Palestine Journal of Mathematics

Vol. 11(2)(2022) , 256–263 © Palestine Polytechnic University-PPU 2022

Multipliers on class of Dirichlet series having vector valued frequencies

Nibha Dua and Niraj Kumar

Communicated by V. Lokesha

MSC 2010 Classifications: 30B50, 30D15.

Keywords and phrases: Dirichlet Series, Dual space.

Abstract Several investigations have been made on class of Dirichlet series with real and complex frequencies in the past. In the present paper, we consider functions represented by Dirichlet series with vector valued frequencies and study the dual nature of the class of such series converging in the complex plane. We also obtain coefficient multipliers for some classes of such series. No such series has yet been considered before.

1 Introduction

An infinite series of the form

$$\sum_{n=1}^{\infty} a_n e^{\lambda_n s} \; ; \quad s = \sigma + it(\sigma, t \in \mathbb{R}) \tag{1.1}$$

where (a_n) is a sequence of complex numbers and (λ_n) is a strictly increasing sequence of positive real numbers tending to infinity is called Dirichlet series. S. Mandelbrojt[6] proved that if the coefficients (a_n) and the sequence of exponents (λ_n) satisfy

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} < \infty$$

and

$$\limsup_{n\to\infty} \frac{\log |a_n|}{\lambda_n} = -\infty$$

then the series (1.1) converges in the whole complex plane. Many researchers in the past have worked on class of entire functions represented by series (1.1). Srivastava[9] provided Banach algebraic structure to class of such series. He[10] also studied growth properties of space of entire Dirichlet series. Kamthan[2] proved the class of entire Dirichlet series to be an FK Space.

B.L. Srivastava[8] replaced the coefficients (a_n) in (1.1) with elements from a commutative Banach Algebra and thus introduced the concept of vector valued Dirichlet series. He showed that if the function $f : \mathbb{C} \to \mathbb{E}$ represented by

$$f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s} \tag{1.2}$$

satisfies

$$\limsup_{n\to\infty} \frac{\log n}{\lambda_n} = D < \infty$$

and

$$\lim_{n\to\infty} \frac{\log \|a_n\|}{\lambda_n} = -\infty$$

then the series (1.2) becomes entire where (a_n) represents a sequence of elements from a complex commutative Banach Algebra, $(\mathbb{E}, \|.\|)$. Kumar and Manocha[5] established some results on class of vector valued Dirichlet series for which the sequence

$${\lambda_n}^{c_1\lambda_n} e^{(c_2 n - c_1)\lambda_n} \|a_n\|$$

is bounded where $c_1, c_2 \geq 0$ and c_1, c_2 are simultaneously not zero. Akanksha and Srivastava[1] studied the dual space of a sequence space which depends upon the order of an entire function represented by vector valued Dirichlet series and also obtained coefficient multipliers for some classes of such series.

Khoi in [3] considered Dirichlet series with complex coefficients and complex sequence of exponents. He showed that if the conditions

$$\limsup_{n\to\infty} \frac{\log n}{|\lambda_n|} = D < \infty$$

and

$$\lim_{n\to\infty} \frac{\log |a_n|}{|\lambda_n|} = -\infty$$

hold then the series

$$f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s}$$

becomes entire. Here (a_n) denotes a sequence of complex coefficients and (λ_n) denotes the complex sequence of exponents satisfying $\lim_{n\to\infty} |\lambda_n| = \infty$. He in the same paper studied the concept of duality and obtained the coefficient multipliers for some classes of such series. Kumar and Manocha[4] studied several properties on class of entire functions defined by multiple Dirichlet series with complex frequencies.

Through this paper, we make a breakthrough in the direction of Dirichlet Series with vector valued frequencies. No such series has yet been considered before.
Consider $f : \mathbb{C} \to \mathbb{E}$ defined by

$$f(s) = \sum_{n=1}^{\infty} a_n e^{\lambda_n s} \ ; s = \sigma + it(\sigma, t \in \mathbb{R}) \tag{1.3}$$

where a_n is the sequence of complex numbers and ${\lambda_n}^{'} s$ are the elements of a commutative complex Banach Algebra, say $(\mathbb{E}, \|.\|)$ having w as an identity element with $\|w\| = 1$.
Let $\lim_{n\to\infty} \|\lambda_n\| = \infty$.
Alvaro H. Salas[7] defined an exponential function from $\mathbb{E}$ to $\mathbb{E}$ as

$$\begin{aligned} exp(x) &= \sum_{n=1}^{\infty} \frac{x^n}{n!} \\ &= w + x + \frac{x^2}{2!} + \frac{x^3}{3!} + ... \end{aligned}$$

where w denotes an identity element of $\mathbb{E}$. Clearly $\|e^x\| \leq e^{\|x\|}$ for every x $\in \mathbb{E}$.

Theorem 1.1. *If the coefficient ${a_n}^{'} s$ and the sequence of exponents λ_n in the series represented by (1.3) satisfy*

$$\limsup_{n\to\infty} \frac{n}{\|\lambda_n\|} < \infty \tag{1.4}$$

and

$$\limsup_{n\to\infty} \frac{\log |a_n|}{\|\lambda_n\|} = -\infty \tag{1.5}$$

then the series (1.3) converges absolutely for all $s \in \mathbb{C}$.

Proof. Take an arbitrary element s $\in \mathbb{C}$ and let $|s| = R$. Then for large 'n' and for $M > R$, we have

$$\log |a_n| < -M\|\lambda_n\| \ (from\,(1.5))$$

Now

$$\begin{aligned}\|a_n e^{\lambda_n s}\| &= |a_n| \ \|e^{\lambda_n s}\| \\ &\leq |a_n| \ e^{\|\lambda_n s\|} \\ &= |a_n| \ e^{|s| \ \|\lambda_n\|} \\ &< e^{-(M-R) \ \|\lambda_n\|}\end{aligned}$$

Since the sequence (λ_n) satisfies condition (1.4), the series $\sum_{n=1}^{\infty} e^{-a\|\lambda_n\|}$ converges for all $a > 0$. Thus the series (1.3) converges absolutely for all s $\in \mathbb{C}$ and hence the theorem.

□

In connection with the Theorem 1.1 , given a vector valued sequence (λ_n) of elements in $\mathbb{E}$, we associate to it the following sequence space

$$\mathbb{M} = \{(a_n) : \lim_{n\to\infty} |a_n|^{\frac{1}{\|\lambda_n\|}} = 0\}$$

That is, elements of the class $\mathbb{M}$ are entire functions represented by Dirichlet series with vector valued frequencies.
In the present paper, we try to study the dual nature of the space $\mathbb{M}$ of entire Dirichlet series with vector valued frequencies and also try to obtain the coefficient multipliers for some classes of such series.

Following are some definitions used in the sequel.

Definition 1.2. Let X and Y be two sequence spaces. Then a sequence (r_n) is said to be a multiplier from space X to Y if for all (x_n) in X , $(r_n x_n) \in$ Y and the space of such multipliers from X to Y is denoted by (X,Y).

Definition 1.3. The Köthe dual of the sequence space X is basically (X, l^1) i.e the space of all multipliers from space X to the sequence space l^1. The Köthe dual of the space $\mathbb{M}$ is denoted by $\mathbb{M}^\alpha$, i.e.

$$\mathbb{M}^\alpha = \{(u_n) \subseteq \mathbb{C} : \sum_{n=1}^{\infty} |a_n u_n| < \infty \ \forall \ (a_n) \in \mathbb{M}\}$$

We define a sequence space $\mathbb{M}^\beta$ by

$$\mathbb{M}^\beta = \{(u_n) \subseteq \mathbb{C} : \sum_{n=1}^{\infty} a_n u_n < \infty \ \forall \ (a_n) \in \mathbb{M}\}$$

2 Main Results

Theorem 2.1. *$(u_n) \in \mathbb{M}^\beta$ if and only if*

$$\limsup_{n\to\infty} |u_n|^{\frac{1}{\|\lambda_n\|}} < \infty \tag{2.1}$$

Proof. Let $(u_n) \in \mathbb{M}^{\beta}$. We consider

$$\limsup_{n\to\infty} |u_n|^{\frac{1}{\|\lambda_n\|}} = \infty$$

then there exists an increasing sequence $(n_k), k = 1, 2, ..$ of positive integers such that

$$\lim_{k\to\infty} |u_{n_k}|^{\frac{1}{\|\lambda_{n_k}\|}} = \infty$$

A sequence (a_n) is defined as $\frac{1}{|u_n|}$ when $n = n_k(k = 1, 2..)$ and 0 otherwise.

Then

$$\begin{aligned}\limsup_{n\to\infty} |a_n|^{\frac{1}{\|\lambda_n\|}} &= \lim_{k\to\infty} |a_{n_k}|^{\frac{1}{\|\lambda_{n_k}\|}} \\ &= \lim_{k\to\infty} \frac{1}{|u_{n_k}|^{\frac{1}{\|\lambda_{n_k}\|}}} \\ &= 0\end{aligned}$$

Hence $(a_n) \in \mathbb{M}$.

However $|a_{n_k} u_{n_k}| = 1$ for every $k = 1, 2, ..$ hence $\sum_{n=1}^{\infty} a_n u_n$ is not convergent leading to the contradiction that $(u_n) \in \mathbb{M}^{\beta}$. Hence (2.1) holds.

Conversely let (2.1) holds.
Then there exists a constant L such that whenever $n \geq n_1$

$$|u_n| \leq L^{\|\lambda_n\|}$$

Let (v_n) be some arbitrary sequence in $\mathbb{M}$. Then for some $0 < \epsilon < 1/L$, there exists some n_2 such that

$$|v_n| \leq \epsilon^{\|\lambda_n\|} \ \forall \ n \geq n_2$$

For $N \geq max(n_1, n_2)$ we have

$$\begin{aligned}\sum_{n=1}^{\infty} |v_n u_n| &= \sum_{n=1}^{N-1} |v_n||u_n| + \sum_{n=N}^{\infty} |v_n||u_n| \\ &\leq \sum_{n=1}^{N-1} |v_n||u_n| + \sum_{n=N}^{\infty} (L\epsilon)^{\|\lambda_n\|}\end{aligned}$$

On account of (1.4), the series $\sum_{n=1}^{\infty} r^{\|\lambda_n\|} < \infty$ for all $r \in (0, 1)$. Thus the series on the right side of above inequality is convergent. Hence $(u_n) \in \mathbb{M}^{\alpha} \subset \mathbb{M}^{\beta}$.

□

Remark 2.2. Clearly $\mathbb{M}^{\alpha} \subset \mathbb{M}^{\beta}$. Also if a sequence $(u_n) \in \mathbb{M}^{\beta}$ then from above $(u_n) \in \mathbb{M}^{\alpha}$. Hence $\mathbb{M}^{\alpha} = \mathbb{M}^{\beta}$.

Theorem 2.3. *Space* $\mathbb{M}$ *is perfect i.e.* $\mathbb{M}^{\alpha\alpha} = \mathbb{M}$.

Proof. Clearly $\mathbb{M} \subset \mathbb{M}^{\alpha\alpha}$.
Let (u_n) does not belong to $\mathbb{M}$. Then

$$\limsup_{n\to\infty} |u_n|^{\frac{1}{\|\lambda_n\|}} > 0$$

Let $L = \limsup_{n\to\infty} |u_n|^{\frac{1}{\|\lambda_n\|}}$.

Then there exists an increasing sequence $\{n_k\}, k = 1, 2, ..$ of positive integers such that

$$L = \lim_{k\to\infty} |u_{n_k}|^{\frac{1}{\|\lambda_{n_k}\|}}$$

A sequence (a_n) is defined as $\frac{1}{|u_n|}$ when $n = n_k(k = 1, 2..)$ and 0 otherwise.
Then

$$\begin{aligned}
\limsup_{n\to\infty} |a_n|^{\frac{1}{\|\lambda_n\|}} &= \lim_{k\to\infty} |a_{n_k}|^{\frac{1}{\|\lambda_{n_k}\|}} \\
&= \lim_{k\to\infty} \frac{1}{|u_{n_k}|^{\frac{1}{\|\lambda_{n_k}\|}}} \\
&= \frac{1}{L} \\
&< \infty
\end{aligned}$$

Hence from Theorem 1, $(a_n) \in \mathbb{M}^\alpha$.
However $|a_n u_n| = 1$ hence $\sum_{n=1}^{\infty} |a_n u_n|$ does not converge. Thus (u_n) does not belong to $\mathbb{M}^{\alpha\alpha}$.
Hence the space $\mathbb{M}$ is perfect.

□

Theorem 2.4. $(\mathbb{M}, l^p) = \mathbb{M}^\alpha$ *for* $0 < p \leq \infty$.

Proof. Let (u_n) does not belong to $\mathbb{M}^\alpha$. Then from Theorem 1,

$$\limsup_{n\to\infty} |u_n|^{\frac{1}{\|\lambda_n\|}} = \infty$$

hence there exists an increasing sequence $(n_k), k = 1, 2, ..$ of positive integers such that

$$\lim_{k\to\infty} |u_n|^{\frac{1}{\|\lambda_n\|}} = \infty$$

For $0 < p < \infty$, a sequence (a_n) is defined as $\frac{1}{|u_n|}$ when $n = n_k(k = 1, 2, ..)$ and 0 otherwise.
Then

$$\begin{aligned}
\limsup_{n\to\infty} |a_n|^{\frac{1}{\|\lambda_n\|}} &= \lim_{k\to\infty} |a_{n_k}|^{\frac{1}{\|\lambda_{n_k}\|}} \\
&= \lim_{k\to\infty} \frac{1}{|u_{n_k}|^{\frac{1}{\|\lambda_{n_k}\|}}} \\
&= 0
\end{aligned}$$

Then $(a_n) \in \mathbb{M}$. However $(u_n a_n)$ does not belong to l^p which shows that $(u_n) \notin (\mathbb{M}, l^p)$.

For $p = \infty$ we take (a_n) as $\frac{n}{|u_n|}$ when $n = n_k(k = 1, 2, ..)$ and 0 otherwise.
Then

$$\begin{aligned}
\limsup_{n\to\infty} |a_n|^{\frac{1}{\|\lambda_n\|}} &= \lim_{k\to\infty} \frac{n^{\frac{1}{\|\lambda_n\|}}}{|u_n|^{\frac{1}{\|\lambda_n\|}}} \\
&= 0
\end{aligned}$$

Then $(a_n) \in \mathbb{M}$. However $(u_n a_n)$ does not belong to l^∞ which shows that $(u_n) \notin (\mathbb{M}, l^p)$.
Thus $(\mathbb{M}, l^p) \subset \mathbb{M}^\alpha$.

Conversely let $(u_n) \in \mathbb{M}^{\alpha}$. Then there exists some T such that

$$|u_n| \leq T^{\|\lambda_n\|} \ for \ n \geq n_1$$

Let $(a_n) \in \mathbb{M}$. Then for $0 < \epsilon < 1/T$, there exists n_2 such that

$$|a_n| \leq \epsilon^{\|\lambda_n\|} \ for \ n \geq n_2$$

Now for $N \geq \max\{n_1, n_2\}$, we have

$$|a_N u_N| \leq (T\epsilon)^{\|\lambda_n\|}$$

If $0 < p < \infty$, then

$$\begin{aligned}\sum_{n=1}^{\infty} |a_n u_n|^p &= \sum_{n=1}^{N-1} |a_n u_n|^p + \sum_{n=N}^{\infty} |a_n u_n|^p \\ &\leq \sum_{n=1}^{N-1} |a_n u_n|^p + \sum_{n=1}^{\infty} (\epsilon T)^{p\|\lambda_n\|} \\ &< \infty\end{aligned}$$

Thus, $(a_n u_n) \in l^p$.
Now, for $p = \infty$, $|a_n u_n| \leq 1$ thus $(a_n u_n) \in l^{\infty}$.
Hence $(u_n) \in (\mathbb{M}, l^p)$ for $0 < p \leq \infty$.

Thus the theorem.

□

Theorem 2.5. $(l^p, \mathbb{M}) = \mathbb{M}$ *for* $0 < p \leq \infty$.

Proof. Let (u_n) belongs to $(l^p, \mathbb{M})$ but does not belong to $\mathbb{M}$. Then

$$\limsup_{n\to\infty} |u_n|^{\frac{1}{\|\lambda_n\|}} > 0$$

Let $L = \limsup_{n\to\infty} |u_n|^{\frac{1}{\|\lambda_n\|}}$

Then for $0 < 2\epsilon < L$, we have a subsequence $(u_{n_k})_k$ such that

$$|u_{n_k}|^{\frac{1}{\|\lambda_{n_k}\|}} > L - \epsilon \ whenever \ k \geq 1$$

i.e

$$|u_{n_k}| > (L - \epsilon)^{\|\lambda_{n_k}\|} \ whenever \ k \geq 1 \tag{2.2}$$

For $0 < p < \infty$, a sequence (v_n) is defined as $\dfrac{(L-2\epsilon)^{\|\lambda_n\|}}{|u_n|}$ whenever $n = n_k (k = 1, 2, ..)$ and 0 elsewhere.
Then

$$\begin{aligned}\sum_{n=1}^{\infty} |v_n|^p &= \sum_{k=1}^{\infty} |v_{n_k}|^p \\ &= \sum_{k=1}^{\infty} \frac{(L-2\epsilon)^{p\|\lambda_{n_k}\|}}{|u_{n_k}|^p} \\ &< \sum_{k=1}^{\infty} \frac{(L-2\epsilon)^{p\|\lambda_{n_k}\|}}{(L-\epsilon)^{p\|\lambda_n\|]}} \ (from \ (2.2)) \\ &< \infty\end{aligned}$$

Here,

$$\begin{aligned}\limsup_{n\to\infty}|v_n u_n|^{\frac{1}{\|\lambda_n\|}} &= \lim_{k\to\infty}|v_{n_k}u_{n_k}|^{\frac{1}{\|\lambda_{n_k}\|}}\\ &= L-2\epsilon\\ &> 0\end{aligned}$$

Thus $(v_n)\in l^p$ but $(v_n u_n)$ does not belong to $\mathbb{M}$ which is a contradiction.

Also for $p=\infty$, we define a sequence (w_n) as $\dfrac{(L-2\epsilon)^{\|\lambda_n\|}}{|u_n|}$ when $n=n_k(k=1,2,..)$ and 0 elsewhere.
Clearly $(w_n)\in l^\infty$ as $|w_n|\le 1\ \forall$ n ≥ 1. Here

$$\begin{aligned}\limsup_{n\to\infty}|w_n u_n|^{\frac{1}{\|\lambda_n\|}} &= \lim_{k\to\infty}|w_{n_k}u_{n_k}|^{\frac{1}{\|\lambda_{n_k}\|}}\\ &= L-2\epsilon\\ &> 0\end{aligned}$$

We are here with sequence $(w_n u_n)$ which does not belong to $\mathbb{M}$ but (w_n) belongs to l^∞ which is again a contradiction to the fact that $(u_n)\in(l^p,\mathbb{M})$.
Hence $(l^p,\mathbb{M})\subset\mathbb{M}$ for $0<p\le\infty$.

Conversely let $(u_n)\in\mathbb{M}$. Then

$$\limsup_{n\to\infty}|u_n|^{\frac{1}{\|\lambda_n\|}}=0$$

Let $(r_n)\in l^p(0<p\le\infty)$ be an arbitrary sequence. Then there exists some B such that

$$|r_n|\le B\ \forall\ n\ge 1.$$

Here

$$\limsup_{n\to\infty}|u_n r_n|^{\frac{1}{\|\lambda_n\|}}=\limsup_{n\to\infty}|u_n|^{\frac{1}{\|\lambda_n\|}}|r_n|^{\frac{1}{\|\lambda_n\|}}=0.$$

Thus $(u_n r_n)\in\mathbb{M}$. Hence for $0<p\le\infty$, $\mathbb{M}\subset(l^p,\mathbb{M})$.
Thus the theorem. □

References

[1] Akanksha and Srivastava, G.S., *Mulitpliers on spaces of entire vector valued Dirichlet series*, Journal of Classical Analysis, **4(1)**(2014), 89-95.

[2] Kamthan, P.K., *FK-space for entire Dirichlet functions*, Collect Math., **20**(1969), 271 - 280.

[3] Khoi, L.H., *Coefficient Mutipliers for some classes of Dirichlet Series in several Complex Variables*, Acta Mathematica Vietnamica, **24(2)**(1999), 169-182.

[4] Kumar, N. and Manocha, G., *Certain results on a class of Entire functions represented by Dirichlet series having complex frequencies*, Acta Universitatis Matthiae Belii, series Mathematics, **23**(2015), 95–100.

[5] Kumar, N. and Manocha, G., *On a class of entire functions represented by Dirichlet series*, Journal of the Egyptian Mathematical Society, **21**(2013), 21-24.

[6] Mandelbrojt, S., *Dirichlet Series: Principles and Methods*, D. Riedel Pub. Co., Dordrecht, Holland, 1972.

[7] Salas, A.H., *The exponential and Logarithmic functions on commutative Banach Algebras*, Int. Journal of Math. Analysis, **4(42)**(2010), 2079 - 2088.

[8] Srivastava, B.L., *Study of Spaces of Certain Classes of Vector Valued Dirichlet Series*, Thesis, Indian Institute Of Technology Kanpur, 1983.

[9] Srivastava, R.K., *On a class of entire Dirichlet series*, Ganita, **30**(1979) , 115-119.

[10] Srivastava, R.K., *Some growth properties of a class of entire Dirichlet series*, Proc. Nat. Acad. Sci. India, **61(A) IV**(1991) ,507-517.

Author information

Nibha Dua and Niraj Kumar, Department of Mathematics, Netaji Subhas Institute of Technology, University of Delhi, Sector 3 Dwarka, New Delhi-110078, INDIA.
E-mail: nibhad.phd.16@nsit.net.in and nirajkumar2001@hotmail.com

Received: December 6, 2020
Accepted: July 4, 2021