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RADICAL OF FILTERS OF TRANSITIVE BE-ALGEBRAS

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Abstract The notion of skew-simple BE-algebras is introduced and derived an equivalent assertions for every skew-simple BE-algebra to become semi-simple. The concept of radical of filters is introduced in a BE-algebra and certain properties of these radicals are derived in terms of direct products and homomorphisms. The concept of semi-maximal filters is introduced in BE-algebras. Some equivalent assertions are derived for every semi-maximal filter to become a maximal filter. Properties of semi-maximal filters are derived in terms of homomorphisms and congruences.

1 Introduction

The notion of BE-algebras was introduced and extensively studied by H. S. Kim and Y. H. Kim in [8]. These classes of BE-algebras were introduced as a generalization of the class of BCK-algebras of K. Iseki and S. Tanaka [7]. Some properties of filters of BE-algebras were studied by S. S. Ahn and Y. H. Kim in [1] and by B. L. Meng in [9]. In [16], A. Walendziak discussed some properties of commutative BE-algebras. He also investigated the relationship between BE-algebras, implicative algebras and J-algebras. In 2012, A. Rezaei, and A. Borumand Saeid [11], stated and proved the first, second and third isomorphism theorems in self-distributive BE-algebras. Later, these authors [12] introduced the notion of commutative ideals in a BE-algebra. In 2013, A. Borumand Saeid, A. Rezaei and R. A. Borzooei [3] extensively studied the properties of some types of filters in BE-algebras. In [4], Chajda *et al.*, Characterized the complements and relative complements of the set of all deductive systems as the so-called annihilators of Hilbert algebras. Later, Halaš[6] introduced the concepts of an annihilator and a relative annihilator of a given subset of a BCK-algebra. In [5], Z. Ciloglu and Y. Ceven introduced the notion of bounded BE-algebras and investigated some properties of them. A. Paad [10] introduced the notion of the radical of ideals in BL-algebras and then characterized the notion of the radical of ideals by elements of a BL-algebra.

In this work, we derive some significant properties of maximal filters of a bounded BE-algebra. The notion of skew-simple BE-algebras is introduced and studied its properties. We prove that the condition of self-distributivity is sufficient to satisfy all the properties of a skew-simple BE-algebra. It is observed that every semi-simple BE-algebra is a skew-simple BE-algebra and the converse is not true. However, some equivalent assertions are derived for a skew-simple BE-algebra to become a semi-simple BE-algebra. The concept of a radicals of a filter is introduced in bounded BE-algebras. The elements of a radical of a filter are characterized in self-distributive BE-algebras. Certain properties of these radicals are then derived with respect to set-intersection, direct products, and homomorphic images.

The concept of semi-maximal filters is introduced, in bounded BE-algebras, in terms of radical of filters. Some equivalent assertions are derived for every semi-maximal filter of a BE-algebra to become a maximal filter. Finally, properties of semi-maximal filters are derived with respect to homomorphism, Cartesian products and congruences.

2 Preliminaries

In this section, we present certain definitions and results which are taken mostly from the papers [1], [5], [8], [9], [14] and [15] for the ready reference of the reader.

Definition 2.1. [8] An algebra $(X, *, 1)$ of type $(2, 0)$ is called a BE-algebra if it satisfies the following properties:

(1) $x * x = 1,$
(2) $x * 1 = 1,$
(3) $1 * x = x,$
(4) $x * (y * z) = y * (x * z)$ for all $x, y, z \in X.$

A BE-algebra X is called *self-distributive* if $x * (y * z) = (x * y) * (x * z)$ for all $x, y, z \in X$. A BE-algebra X is called *transitive* if $y * z \leq (x * y) * (x * z)$ for all $x, y, z \in X$. Every self-distributive BE-algebra is transitive. A BE-algebra $(X, *, 1)$ is said to be an *implicative* BE*-algebra*[13] if it satisfies the implicative condition $x = (x * y) * x$ for all $x, y \in X$. We introduce a relation $\leq$ on X by $x \leq y$ if and only if $x * y = 1$ for all $x, y \in X$.

Theorem 2.2. [9] *Let X be a transitive BE-algebra and $x, y, z \in X$. Then*

(1) $1 \leq x$ *implies* $x = 1$,
(2) $y \leq z$ *implies* $x * y \leq x * z$ *and* $z * x \leq y * x$.

Definition 2.3. [8] A non-empty subset F of a BE-algebra X is called a *filter* of X if, for all $x, y \in X$, it satisfies the following properties:

(1) $1 \in F$,
(2) $x \in F$ and $x * y \in F$ imply that $y \in F$.

Theorem 2.4. [1] *If A is a non-empty subset of a transitive BE-algebra X, then*

$$\langle A \rangle = \{x \in X \mid a_1 * (a_2 * (\cdots * (a_n * x) \cdots)) = 1 \text{ for some } a_1, a_2, \ldots, a_n \in A\}.$$

Let F be a filter of a BE-algebra X. For any $a \in X$, $\langle F \cup \{a\} \rangle = \{x \in X \mid a^n * x \in F \text{ for some } n \in \mathbb{N}\}$. For $A = \{a\}$, we will denote $\langle \{a\} \rangle$, briefly by $\langle a \rangle$, we call it a principal filter of X. If X is self-distributive, then $\langle a \rangle = \{x \in X \mid a * x = 1\}$.

A BE-algebra X is called bounded[5], if there exists an element 0 satisfying $0 \leq x$ (or $0 * x = 1$) for all $x \in X$. Define a unary operation N on a bounded BE-algebra X by $xN = x * 0$ for all $x \in X$.

Theorem 2.5. [5] *Let X be a bounded BE-algebra and $x, y, z \in X$. Then*

(1) $1N = 0$ *and* $0N = 1$,
(2) $x \leq xNN$,
(3) $x * yN = y * xN$.

Lemma 2.6. [5] *Let X be a bounded and transitive BE-algebra. For any $x, y, z \in X$, we have*

(1) $x \leq y$ *implies* $yN \leq xN$,
(2) $xNNN \leq xN$,
(3) $x * y \leq yN * xN$,
(4) $(xN * yN)NN \leq xN * yN$.

An element x of a bounded BE-algebra X is called *dense*[15] if $xN = 0$. Let X and Y be two bounded BE-algebras, then a homomorphism $f : X \to Y$ is called bounded if $f(0) = 0$. If f is a bounded homomorphism, then it is easily observed that $f(xN) = f(x)N$ for all $x \in X$.

Definition 2.7. [5] An element x of a bounded BE-algebra X is called an *involutory element* if $xNN = x$. If every element of a BE-algebra X is involutory, then X is called an *involutory* BE-algebra.

Definition 2.8. [15] A non-empty subset I of a bounded BE-algebra X is called an *ideal* of X if it satisfies the following conditions for all $x, y \in X$:

(I1) $0 \in I$,
(I2) $x \in I$ and $(xN * yN)N \in I$ imply that $y \in I$.

Obviously the single-ton set $\{0\}$ is an ideal of a BE-algebra X. For, suppose $x \in \{0\}$ and $(xN * yN)N \in \{0\}$ for $x, y \in X$. Then $x = 0$ and $yNN = (0N * yN)N \in \{0\}$. Hence $y \leq yNN = 0 \in \{0\}$. Thus $\{0\}$ is an ideal of X.

Proposition 2.9. [15] *Let I be an ideal of a bounded and transitive BE-algebra X. Then we have the following:*

(1) *For any $x, y \in X, x \in I$ and $y \leq x$ imply $y \in I$,*
(2) *For any $x \in X, x \in I$ if and only if $xNN \in I$.*

A filter F of a BE-algebra X is called *proper* if $F \neq X$. A proper filter M of a BE-algebra is called *maximal* if there exists no proper filter Q of X such that $M \subset Q$.

Theorem 2.10. [14] *A proper filter M of a transitive BE-algebra X is maximal if and only if for each $x \notin M$, $\langle M \cup \{x\} \rangle = X$.*

Lemma 2.11. [2] *Let F be a filter of a bounded and transitive BE-algebra X. Then*

(1) *F is proper if and only if $0 \notin F$.*
(2) *each proper filter is contained in a maximal filter.*

Theorem 2.12. [2] *Every BE-algebra contains at least one maximal filter.*

Theorem 2.13. [15] *For any filter F of a transitive BE-algebra, the binary relation θ_F on defined on X by*

$$(x, y) \in \theta_F \text{ if and only if } x * y \in F \text{ and } y * x \in F \text{ for all } x, y \in X.$$

is a congruence on X.

From the above theorem, it is easy to see that the quotient algebra $X/F = \{F_x \mid x \in X\}$ (where F_x is the congruence class of x modulo θ_F) is a bounded BE-algebra in which the binary operation $*$ is defined as $F_x * F_y = F_{x*y}$ for $x, y \in X$ and the unary operation N is defined as $(F_x)N = F_{xN}$ for all $x \in X$. Moreover, the quotient algebra X/F contains the greatest element F_1. Throughout this article, X stands for a bounded BE-algebra unless otherwise stated.

3 Properties of maximal filters

In this section, some important properties of maximal filters are given. The concept of skew-simple BE-algebra is introduced in a BE-algebra and proved that every self-distributive BE-algebra is skew-simple. A set of equivalent conditions is given for every skew-simple BE-algebra to become a semi-simple.

Definition 3.1. Let X be a BE-algebra and F is any subset of X. Then the set $N(F)$ is defined as $N(F) = \{x \in X \mid xN \in F\}$.

Proposition 3.2. *The following implications hold in a transitive BE-algebra X:*

(1) *For any filter F of X, $x \in F$ implies $xN \in N(F)$,*
(2) *For any ideal I of X, $x \in I$ implies $xN \in N(I)$,*
(3) *For any filter F of X, $N(F)$ is an ideal of X,*
(4) *For any ideal I of X, $N(I)$ is a filter of X.*

Proof. (1) Let F be a filter of X and suppose $x \in F$. Since $x \leq xNN$, we get $xNN \in F$. Hence $xN \in N(F)$.
(2) Let I be an ideal of X and suppose $x \in I$. Since I is an ideal, we get $xNN \in I$. Hence

$xN \in N(I)$.
(3) Let F be a filter of X. Since $0N = 1 \in F$, we get $0 \in N(F)$. Let $x, y \in X$ be such that $x \in N(F)$ and $(xN * yN)N \in N(F)$. Then $xN \in F$ and $(xN * yN)NN \in F$. By Lemma 2.6(4), we get $(xN * yN)NN \leq xN * yN$ and hence $xN * yN \in F$. Since $xN \in F$ and F is a filter, we get $yN \in F$. Thus $y \in N(F)$. Therefore $N(F)$ is an ideal of X.
(4) Let I be an ideal of X. Since $1N = 0 \in I$, we get $1 \in N(I)$. Let $x, y \in X$ be such that $x \in N(I)$ and $x * y \in N(I)$. Then $xN \in I$ and $(x * y)N \in I$. Since $x * y \leq xNN * yNN$, we get $(xNN * yNN)N \leq (x * y)N$. Hence $(xNN * yNN)N \in I$. Since $xN \in I$ and I is an ideal, we get $yN \in I$. Thus $y \in N(I)$. Therefore $N(I)$ is a filter of X. □

Theorem 3.3. *A proper filter F of a self-distributive BE-algebra X is maximal if and only if it satisfies the following condition:*

$$x \notin F \text{ implies } xN \in F \quad \text{for all } x \in X.$$

Proof. Let F be a proper filter of X. Assume that F is maximal. Let $x \notin F$. Then $\langle F \cup \{x\} \rangle = X$. Hence $0 \in \langle F \cup \{x\} \rangle$. Since X is self-distributive, we get $xN = x * 0 \in F$.

Conversely, assume that F satisfies the condition. Suppose F is not maximal. Then there exists a proper filter Q of X such that $F \subset Q$. Choose $x \in Q - F$. Then $x \notin F$. By the assumed condition, we get $x * 0 = xN \in F \subset Q$. Since $x \in Q$ and Q is a filter, we get $0 \in Q$ which is contradiction. Therefore F is a maximal filter of X. □

Proposition 3.4. *The following conditions are hold in a self-distributive BE-algebra X:*

(1) *For any maximal filter F of X, $N(F)$ is a maximal ideal of X,*
(2) *For any maximal ideal I of X, $N(I)$ is a maximal filter of X.*

Proof. (1) Let F be a maximal filter of X. By Proposition 3.2, we get that $N(F)$ is an ideal of X. Clearly F is proper filter of X. Suppose $1 \in N(F)$. Then $0 = 1N \in F$ which is a contradiction. Hence $N(F)$ is proper ideal of X. Let $x \in X$. Suppose $x \notin N(F)$. By the definition of $N(F)$, we get $xN \notin F$. Since F is maximal, we get $xNN \in F$. Hence $xN \in N(F)$. Therefore $N(F)$ is a maximal ideal of X.
(2) Let I be a maximal ideal of X. By Proposition 3.2, we get that $N(I)$ is a filter of X. Clearly I is proper ideal of X. Suppose $0 \in N(I)$. Then $1 = 0N \in I$ which is a contradiction. Hence $N(I)$ is proper filter of X. Let $x \in X$. Suppose $x \notin N(I)$. By the definition of $N(F)$, we get $xN \notin I$. Since I is maximal, we get $xNN \in I$. Hence $xN \in N(I)$. Therefore $N(I)$ is a maximal filter of X. □

Definition 3.5. Let X be a bounded BE-algebra. Then the *radical* of X, written $rad(X)$, is defined as

$$rad(X) = \cap\{F \mid F \in max(X)\}$$

where $max(X)$ is the collection of all maximal filters of X.

It is clear that $rad(X)$ always exists for a bounded BE-algebra. In the contemporary algebra, it is observed that a BE-algebra X is called *semi-simple* if $rad(X) = \{1\}$. In [2], authors proved that every involutory BE-algebra is semi-simple.

Proposition 3.6. *In a transitive BE-algebra X, we have $D(X) \subseteq rad(X)$, where $D(X)$ is the set of all dense elements of X.*

Proof. It is enough to prove that $D(X)$ is contained in every maximal filter of X. Let M be a maximal filter of X such that $D(X) \nsubseteq M$. Then there exists $x \in D(X)$ such that $x \notin M$. Since M is maximal, we get $\langle M \cup \{x\} \rangle = X$. Hence $0 \in \langle M \cup \{x\} \rangle$. Hence $x^n * 0 \in M$ for some positive integer n. Hence

$$\begin{aligned} x^n * 0 \in M &\Rightarrow \underbrace{x * (x * (\cdots (x}_{\text{n times}} *0) \cdots)) \in M \\ &\Rightarrow \underbrace{x * (x * (\cdots (x}_{\text{n-1 times}} *(x * 0)) \cdots)) \in M \\ &\Rightarrow \underbrace{x * (x * (\cdots (x}_{\text{n-1 times}} *0) \cdots)) \in M \\ &\Rightarrow \underbrace{x * (x * (\cdots (x}_{\text{n-2 times}} *(x * 0))) \cdots) \in M \\ &\quad \cdots \\ &\quad \cdots \\ &\Rightarrow x * (x * 0) \in M \\ &\Rightarrow x * 0 \in M. \end{aligned}$$

which means $xN \in M$. Since $x \in D(X)$, we get $0 = xN \in M$, which is a contradiction. Hence $D(X) \subseteq M$ for all maximal filters M of X. Thus $D(X) \subseteq \cap\{M \mid M \in max(X)\}$. □

Since $1 \in D(X)$, the following corollary is a direct consequence of the above.

Corollary 3.7. *For any BE-algebra X, we have $\{1\} \subseteq D(X) \subseteq rad(X)$.*

Definition 3.8. A bounded BE-algebra X is called *skew-simple* if $rad(X) = D(X)$.

We first observe the non-trivial example:

Example 3.9. Let $X = \{0, a, b, c, 1\}$. Define an operation $*$ on X as follows:

$*$	1	a	b	c	0
1	1	a	b	c	0
a	1	1	b	c	b
b	1	a	1	c	a
c	1	a	b	1	0
0	1	1	1	1	1

Clearly $(X, *, 0, 1)$ is a bounded BE-algebra and $D(X) = \{c, 1\}$. It is easy to check that $F_1 = \{1\}$, $F_2 = \{1, a\}$, $F_3 = \{1, b\}$, $F_4 = \{1, c\}$, $F_5 = \{1, a, c\}$ and $F_6 = \{1, b, c\}$ are proper filters of X in which F_5 and F_6 are maximal filters of X. Hence $rad(X) = F_5 \cap F_6 = \{1, c\}$, which shows that $rad(X) = D(X)$. Therefore X is skew-simple.

Proposition 3.10. *Every semi-simple BE-algebra is skew-simple.*

Proof. Let X be a semi-simple BE-algebra. Then $rad(X) = \{1\}$. By Proposition 3.6, we have $D(X) \subseteq rad(X)$. Now, let $x \in rad(X)$. Then $x \in \{1\}$ and hence $x = 1 \in D(X)$. Thus $rad(X) = D(X)$. Therefore X is a skew-simple BE-algebra. □

The converse of the above proposition is not true. That is every skew-simple BE-algebra need not be semi-simple. For, consider the following example:

Example 3.11. Let $X = \{0, a, b, 1\}$. Define an operation $*$ on X as follows:

$*$	1	a	b	0
1	1	a	b	0
a	1	1	b	0
b	1	a	1	0
0	1	1	1	1

Clearly $(X, *, 0, 1)$ is a bounded BE-algebra and $D(X) = \{a, b, 1\}$. It is easy to check that $F_1 = \{1\}$, $F_2 = \{1, a\}$, $F_3 = \{1, b\}$ and $F_4 = \{1, a, b\}$ are proper filters of X in which F_4 is the only maximal filters of X. Hence $rad(X) = F_4 = \{1, a, b\} = D(X)$. Hence X is a skew-simple BE-algebra. Since $rad(X) \neq \{1\}$, we conclude that X is not a skew-simple BE-algebra.

In the following theorem, a set of equivalent assertions is derived for every skew-simple BE-algebra to become a semi-simple BE-algebra.

Theorem 3.12. *Let X be a skew-simple BE-algebra. Then the following assertions are equivalent:*

(1) *X is semi-simple;*
(2) *every filter contains $D(X)$;*
(3) *X possesses a unique dense element.*

Proof. (1) $\Rightarrow$ (2): Assume that X is semi-simple. Then $rad(X) = \{1\}$. Since $D(X) \subseteq rad(X)$, we get $D(X) = \{1\}$. Hence $D(X) = \{1\} \subseteq F$ for every filter F of X.
(2) $\Rightarrow$ (3): Assume that every filter contains $D(X)$. Hence $D(X) \subseteq \{1\}$. Thus $D(X) = \{1\}$, which means X possesses a unique dense element, precisely 1.
(3) $\Rightarrow$ (1): Assume that X possesses a unique dense element 1. Hence $D(X) = \{1\}$. Since X is skew-simple, we get $rad(X) = D(X) = \{1\}$. Therefore X is semi-simple. □

Theorem 3.13. *A bounded BE-algebra X is skew-simple if and only if for each $x \in X$ with $xN \neq 0$, there exists a proper filter F of X such that $\langle F \cup \{x\} \rangle = X$.*

Proof. Assume that X is skew-simple. Then $rad(X) = D(X)$. Hence $\bigcap_{F \in Max(X)} F = D(X)$. Let $x \in X$ and $xN \neq 0$. Hence $x \notin D(X)$. Then there exists a maximal filter F of X such that $x \notin F$ (otherwise, if every maximal filter contains x, then $x \in \bigcap_{F \in max(X)} F = D(X)$). Since F is maximal, we get $\langle F \cup \{x\} \rangle = X$.

Conversely, assume the condition. Suppose $rad(X) \neq D(X)$. Choose $x \in rad(X)$ with $x \notin D(X)$. By the assumed condition, there exists a proper filter F of X such that $\langle F \cup \{x\} \rangle = X$. Hence $x \notin F$. Consider $\mathfrak{T} = \{G \mid G \text{ is a filter of } X, x \notin G \text{ and } F \subseteq G\}$. Clearly $F \in \mathfrak{T}$ and $\mathfrak{T} \neq \emptyset$. Clearly, $\mathfrak{T}$ is a partially ordered set, with the set inclusion, in which every chain has an upper bound. By the Zorn's lemma, $\mathfrak{T}$ has a maximal element say F_0. Then $x \notin F_0$ and $F \subseteq F_0$. Suppose there exists a proper filter M of X such that $F \subseteq F_0 \subset M \subseteq X$. By the maximality of M, we get $x \in M$. Hence $X = \langle F \cup \{x\} \rangle \subset \langle M \cup \{x\} \rangle = M$. Thus F_0 is a maximal filter of X and $x \notin F_0$, which is a contradiction. Therefore $rad(X) = D(X)$, which means that X is skew-simple. □

Theorem 3.14. *Let X be a self-distributive BE-algebra. For every $x \in X$ with $xN \neq 0$, there exists a maximal filter F of X such that $x \notin F$.*

Proof. Let $x \in X$ and $xN \neq 0$. We first claim that $\langle xN \rangle$ is a proper filter of X. Suppose $0 \in \langle xN \rangle$. Since X is self-distributive, we get $xNN = xN * 0 = 1$. Hence $xN \leq xNNN = 0$. Thus $xN = 0$, which is a contradiction. Hence $\langle xN \rangle$ is a proper filter of X. Then there exists a maximal filter F of X such that $\langle xN \rangle \subseteq F$. Suppose $x \in F$. Then $x * 0 = xN \in \langle xN \rangle \subseteq F$. Since $x \in F$, we get $0 \in F$ which is a contradiction. Hence F is a maximal filter of X such that $x \notin F$. □

Theorem 3.15. *Let X be a self-distributive BE-algebra. Then $D(X) = \bigcap_{M \in max(X)} M$.*

Proof. Clearly $D(X) \subseteq rad(X)$. Conversely, suppose that $x \notin D(X)$. Then $xN \neq 0$. By the above theorem, there exists a maximal filter M such that $x \notin M$. Hence $x \notin \bigcap_{M \in max(X)} M$. □

Corollary 3.16. *Every self-distributive BE-algebra is skew-simple.*

4 Radical of filters in BE-algebras

In this section, the notion of radical of a filter of a BE-algebra is introduced. Some properties of these radicals are studied in bounded BE-algebras.

Definition 4.1. Let F be a proper filter of a BE-algebra X. The intersection of all maximal filters of X that contain F is called the radical of F and is denoted by $rad(F)$. That is

$$rad(F) = \cap\{M \mid M \in Max(X) \text{ such that } F \subseteq M\}$$

Note that $rad(F)$ is a filter of X and $F \subseteq rad(F)$.

Proposition 4.2. *Let M be maximal filter of a BE-algebra X. Then $rad(M) = M$.*

Proof. By Definition 4.1, the proof is clear. □

Example 4.3. Let $X = \{0, a, b, c, d, 1\}$. Define an operation $*$ on X as follows:

$*$	0	a	b	c	d	1
0	1	1	1	1	1	1
a	0	1	b	b	d	1
b	0	a	1	a	d	1
c	0	1	1	1	d	1
d	d	1	1	1	1	1
1	0	a	b	c	d	1

Clearly $(X, *, 0, 1)$ is a bounded BE-algebra. Clearly $F = \{1\}$, $G = \{1, a\}$, $H = \{1, b\}$ and $I = \{1, a, b, c\}$ are proper filters of X. Note that $rad(F) = \{1, a, b, c\}$, $rad(G) = \{1, a, b, c\}$, $rad(H) = \{1, a, b, c\}$ and $rad(I) = \{1, a, b, c\} = I$.

In the following theorem, we characterize the elements of $rad(F)$ of a bounded BE-algebra X, where F is a proper filter of X.

Theorem 4.4. *Let F be a proper filter of a self-distributive BE-algebra X. Then*

$$rad(F) = \{x \in X \mid xN * x \in F\}.$$

Proof. Given that F is a proper filter of X. Put $\mathcal{K} = \{x \in X \mid xN * x \in F\}$. Let $x \in \mathcal{K}$. Suppose $x \notin rad(F)$. Then there exists a maximal filter M of X such that $F \subseteq M$ and $x \notin M$. Since M is maximal, by Theorem 3.3, we get $xN \in M$. Since $x \in \mathcal{K}$, we get $xN * x \in F \subseteq M$. Since $xN \in M$ and M is a filter, we get $x \in M$ which is a contradiction. Hence $x \in rad(F)$. Therefore $\mathcal{K} \subseteq rad(F)$.

Conversely, let $x \in rad(F)$. By the definition of $rad(F)$, we get $x \in M$ for each maximal filter M of X with $F \subseteq M$. Suppose $xN * x \notin F$. Then there exists a maximal filter M' of X such that $F \subseteq M'$ and $xN * x \notin M'$. Since $F \subseteq M'$ and $x \in rad(F)$, we get $x \in M'$. Since $x \in M'$, then by the property of filters, we get $xN * x \in M'$ which is a contradiction. Hence $xN * x \in F$, which means $x \in \mathcal{K}$. Therefore $rad(F) \subseteq \mathcal{K}$ and hence $rad(F) = \mathcal{K} = \{x \in X \mid xN * x \in F\}$. □

Proposition 4.5. *Let F be a filter of a self-distributive BE-algebra X. Then F is a proper filter of X if and only if $rad(F)$ is a proper filter of X.*

Proof. It follows form Definition 4.1 and Theorem 4.4. □

Lemma 4.6. *Let F and G be proper filters of a self-distributive BE-algebra X. Then*

(1) *$F \subseteq G$ implies $rad(F) \subseteq rad(G)$. Moreover, if X is an implicative BE-algebra, then $rad(F) \subseteq rad(G)$ implies $F \subseteq G$.*

(2) *$rad(rad(F)) = rad(F)$.*

(3) *$rad(F) = X$ if and only if $F = X$.*

Proof. (1) Suppose $F \subseteq G$ and $x \in rad(F)$. Then $xN * x \in F \subseteq G$, which means $x \in rad(G)$. Therefore $rad(F) \subseteq rad(G)$. Now, let X be an implicative BE-algebra. Suppose $rad(F) \subseteq rad(G)$. Let $x \in F$. Since $F \subseteq rad(F)$, we get $x \in rad(F) \subseteq rad(G)$. Hence, by Theorem 4.4, we get $xN*x \in G$. Thus $(x*0)*x \in G$. Since X is implicative, we get $x \in G$. Therefore $F \subseteq G$.
(2) Since $F \subseteq rad(F)$, by (1), we get $rad(F) \subseteq rad(rad(F))$. Conversely, let $x \in rad(rad(F))$. Then $x \in M$ for any maximal filter M such that $rad(F) \subseteq M$. Let M' be an any maximal filter of X such that $F \subseteq M'$. Then by (1), we get $rad(F) \subseteq rad(M') = M'$ because of M' is maximal. Hence $x \in M'$. That is $x \in M'$ for each maximal filter M' such that $F \subseteq M'$. By the definition of $rad(F)$, it yields $x \in rad(F)$. Therefore $rad(rad(F)) \subseteq rad(F)$.
(3) It follows form Proposition 4.5. □

Proposition 4.7. *Let $\{F_\alpha\}_{\alpha\in\Delta}$ be a family of filters of a self-distributive BE-algebra X. Then*

$$rad(\bigcap_{\alpha\in\Delta} F_\alpha) = \bigcap_{\alpha\in\Delta} rad(F_\alpha).$$

Proof. It is clear that $\bigcap_{\alpha\in\Delta} F_\alpha \subseteq F_\alpha$ for each $\alpha \in \Delta$. By Lemma 4.6, we get $rad(\bigcap_{\alpha\in\Delta} F_\alpha) \subseteq rad(F_\alpha)$ for each $\alpha \in \Delta$. Hence $rad(\bigcap_{\alpha\in\Delta} F_\alpha) \subseteq \bigcap_{\alpha\in\Delta} rad(F_\alpha)$. Conversely, let $x \in \bigcap_{\alpha\in\Delta} rad(F_\alpha)$. Then $x \in rad(F_\alpha)$ for each $\alpha \in \Delta$. Hence $xN * x \in F_\alpha$ for each $\alpha \in \Delta$. Thus $xN * x \in \bigcap_{\alpha\in\Delta} F_\alpha$. Hence $x \in rad(\bigcap_{\alpha\in\Delta} F_\alpha)$ and so $\bigcap_{\alpha\in\Delta} rad(F_\alpha) \subseteq rad(\bigcap_{\alpha\in\Delta} F_\alpha)$. Therefore $\bigcap_{\alpha\in\Delta} rad(F_\alpha) = rad(\bigcap_{\alpha\in\Delta} F_\alpha)$. □

Let $I_n = \{1, 2, \ldots, n\}$ and $\{X_i = (X_i, *, 0, 1) \mid i \in I_n\}$ be a finite family of bounded BE-algebras. Define the direct product of bounded BE-algebras $X_1, X_2, \ldots, X_n$ as an algebraic structure

$$\Big(\prod_{i=1}^{n} X_i, \circledast, (0_1, 0_2, \ldots, 0_n), (1_1, 1_2, \ldots, 1_n)\Big)$$

where $\prod_{i=1}^{n} X_i = \{(a_i) = (a_1, a_2, \ldots, a_n) \mid a_i \in X_i \text{ and } i \in I_n\}$ and whose operations are defined as

$$(x_i) \circledast (y_i) = (x_i * y_i)$$

for any $(x_i), (y_i) \in \prod_{i=1}^{n} X_i$ and $(x_i)N = (x_iN)$. Then clearly the direct product is a bounded BE-algebra with largest element (1_i) and the smallest element (0_i). Note that $\prod_{i=1}^{n} X_i$ is self-distributive whenever each X_i is self-distributive.

Proposition 4.8. *Let $\{F_i\}_{i\in I_n}$, where $I_n = \{1, 2, \ldots, n\}$, be a finite family of filters of a self-distributive BE-algebra X. Then*

$$rad\Big(\prod_{i=1}^{n} F_i\Big) = \prod_{i=1}^{n} rad(F_i).$$

Proof. Let $\{F_i\}_{i\in I_n}$, where $I_n = \{1, 2, \ldots, n\}$, be a finite family of filters of a self-distributive BE-algebra X. It is easy to check that $\prod_{i=1}^{n} F_i$ is a filter of the product algebra X^n. Let $(x_i) \in \prod_{i=1}^{n} F_i$ where $x_i \in F_i$ for $i \in I_n$. Then

$$\begin{aligned}
(x_i) \in rad\Big(\prod_{i=1}^{n} F_i\Big) \quad &\Leftrightarrow \quad (x_i)N \circledast (x_i) \in \prod_{i=1}^{n} F_i \\
&\Leftrightarrow \quad (x_iN * x_i) \in \prod_{i=1}^{n} F_i \\
&\Leftrightarrow \quad x_iN * x_i \in F_i \quad \text{for } i \in I_n \\
&\Leftrightarrow \quad x_i \in rad(F_i) \quad \text{for } i \in I_n \\
&\Leftrightarrow \quad (x_i) \in \prod_{i=1}^{n} rad(F_i)
\end{aligned}$$

Therefore $rad\Big(\prod_{i=1}^{n} F_i\Big) = \prod_{i=1}^{n} rad(F_i)$. □

Definition 4.9. [2] Let X and Y be two bounded BE-algebras. A mapping $f : X \to Y$ is called a homomorphism from X to Y if it satisfies the following conditions:

(1) $f(x * y) = f(x) * f(y)$,
(2) $f(xN) = f(x)N$,
(3) $f(0) = 0$, for all $x, y \in X$.

Clearly $ker(f) = \{x \in X \mid f(x) = 1\}$ is a filter of X. Moreover, the set $Dker(f) = \{x \in X \mid f(x) = 0\}$ is called *dual kernel* of the BE-homomorphism f.

Proposition 4.10. *Let X and Y be two bounded BE-algebras and $f : X \to Y$ be a BE-homomorphism. Then $Dker(f) = N(ker(f))$.*

Proof. Clearly $Dker(f)$ is an ideal of X. Let $x \in Dker(f)$. Then $f(x) = 0$. Thus $f(xN) = (f(x))N = 0N = 1$. So, $xN \in ker(f)$. Hence $x \in N(ker(f))$. Conversely, Let $x \in N(ker(f))$. Then $xN \in ker(f)$. Thus $f(xN) = 1$. Now, $f(xNN) = (f(xN))N = 1N = 0$. Since $x \leq xNN$, we get $f(x) \leq f(xNN)$. So $f(x) = 0$. Hence $x \in Dker(f)$. Therefore $Dker(f) = N(ker(f))$. □

Proposition 4.11. *Let X and Y be two bounded BE-algebras where X is self-distributive. Suppose $\psi : X \to Y$ be a BE-homomorphism. Then*

$$rad(Dker(\psi)) = \psi^{-1}(rad(\{0\}))$$

Proof. By Theorem 4.4, we get $x \in rad(Dker(\psi))$ if and only if $xN * x \in Dker(\psi)$ if and only if $\psi(xN * x) = 0$ if and only if $\psi(xN) * \psi(x) = 0$ if and only if $(\psi(x))N * \psi(x) = 0$ if and only if $\psi(x) \in rad(\{0\})$ if and only if $x \in \psi^{-1}(rad(\{0\}))$. □

Proposition 4.12. *Let X and Y be two bounded BE-algebras where X is self-distributive. Suppose that $\psi : X \to Y$ be a BE-homomorphism. If F_1 is a filter of X and F_2 is a filter of Y. Then we have*

(1) *$rad(\psi^{-1}(F_2)) = \psi^{-1}(rad(F_2))$,*
(2) *If ψ is a BE-isomorphism, then $rad(\psi(F_1)) = \psi(rad(F_1))$.*

Proof. (1). Since F_2 is a filter of Y, it is easy to check that $\psi^{-1}(F_2)$ is a filter of X. Let $x \in X$. Then by Theorem 4.4, we get

$$\begin{aligned} x \in rad(\psi^{-1}(F_2)) &\Rightarrow xN * x \in \psi^{-1}(F_2) \\ &\Rightarrow \psi(xN * x) \in F_2 \\ &\Rightarrow \psi(xN) * \psi(x) \in F_2 \\ &\Rightarrow (\psi(x))N * \psi(x) \in F_2 \\ &\Rightarrow \psi(x) \in rad(F_2) \\ &\Rightarrow x \in \psi^{-1}(rad(F_2)) \end{aligned}$$

Hence $rad(\psi^{-1}(F_2)) \subseteq \psi^{-1}(rad(F_2))$ and $\psi^{-1}(rad(F_2)) \subseteq rad(\psi^{-1}(F_2))$.
(2). Let F_1 be a filter of X and ψ be a BE-isomorphism. Then $\psi(F_1)$ is a filter of Y. Now, let $y \in \psi(rad(F_1))$. Then there exists $x \in rad(F_1)$, such that $y = \psi(x)$. Now

$$\begin{aligned} x \in rad(F_1) &\Rightarrow xN * x \in F_1 \\ &\Rightarrow \psi(xN * x) \in \psi(F_1) \\ &\Rightarrow \psi(xN) * \psi(x) \in \psi(F_1) \\ &\Rightarrow (\psi(x))N * \psi(x) \in \psi(F_1) \\ &\Rightarrow \psi(x) \in rad(\psi(F_1)) \end{aligned}$$

which means $y = \psi(x) \in rad(\psi(F_1))$. Therefore $\psi(rad(F_1)) \subseteq rad(\psi(F_1))$.

Conversely, let $y \in rad(\psi(F_1))$. Then by Theorem 4.4, we get $yN * y \in \psi(F_1)$. Since ψ is a BE-epimorphism, there exists $x \in F_1$ such that $\psi(x) = yN * y$. Thus

$$(\psi^{-1}(y))N * \psi^{-1}(y) = \psi^{-1}(yN) * \psi^{-1}(y) = \psi^{-1}(yN * y) = x \in F_1.$$

Thus, by Theorem 4.4, we get $\psi^{-1}(y) \in rad(F_1)$. Hence $y \in \psi(rad(F_1))$. Therefore, $rad(\psi(F_1)) \subseteq \psi(rad(F_1))$ and so $rad(\psi(F_1)) = \psi(rad(F_1))$. □

5 Semi-maximal filters of BE-algebras

In this section, we introduce the concept of semi-maximal filters in BE-algebras through the radical of a filter. Some equivalent assertions are derived for every filter of a BE-algebra to become semi-maximal. Finally, properties of semi-maximal filters are derived with respect to homomorphism, Cartesian products and congruences.

Definition 5.1. A proper filter F of a bounded BE-algebra X is called a *semi-maximal filter* of X if $rad(F) = F$.

Example 5.2. Let $X = \{0, a, b, c, 1\}$. Define an operation $*$ on X as follows:

$*$	0	a	b	c	1
0	1	1	1	1	1
a	0	1	b	c	1
b	0	a	1	c	1
c	0	1	b	1	1
1	0	a	b	c	1

Clearly $(X, *, 0, 1)$ is a bounded BE-algebra. Clearly $F = \{1\}$, $G = \{1, a\}$, $H = \{1, b\}$, $I = \{1, a, b\}$, $J = \{1, a, c\}$ and $K = \{1, a, b, c\}$ are proper filters of X. Note that $rad(F) = \{1, a, b, c\} \neq F$, $rad(G) = \{1, a, b, c\} \neq G$, $rad(H) = \{1, a, b, c\} \neq H$, $rad(I) = \{1, a, b, c\} \neq I$, $rad(J) = \{1, a, b, c\} \neq J$ and $rad(K) = \{1, a, b, c\} = K$. Therefore K is a semi-maximal filter and F, G, H, I, J are not semi-maximal.

Proposition 5.3. *Every maximal filter of a self-distributive BE-algebra is semi-maximal.*

Proof. Let X be a self-distributive BE-algebra and M a maximal filter of X. Clearly $M \subseteq rad(M)$. Conversely, let $x \in rad(M)$. Then $xN * x \in M$. Suppose $x \notin M$. By Theorem 3.3, we ge $xN \in M$. Since $xN * x \in M$ and M is a filter, we get $x \in M$ which is a contradiction. Hence $x \in M$. Thus $rad(M) \subseteq M$ which yields $rad(M) = M$. Therefore M is semi-maximal. □

The converse of the above proposition is not true. For, consider the following:

Example 5.4. Let $X = \{0, a, b, c, 1\}$. Define an operation $*$ on X as follows:

$*$	0	a	b	c	1
0	1	1	1	1	1
a	b	1	b	1	1
b	a	a	1	1	1
c	0	a	b	1	1
1	0	a	b	c	1

Clearly $(X, *, 0, 1)$ is a bounded BE-algebra. It can be easily verified that $F_1 = \{1, c\}$; $F_2 = \{1, a, c\}$ and $F_3 = \{1, b, c\}$ are proper filters of X. Moreover, we can see that F_2 and F_3 are maximal filters of X such $F_1 \subset F_2$ and $F_1 \subset F_3$. Now

$$rad(F_1) = F_2 \cap F_3 = \{1, c\} = F_1.$$

Hence F_1 is a semi-maximal filter of X but not maximal because of $F_1 \subset F_2, F_3$.

In the following theorem, we derive a set of equivalent assertions for a semi-maximal filter of a BE-algebra to become maximal.

Theorem 5.5. *Let F be a semi-maximal filter of a self-distributive BE-algebra X. Then the following assertions are equivalent:*

(1) *F is maximal;*
(2) *$rad(F)$ is maximal;*
(3) *for any $x \in X$, $x \notin rad(F)$ implies $xN \in F$.*

Proof. (1) ⇒ (2): Assume that F is a maximal filter of X. Since F semi-maximal, we get $rad(F) = F$. Since F is maximal, it yields that $rad(F)$ is a maximal filter of X.
(2) ⇒ (3): Let $x \in X$. Suppose $x \notin rad(F)$. Since $rad(F)$ is maximal, by Theorem 3.3, we get $xN \in rad(F)$. Since F is semi-maximal, we get $xN \in F$.
(3) ⇒ (1): Let $x \notin F$. Since F is semi-maximal, we get $rad(F) = F$. Hence $x \notin rad(F)$. By (3), we get $xN \in F$. Therefore F is maximal. □

Proposition 5.6. *Let $\{F_\alpha\}_{\alpha\in\Delta}$ be a family of semi-maximal filters of a self-distributive BE-algebra X. Then $\bigcap_{\alpha\in\Delta} F_\alpha$ is a semi-maximal filter of X.*

Proof. Since F_α is a semi-maximal filters of X for each $\alpha \in \Delta$, we get $rad(F_\alpha) = F_\alpha$ for all $\alpha \in \Delta$. Since X is self-distributive, by Proposition 4.7, we get

$$rad(\bigcap_{\alpha\in\Delta} F_\alpha) = \bigcap_{\alpha\in\Delta} rad(F_\alpha) = \bigcap_{\alpha\in\Delta} F_\alpha.$$

Therefore $\bigcap_{\alpha\in\Delta} F_\alpha$ is a semi-maximal filter of X. □

Proposition 5.7. *Let $\{F_i\}_{i\in I_n}$, where $I_n = \{1, 2, \ldots, n\}$, be a finite family of semi-maximal filters of a self-distributive BE-algebra X. Then $\prod_{i=1}^{n} F_i$ is a semi-maximal filter of X.*

Proof. Since F_i is a semi-maximal filters of X for each $i \in I_n$, we get $rad(F_i) = F_i$ for all $i \in I_n$. Since X is self-distributive, by Proposition 4.8, we get

$$rad\Big(\prod_{i=1}^{n} F_i\Big) = \prod_{i=1}^{n} rad(F_i).$$

Therefore $\prod_{i=1}^{n} F_i$ is a semi-maximal filter of X. □

Proposition 5.8. *Let X and Y be two self-distributive BE-algebras, and $f : X \to Y$ be a BE-homomorphism. If F and G are proper filters of X and Y respectively, then*

(1) *If G is semi-maximal filter of Y, then $f^{-1}(G)$ is semi-maximal of X.*
(2) *If $\{1\}$ is semi-maximal of X, then $ker(f)$ is semi-maximal of X.*
(3) *If f is BE-isomorphism and F is a semi-maximal filter of X, then $f(F)$ is a semi-maximal filter of Y.*

Proof. (1). Clearly $f^{-1}(G)$ is a filter of X and hence $f^{-1}(G) \subseteq rad(f^{-1}(G))$. Suppose G is semi-maximal of Y. Then $rad(G) = G$. Let $x \in X$ be such that $x \in rad(f^{-1}(G))$. Then $xN * x \in f^{-1}(G)$. Then $f(x)N * f(x) = f(xN * x) \in G$. Hence $f(x) \in rad(G) = G$, which means $x \in f^{-1}(G)$. Thus $rad(f^{-1}(G)) \subseteq f^{-1}(G)$. Therefore $f^{-1}(G)$ is a semi-maximal filter of X.
(2). Assume that $\{1\}$ is a semi-maximal filter of X. Then $rad(\{1\}) = \{1\}$. Clearly $ker(f)$ is a filter of X and hence $ker(f) \subseteq rad(ker(f))$. Again, let $x \in X$ be such that $x \in rad(ker(f))$. Then $xN * x \in ker(f)$. Hence $f(x)N * f(x) = f(xN * x) = 1 \in \{1\}$, which gives $f(x) \in rad(\{1\}) = \{1\}$. Hence $f(x) = 1$. Thus $x \in ker(f)$, which concludes that $rad(ker(f)) \subseteq ker(f)$. Hence $rad(ker(f)) = ker(f)$. Therefore $ker(f)$ is a semi-maximal filter of X.
(3). Let F be a semi-maximal filter of X. Since f is BE-isomorphism, we get $f(F)$ is a filter of Y. Since F is semi-maximal, we get $rad(F) = F$. By Proposition 4.12(2), we get $rad(f(F)) = f(rad(F)) = f(F)$. Hence $f(F)$ is a semi-maximal filter of Y. □

Proposition 5.9. *Let F be a proper filter of a self-distributive BE-algebra X. Then $rad(\{1\}/F) = rad(F)/F$.*

Proof. Let F be a proper filter of X. By Theorem 4.4, we get

$$\begin{aligned} rad(\{1\}/F) &= \{F_x \in X/F \mid (F_x)N * F_x \in \{1\}/F\} \\ &= \{F_x \in X/F \mid F_{xN*x} \in \{1\}/F\} \\ &= \{F_x \in X/F \mid (xN * x, 1) \in \theta_F\} \\ &= \{F_x \in X/F \mid xN * x \in F\} \\ &= \{F_x \in X/F \mid x \in rad(F)\} \\ &= rad(F)/F. \end{aligned}$$

□

Theorem 5.10. *Let F be a proper filter of a self-distributive BE-algebra X. Then $rad(F)$ the smallest semi-maximal filter of X such that $F \subseteq rad(F)$.*

Proof. Since $rad(rad(F)) = rad(F)$, we have $rad(F)$ is a semi-maximal filter of X. Now, let G be a semi-maximal filter of X such that $F \subseteq G$. Then $rad(F) \subseteq rad(G) = G$. Thus $rad(F)$ the smallest semi-maximal filter of X such that $F \subseteq rad(F)$. □

Lemma 5.11. *Let F be a proper filter of a self-distributive BE-algebra X and θ_F be the congruence on X. Then*

(1) *$\{1\}/F$ is a filter of X/F where $\{1\}/F = \{F_x \mid (x, 1) \in \theta_F\}$.*

(2) *$F_x \in rad(F)/F$ implies $x \in rad(F)$.*

(3) *$rad(F)/F$ is a semi-maximal filter of X/F.*

Proof. (1) Clearly $F_1 \in \{1\}/F$. Let $F_x, F_x * F_y \in \{1\}/F$. Then $F_{x*y} \in \{1\}/F$. Hence $(x, 1) \in \theta_F$ and $(x * y, 1) \in \theta_F$. Thus $x = 1 * x \in F$ and $x * y = 1 * (x * y) \in F$. Since F is a filter, we get $y \in F$. Thus $1 * y = y \in F$ and $y * 1 = 1 \in F$. Hence $(y, 1) \in \theta_F$, which gives $F_y \in \{1\}/F$. Therefore $\{1\}/F$ is a filter of X/F.
(2). Let $x \in X$ and $F_x \in rad(F)/F$. Then $F_x = F_a$ for some $a \in rad(F)$. Hence $(x, a) \in \theta_F$, which provides $a*x \in F \subseteq rad(F)$. Since $a \in rad(F)$ and $rad(F)$ is a filter, we get $x \in rad(F)$.
(3) Since $1 \in F \subseteq rad(F)$, we get $F_1 \in rad(F)/F$. Let $F_x, F_x * F_y \in rad(F)/F$. Then $F_{x*y} \in rad(F)/F$. By (2), we get $x \in rad(F)$ and $x * y \in rad(F)$. Since $rad(F)$ is a filter of X, we get $y \in rad(F)$. Hence $F_y \in rad(F)/F$. Therefore $rad(F)/F$ is a filter of X/F. We now show that $rad(F)/F$ is semi-maximal in X/F. Clearly $rad(F)/F \subseteq rad(rad(F)/F)$. Conversely, let $F_x \in rad(rad(F)/F)$. Then by Theorem 4.4, we get $F_{xN*x} = F_xN * F_x \in rad(F)/F$. Then by (2), we get $xN * x \in rad(F)$. So by Theorem 4.4, we get $x \in rad(rad(F)) = rad(F)$. Hence $F_x \in rad(F)/F$. Therefore $rad(rad(F)/F) \subseteq rad(F)/F$, which gives $rad(F)/F$ is semi-maximal of X/F. □

Theorem 5.12. *Let X be a self-distributive BE-algebra and F be a proper filter of X. Then F is a semi-maximal filter of X if and only if $\{1\}/F$ is a semi-maximal filter of the quotient algebra X/F.*

Proof. Assume that F is a semi-maximal filter of X. Then $rad(F) = F$. By Lemma 5.11(1), we have $\{1\}/F$ is a filter of X/F. Clearly $\{1\}/F \subseteq rad(\{1\}/F)$. Let $F_x \in rad(\{1\}/F)$. Then $F_{xN*x} = F_xN * F_x \in \{1\}/F$. Now

$$\begin{aligned} F_{xN*x} \in \{1\}/F &\Rightarrow (xN * x, 1) \in \theta_F \\ &\Rightarrow 1 * (xN * x) \in F \\ &\Rightarrow xN * x \in F \\ &\Rightarrow x \in rad(F) = F \end{aligned}$$

which gives $1 * x \in F$. Since F is a filter, we get $x * 1 = 1 \in F$. Hence $(x, 1) \in \theta_F$, which means $F_x \in \{1\}/F$. Therefore $rad(\{1\}/F) = \{1\}/F$.

Conversely, assume that $\{1\}/F$ is a semi-maximal filter of X/F. Then $rad(\{1\}/F) = \{1\}/F$.

Clearly $F \subseteq rad(F)$. Again, let $x \in rad(F)$. Then $xN * x \in F$. Since $1 \in F$, we get $(xN * x, 1) \in \theta_F$. Hence

$$\begin{aligned} F_{xN*x} \in \{1\}/F &\Rightarrow (F_x)N * F_x \in \{1\}/F \\ &\Rightarrow F_x \in rad(\{1\}/F) \\ &\Rightarrow F_x \in \{1\}/F \end{aligned}$$

which gives $(x, 1) \in \theta_F$. Hence $x = 1*x \in F$. Thus $rad(F) \subseteq F$. Therefore F is semi-maximal of X. □

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