\gamma}}ds \\
&= M^2Lr\left\|\omega-\widetilde{\omega}\right\|_{C_{1-\gamma}}\sum_{k=1}^{p}\left|\lambda_k\right|\left\|\mathbf{B}\right\|\left(t_k-t_0\right)+L\,M\,r\left\|\omega-\widetilde{\omega}\right\|_{C_{1-\gamma}}\left(t-t_0\right) \\
&\le M^2aLr\left\|\omega-\widetilde{\omega}\right\|_{C_{1-\gamma}}\left\|\mathbf{B}\right\|\sum_{k=1}^{p}\left|\lambda_k\right|+LMra\left\|\omega-\widetilde{\omega}\right\|_{C_{1-\gamma}} \\
&= LMra\left(1+M\left\|\mathbf{B}\right\|\sum_{k=1}^{p}\left|\lambda_k\right|\right)\left\|\omega-\widetilde{\omega}\right\|_{C_{1-\gamma}} \\
&\le (1+r)MLa\left(1+M\left\|\mathbf{B}\right\|\sum_{k=1}^{p}\left|\lambda_k\right|\right)\left\|\omega-\widetilde{\omega}\right\|_{C_{1-\gamma}}.
\end{aligned}
$$

Then, we have

$$\left\|F\omega-F\widetilde{\omega}\right\|_{C_{1-\gamma}}\le\widetilde{q}\left\|\omega-\widetilde{\omega}\right\|_{C_{1-\gamma}} \tag{3.4}$$

where

$$\widetilde{q} := (1+r)MLa\left(1 + M\left\|\mathbf{B}\right\|_{C_{1-\gamma}}\sum_{k=1}^{p}|\lambda_k|\right)$$

with $0 < \widetilde{q} < 1$.

Thus, we have that inequality (3.4) satisfies the conditions of the Banach contraction theorem (see Theorem 2.2). Therefore, we can ensure that there is a single F operator fixed point in the space Λ and this point is the mild solution of the problem Eq.(1.1) satisfying the condition Eq.(1.2). □

4 Existence and uniqueness of classical solutions

Before beginning to investigate the results of this section, let us present the definition of the classical solution to the problem Eq.(1.1) satisfying the condition Eq.(1.2).

Definition 4.1. A function $u : J \to \Omega$ is said to be a classical solution of the fractional functional differential nonlocal evolution Cauchy problem Eq.(1.1) if

(i) u is continuous on J and continuously differentiable on $J/\{t_0^+\}$;

(ii) ${}^{H}\mathbb{D}_{t_0^+}^{\alpha,\beta}u(t) + \mathcal{A}u(t) = f(t, u(t), u(b_1(t)), \cdots, u(b_r(t)))$, for $t \in J/\{t_0^+\}$;

(iii) $I_{t_{0+}}^{1-\gamma}u(t_0) + \sum_{k=1}^{p}\lambda_k I_{t_{0+}}^{1-\gamma}u(t_k) = u_0$.

The first result of this section, is a direct consequence of the u solution being said classical, i.e., we investigate that if u is classical from the problem Eq.(1.1) satisfying the condition Eq.(1.2), so it's mild.

Theorem 4.2. *Assume that $f : J \times \Omega^{r+1} \to \Omega$ is Lipschitz continuous on $J \times \Omega^{r+1}$. If u is a classical solution to the problem* Eq.(1.1) *then u is a mild solution of this problem.*

Proof. Let u be a classical solution of the problem Eq.(1.1). Then u satisfies Eq.(1.1) and consequently,

$$u(t) = \mathbb{S}_{\alpha,\beta}(t-t_0)\, I_{t_{0+}}^{1-\gamma}u(t_0) + \int_{t_{0+}}^{t}\mathbf{K}_\alpha(t-s)\, f(s, u(s), u(b_1(s)), ..., u(b_r(s)))\, ds \tag{4.1}$$

with $t \in J$.

From Eq.(4.1), we have

$$u(t_k) = \mathbb{S}_{\alpha,\beta}(t_k-t_0)\, I_{t_{0+}}^{1-\gamma}u(t_0) + \int_{t_{0+}}^{t_k}\mathbf{K}_\alpha(t_k-s)\, f(s, u(s), u(b_1(s)), ..., u(b_r(s)))\, ds. \tag{4.2}$$

By condition 3 and Eq.(4.2), we obtain

$$\begin{aligned} & I_{t_{0+}}^{1-\gamma}u(t_0) + \sum_{k=1}^{p}\lambda_k I_{t_{0+}}^{1-\gamma}\mathbb{S}_{\alpha,\beta}(t_k-t_0)\, I_{t_{0+}}^{1-\gamma}u(t_0) + \\ & + \int_{t_{0+}}^{t_k}\mathbf{K}_\alpha(t_k-s)\, f(s, u(s), u(b_1(s)), ..., u(b_r(s)))\, ds \\ = \; & u_0 \end{aligned}$$

or it can also be written as

$$\begin{aligned} u_0 \;=\; & \left(I + \sum_{k=1}^{p}\lambda_k I_{t_{0+}}^{1-\gamma}\mathbb{S}_{\alpha,\beta}(t_k-t_0)\right) I_{t_{0+}}^{1-\gamma}u(t_0) \\ & + \sum_{k=1}^{p}\lambda_k I_{t_{0+}}^{1-\gamma}\int_{t_{0+}}^{t_k}\mathbf{K}_\alpha(t_k-s)\, f(s, u(s), u(b_1(s)), ..., u(b_r(s)))\, ds. \end{aligned} \tag{4.3}$$

Applying the operator $\mathbf{B}$ on both sides of Eq.(4.3), we obtain

$$I_{t_{0+}}^{1-\gamma}u\left(t_0\right)=\mathbf{B}u_0-\mathbf{B}\sum_{k=1}^{p}\lambda_k I_{t_{0+}}^{1-\gamma}\int_{t_{0+}}^{t_k}\mathbf{K}_\alpha\left(t_k-s\right)f\left(s,u\left(s\right),u\left(b_1\left(s\right)\right),...,u\left(b_r\left(s\right)\right)\right)ds. \tag{4.4}$$

Then, we have by means of Eq..(4.1) and Eq.(4.4) imply that u satisfies Eq.(2.6), which completes the proof. □

To finish the section, we will investigate the uniqueness of classical solutions of the problem Eq.(1.1) satisfying the condition Eq.(1.2), through the following result.

Theorem 4.3. *Suppose that*

(i) $f:J\times\Omega^{r+1}\to\Omega$, $b_i:J\to J(i=1,2,\cdots,r)$ *are continuous on* J *and there is* $\eta>0$ *such that*

$$\|f(s,z_0,z_1,\cdots,z_r)-f(s,\overline{z}_0,\overline{z}_1,\cdots,\overline{z}_r\|\leq\eta\left(|s-\overline{s}|+\sum_{i=0}^{r}\|z_i-\overline{z}_i\|_{C_{1-\gamma}}\right); \tag{4.5}$$

for $s,\overline{s}\in J$, $z_i,\overline{z}_i\in\Omega$, $i=0,1,\cdots,r$*;*

(ii) $(r+1)M\eta a\left(1+M\|\mathbf{B}\|\sum_{k=1}^{p}|\lambda_k|\right)<1$*;*

(iii) $u_0\in\Omega$.

(iv) Then the fractional functional differential nonlocal evolution problem Eq.(1.1) *has a unique mild solution denoted by* u*. Moreover, if* $\mathbf{B}u_0\in D(\mathcal{A})$ *and*

$$\mathbf{B}\int_{t_0^+}^{t_k}\mathbf{K}_\alpha(t_k-s)f(s,u(s),u(b_1(s)),\cdots,u(b_r(s)))\,\mathrm{d}s\in D(\mathcal{A}) \tag{4.6}$$

$k=1,2,\cdots,p$ *and if there is* $K>0$ *such that*

$$\|u(b_i(s))-u(b_i(\overline{s}))\|_{C_{1-\gamma}}\leq\|u(s)-u(\overline{s})\|\,,\text{ for }s,\overline{s}\in J \tag{4.7}$$

then u *is the unique classical solution to the problem* Eq.(1.1).

Proof. Note that, Eq.(1.1) satisfying Eq.(1.2), admits a unique solution, since the conditions of Theorem 3.1 are satisfied. On the other hand, it remains to be proved that u is indeed a unique classical solution for Eq.(1.1). Consider

$$N:=\max_{s\in J}\|f(s,u(s),u(b_1(s)),\cdots,u(b_r(s)))\| \tag{4.8}$$

and note that

$$\begin{aligned}
&u(t+h)-u(t)\\
=\;&\left(\mathbb{S}_{\alpha,\beta}(t+h-t_0)-\mathbb{S}(t-t_0)\right)\mathbf{B}u_0-\left(\mathbb{S}_{\alpha,\beta}(t+h-t_0)-\mathbb{S}(t-t_0)\right)\mathbf{B}\times\\
&\times\sum_{k=1}^{p}\lambda_k I_{t_{0^+}}^{1-\gamma}\int_{t_0^+}^{t_k}\mathbf{K}_\alpha(t_k-s)f(s,u(s),u(b_1(s)),\cdots,u(b_r(s)))ds\\
&+\int_{t_0^+}^{t_0+h}\mathbf{K}_\alpha(t+h-s)f(s,u(s),u(b_1(s)),\cdots,u(b_r(s)))ds\\
&+\int_{t_0^+}^{t}\mathbf{K}_\alpha(t-s)\left[\begin{array}{c}f(s,u(s+h),u(b_1(s+h)),\cdots,u(b_r(s+h)))\\-f(s,u(s),u(b_1(s)),\cdots,u(b_r(s)))\end{array}\right]ds
\end{aligned} \tag{4.9}$$

for $t\in[t_0,t_0+a]$, $h>0$ and $t+h\in(t_0,t_0+a)$.

Consequently by Eq.(4.7), Eq.(4.8), Eq.(4.9) and a condition 4, we obtain

$$\begin{aligned}
& \|u(t+h)-u(t)\| \\
= & \left\| \begin{array}{c} (\mathbb{S}_{\alpha,\beta}(t+h-t_0)-\mathbb{S}_{\alpha,\beta}(t-t_0))\,\mathbf{B}u_0-(\mathbb{S}_{\alpha,\beta}(t+h-t_0)-\mathbb{S}_{\alpha,\beta}(t-t_0))\mathbf{B}\times \\ \times\sum_{k=1}^{p}\lambda_k I_{t_0+}^{1-\gamma}\displaystyle\int_{t_0^+}^{t_k}\mathbf{K}_\alpha(t_k-s)f(s,u(s),u(b_1(s)),\cdots,u(b_r(s)))ds \\ +\displaystyle\int_{t_0^+}^{t_0+h}\mathbf{K}_\alpha(t+h-s)f(s,u(s),u(b_1(s)),\cdots,u(b_r(s)))ds \\ +\displaystyle\int_{t_0^+}^{t}\mathbf{K}_\alpha(t-s)[f(s,u(s+h),u(b_1(s+h)),\cdots,u(b_r(s+h))) \\ -f(s,u(s),u(b_1(s)),\cdots,u(b_r(s)))]ds \end{array} \right\| \\
\leq & \left(\|\mathbb{S}_{\alpha,\beta}(t+h-t_0)\|+\|\mathbb{S}_{\alpha,\beta}(t-t_0)\|\right)\|\mathbf{B}u_0\|+\left(\|\mathbb{S}_{\alpha,\beta}(t+h-t_0)\|+\|\mathbb{S}_{\alpha,\beta}(t-t_0)\|\right)\times \\
& \times\|\mathbf{B}\|\sum_{k=1}^{p}\lambda_k\left\|I_{t_0+}^{1-\gamma}\right\|\int_{t_0^+}^{t_k}\|\mathbf{K}_\alpha(t_k-s)\|\,\|f(s,u(s),u(b_1(s)),\cdots,u(b_r(s)))\|\,ds \\
& +\int_{t_0^+}^{t_0+h}\|\mathbf{K}_\alpha(t+h-s)\|\,\|f(s,u(s),u(b_1(s)),\cdots,u(b_r(s)))\|\,ds \\
& +\int_{t_0^+}^{t}\|\mathbf{K}_\alpha(t-s)\|\left\| \begin{array}{c} f(s,u(s+h),u(b_1(s+h)),\cdots,u(b_r(s+h))) \\ -f(s,u(s),u(b_1(s)),\cdots,u(b_r(s))) \end{array} \right\| ds \\
\leq & 2Mh\|\mathbf{B}u_0\|+2Mh\|\mathbf{B}\|\sum_{k=1}^{p}\lambda_k\int_{t_0^+}^{t_k}\|\mathbf{K}_\alpha(t_k-s)f(s,u(s),u(b_1(s)),\cdots,u(b_r(s)))\|_{C_{1-\gamma}}ds \\
& +MNh+M\eta\int_{t_0^+}^{t}\|u(s+h)-u(s)\|_{C_{1-\gamma}}ds+M\eta k\int_{t_0^+}^{t}\|u(s+h)-u(s)\|\,ds \\
& +\cdots+M\eta kr\int_{t_0^+}^{t}\|u(s+h)-u(s)\|\,ds \\
= & 2Mh\|\mathbf{B}u_0\|+2Mh\|\mathbf{B}\|\sum_{k=1}^{p}\lambda_k\times \\
& \times\int_{t_0^+}^{t_k}\|\mathbf{K}_\alpha(t_k-s)f(s,u(s),u(b_1(s)),\cdots,u(b_r(s)))\|\,ds \\
& +MNh+M\eta(1+k+kr)\int_{t_0^+}^{t}\|u(s+h)-u(s)\|\,ds \\
= & \left[\begin{array}{c} 2M\|\mathbf{B}u_0\|+2M\|\mathbf{B}\|\sum_{k=1}^{p}\lambda_k\times \\ \times\displaystyle\int_{t_0^+}^{t_k}\|\mathbf{K}_\alpha(t_k-s)f(s,u(s),u(b_1(s)),\cdots,u(b_r(s)))\|\,ds+MN \end{array} \right] h \\
& +M\eta(1+k+kr)\int_{t_0^+}^{t}\|u(s+h)-u(s)\|\,ds \\
= & \widetilde{\delta}h+M\eta(1+k+kr)\int_{t_0^+}^{t}\|u(s+h)-u(s)\|\,ds \qquad (4.10)
\end{aligned}$$

for $t\in[t_0,t_0+a)$, $h>0$ and $t+h\in(t_0,t_0+a)$, where

$$\widetilde{\delta}:=2M\|\mathbf{B}u_0\|+M^2N2\|\mathbf{B}\|\sum_{k=1}^{p}\lambda_k\int_{t_0^+}^{t_k}\|\mathbf{K}_\alpha(t_k-s)f(s,u(s),u(b_1(s)),\cdots,u(b_r(s)))\|\,ds.$$

Note that, for $0 < \alpha < 1$, $t \in [t_0, t_0 + a)$, Eq.(4.10) can rewritten in the following form

$$\begin{aligned} \|u(t+h)-u(t)\| &\leq \widetilde{\delta}h + M\eta(1+k+rk)\int_{t_0^+}^{t} \|u(s+h)-u(s)\|\, ds \\ &\leq \widetilde{\delta}h + M\eta(1+rk)t^{1-\alpha}\int_{t_0^+}^{t} (t-s)^{\alpha-1}\|u(s+h)-u(s)\|\, ds \\ &\leq \widetilde{\delta}h + M\eta(1+rk)p\int_{t_0^+}^{t} (t-s)^{\alpha-1}\|u(s+h)-u(s)\|\, ds. \end{aligned} \tag{4.11}$$

By means of inequality (4.11) and of Gronwall inequality (see Lemma 2.4) we get

$$\begin{aligned} \|u(t+h)-u(t)\|_{C_{1-\gamma}} &\leq \widetilde{\delta}h\mathbb{E}_{\alpha}\left[M\eta(1+rk)\Gamma(\alpha)(t-s)^{\alpha}\right] \\ &\leq \widetilde{\delta}h\mathbb{E}_{\alpha}\left[M\eta(1+rk)\Gamma(\alpha)a^{\alpha}\right] \end{aligned} \tag{4.12}$$

for $t \in [t_0, t_0 + a)$, $h > 0$, $t + h \in (t_0, t_0 + a)$ and $\mathbb{E}_{\alpha}(\cdot)$ is the classical Mittag-Leffler function of a parameter.

Note that the continuity of u and of f on J and $J \times \Omega^{k+1}$, respectively, implies that $t \to f(t, u(t), u(b_1(t)), \cdots, u(b_r(t)))$ is Lipschitz continuous in the interval J. In this sense, we have to from this fact and the conditions of Theorem 3.1 imply that, by Theorem 2.6 the linear fractional Cauchy problem

$${}^{H}\mathbb{D}_{t_0^+}^{\alpha,\beta} v(t) + \mathcal{A}v(t) = f\left(t, u(t), u(b_1(t)), \cdots, u(b_r(t))\right),\ t \in J/\{t_0\} \tag{4.13}$$

and

$$I_{t_{0^+}}^{1-\gamma} v(t_0) = u_0 - \sum_{k=1}^{p} \lambda_k I_{t_{0^+}}^{1-\gamma} u(t_k) \tag{4.14}$$

has a unique solution v such that

$$v(t) = \mathbb{S}_{\alpha,\beta}(t-t_0) I_{t_{0^+}}^{1-\gamma} v(t_0) + \int_{t_0^+}^{t} \mathbf{K}_{\alpha}(t-s) f(s, u(s), u(b_1(s)), \cdots, u(b_r(s))) ds \tag{4.15}$$

with $t \in J$. Now, we shall show that

$$u(t) = v(t),\ t \in J. \tag{4.16}$$

Note that by means of Eq.(4.14), Remark 2.7 and Eq.(2.6), we have

$$\begin{aligned} I_{t_{0^+}}^{1-\gamma} v(t_0) = I_{t_{0^+}}^{1-\gamma} u(t_0) = \mathbf{B}u_0 - \mathbf{B}\sum_{k=1}^{p} \lambda_k I_{t_{0^+}}^{1-\gamma} \times \\ \times \int_{t_0^+}^{t_k} \mathbf{K}_{\alpha}(t_k - s) f(s, u(s), u(b_1(s)), \cdots, u(b_r(s))) ds. \end{aligned} \tag{4.17}$$

Thus,

$$\begin{aligned} & \mathbb{S}_{\alpha,\beta}(t-t_0) I_{t_{0^+}}^{1-\gamma} v(t_0) \\ = \ & \mathbb{S}_{\alpha,\beta}(t-t_0)\mathbf{B}u_0 - \mathbb{S}_{\alpha,\beta}(t-t_0)\mathbf{B} \times \\ & \times \sum_{k=1}^{p} \lambda_k I_{t_{0^+}}^{1-\gamma} \int_{t_0^+}^{t_k} \mathbf{K}_{\alpha}(t_k - s) f(s, u(s), u(b_1(s)), \cdots, u(b_r(s))) ds \end{aligned} \tag{4.18}$$

$t \in J$.

Next, from Eq.(4.15), Eq.(4.18) and Eq.(2.6), we obtain

$$\begin{aligned} v(t) &= \mathbb{S}_{\alpha,\beta}(t-t_0)\mathbf{B}u_0 - \mathbb{S}_{\alpha,\beta}(t-t_0)\mathbf{B}\times \\ &\times \sum_{k=1}^{p} \lambda_k I_{t_{0+}}^{1-\gamma} \int_{t_0^+}^{t_k} \mathbf{K}_\alpha(t_k-s) f(s,u(s),u(b_1(s)),\cdots,u(b_r(s)))ds \\ &+ \int_{t_0^+}^{t} \mathbf{K}_\alpha(t-s) f(s,u(s),u(b_1(s)),\cdots,u(b_r(s)))ds \\ &= u(t) \end{aligned} \tag{4.19}$$

$t \in J$ and, therefore, Eq.(4.16) holds.

Thus, we conclude that u is the classical solution of Eq.(1.1)-Eq.(1.2). Now suppose $u^* \neq u$ another classical solution of Eq.(1.1)-Eq.(1.2) in the interval J. Through the Theorem 4.2, we have that u^* is a solution to the problem Eq.(1.1)-Eq.(1.2). On the other hand, the Theorem 3.1 guarantees the uniqueness of a solution for Eq.(1.1)-Eq.(1.2), so we conclude that $u = u^*$. Therefore, we conclude the result. □

5 Concluding remarks

In this paper, we investigate the existence and uniqueness of mild and classical solutions to the fractional functional differential evolution equation through the Cauchy contraction theorem and Gronwall inequality. We were able to present new results that actually contribute to the advancement and growth of the area. However, there are still some issues that deserve special mention, namely, investigating mild, strong, classical solutions to fractional differential equations introduced by means of the ψ-Hilfer fractional derivative [19]. But for such a success, it is necessary and sufficient condition to obtain an integral transform to investigate such solutions. Studies in this direction have been developed and certainly contributed a lot to the discussion of open problems, in particular, problems of controllability and regularity of mild solutions for Eq.(1.1) of the problem investigated here.

Acknowledgment

J. Vanterler acknowledges the financial support of a PNPD-CAPES (nº88882.305834/2018-01) scholarship of the Postgraduate Program in Applied Mathematics of IMECC-Unicamp and we thank anonymous referees by the suggestions that improved the paper.

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Author information

J. Vanterler da C. Sousa [1], Thabet Abdeljawad [*] and D. S. Oliveira [2], [1] Department of Applied Mathematics, Imecc-State University of Campinas, 13083-859, Campinas, SP, Brazil; Department of Mathematics and General Sciences, [*] Prince Sultan University,P. O. Box 66833, 11586 Riyadh, Saudi Arabia and [2] Coordination of Civil Engineering, Technological Federal University of Parana, 85053-525, Guarapuava, PR,, Brazil.
E-mail: vanterler@ime.unicamp.br,tabdeljawad@psu.edu.sa,oliveiradaniela@utfpr.edu.br

Received: December 1, 2020
Accepted: November 8, 2021

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