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TRIHARMONIC CURVES IN SOL$_3$ SPACE

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Abstract In this paper, we study triharmonic curves and bi -f- harmonic curves in the standard three-dimensional geometry Sol_3 with the left-invariant metric $g_{Sol_3} = ds^2 = (e^z dx)^2 + (e^{-z} dy)^2 + (dz)^2$. We characterize the triharmonic curves in terms of their curvature and torsion.

1 Introduction

Let $\psi : I \to Sol_3$ be a differentiable curve parametrized by arc length and let $\{T, N, B\}$ be the orthonormal moving Frenet frame along the curve ψ in Sol_3 such that $T = \psi^{'}$ is the unit vector field tangent to ψ, N is the unit vector field in the direction $\nabla_T T$ normal to ψ (principal normal) and $B = T \wedge N$ (binormal vector). Then we have the following Frenet equations

$$\begin{pmatrix} \nabla_T T \\ \nabla_T N \\ \nabla_T B \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}, \tag{1.1}$$

where

$$k^2 = g_{Sol_3}(\nabla_T T, \nabla_T T),$$

is the curvature of ψ and τ is its torsion.

The planes spanned by $\{T, N\}$, $\{T, B\}$ and $\{N, B\}$ are respectively known as the osculating, the rectifying and the normal plane.

Now curves with position vectors lie in the above defined three planes are respectively called osculating, rectifying and normal curves.
A. A. Shaikh, M. S. Lone and P. R. Ghosh in [13], [14], [15] studied rectifying, osculating and normal curves on a smooth immersed surface in the Euclidean space $\mathbb{R}^3$ and obtained their characterizations under isometry of surfaces.
First we should recall some notions and results related to the harmonic and the Polyharmonic ($r-$ harmonic $r \geq 1$) maps between Riemannian manifolds.

Harmonic maps $\psi : (M, g) \to (N, \tilde{g})$ between Riemannian manifolds are the critical points of the energy functional

$$E_1 : C^\infty(M, N) \to \mathbb{R}, \;\; E_1(\psi) = \frac{1}{2} \int_M |\mathrm{d}\psi|^2 \, v_g,$$

and is characterized by the vanishing of the first tension field

$$\tau_1(\psi) = -\mathrm{d}^* \mathrm{d}\psi = trace \nabla \mathrm{d}\psi,$$

where d is the exterior differentiation and d* is the codifferentiation.

We remind that the bienergy of ψ is given by

$$E_2 : C^\infty(M, N) \to \mathbb{R}, \;\; E_2(\psi) = \frac{1}{2} \int_M |\tau(\psi)|^2 \, v_g,$$

and the bitension field $\tau_2(\psi)$ has the expression

$$\tau_2(\psi) = -\Delta^{\psi}\tau(\psi) - trace_g R^N(\mathrm{d}\psi, \tau(\psi))\mathrm{d}\psi,$$

where $\Delta^{\psi} = -trace(\nabla^{\psi})^2 = -trace(\nabla^{\psi}\nabla^{\psi} - \nabla^{\psi}_{\nabla})$.

A smooth map ψ is *biharmonic* if it satisfies the following biharmonic equation

$$\tau_2(\psi) = 0.$$

Biharmonic maps are the critical points of the bienergy functional E_2. We call proper biharmonic the non-harmonic biharmonic maps. Biharmonic curves ψ of a Riemannian manifold are the solutions of the fourth order differential equation

$$\nabla^3_{\phi'}\phi' - R(\phi', \nabla_{\phi'}\phi')\phi' = 0. \tag{1.2}$$

Eells and Lemaire [5] proposed the problem to consider the polyharmonic ($r-$ harmonic $r \geq 1$) maps of order r, these are critical points of the $r-$ energy functional defined by

$$E_r(\psi) = \int_M e_r(\psi)v_g, \quad r \geq 1, \tag{1.3}$$

where $e_r(\psi) = \frac{1}{2}\|(\mathrm{d} + \mathrm{d}^*)^r\psi\|^2$ for smooth maps ψ.

A map ψ is $r-$ harmonic if it is a critical point of the functional $E_r(\psi)$ defined in (1.3).

Every harmonic map is a solution of the polyharmonic map, see [1] for a recent classification result. In [19], S.B. Wang studied the first variation formula of the k- energy E_k , whose critical maps are called k- harmonic maps. In [8], S. Maeta showed the second variation formula of the k- energy. Triharmonic curves with constant curvature in space forms were studied by Maeta in [8].

In this paper, we study triharmonic curves and bi -f- harmonic curves in the standard three-dimensional geometry Sol$_3$. We characterize the triharmonic curves in terms of their curvature and torsion.

2 Preliminaries

The space Sol$_3$ is one of the eight models of geometry of Thurston [17]. The space Sol$_3$ is the space $\mathbb{R}^3$ equipped with the metric

$$g_{Sol_3} = ds^2 = (e^z dx)^2 + (e^{-z}dy)^2 + (dz)^2,$$

where $(x,\, y,\, z)$ are usual coordinates of $\mathbb{R}^3$ (see for instance [6], [18]).

The space Sol_3 is a Lie group with the multiplication

$$(x, y, z) * (x', y', z') = (x + e^{-z}x', y + e^z y', z + z'),$$

where $*$ denotes the group operation of Sol_3. A left-invariant orthonormal frame $\{e_1,\ e_2,\ e_3\}$ in Sol_3 is given by

$$e_1 = e^{-z}\frac{\partial}{\partial x}, \quad e_2 = e^z\frac{\partial}{\partial y}, \quad e_3 = \frac{\partial}{\partial z}.$$

Proposition 2.1 ([18]). *The Levi Civita connection ∇ of Sol_3 with respect to this* *frame is*

$$\begin{pmatrix} \nabla_{e_1}e_1 \\ \nabla_{e_1}e_2 \\ \nabla_{e_1}e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

$$\begin{pmatrix} \nabla_{e_2}e_1 \\ \nabla_{e_2}e_2 \\ \nabla_{e_2}e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix} \tag{2.1}$$

$$\begin{pmatrix} \nabla_{e_3}e_1 \\ \nabla_{e_3}e_2 \\ \nabla_{e_3}e_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}.$$

Also, we obtain the bracket relations

$$[e_1,\ e_2] = 0, \quad [e_2,\ e_3] = -e_2, \quad [e_1,\ e_3] = e_1. \tag{2.2}$$

We shall adopt the following notation and sign convention. The Riemannian curvature operator is given by

$$R(X,\ Y)Z = \nabla_X\nabla_Y Z - \nabla_Y\nabla_X Z - \nabla_{[X,\ Y]}Z. \tag{2.3}$$

The Riemannian curvature tensor is given by

$$R(X,\ Y,\ Z,\ W) = g_{Sol_3}(R(Y,\ X)Z,\ W) = -g_{Sol_3}(R(X,\ Y)Z,\ W), \tag{2.4}$$

where X, Y, Z, W are smooth vector fields on Sol_3.
Moreover we put

$$R_{ijk} = R(e_i,\ e_j)e_k, \quad R_{ijkl} = R(e_i,\ e_j,\ e_k,\ e_l), \tag{2.5}$$

where $i, j, k, l \in \{1, 2, 3\}$.
The non vanishing components of the above tensor fields are

$$R_{121} = R_{233} = -e_2,\ R_{131} = R_{232} = e_3,\ R_{122} = -R_{133} = e_1, \tag{2.6}$$

and

$$R_{1212} = -R_{1313} = -R_{2323} = 1. \tag{2.7}$$

3 Polyharmonic curves in Sol_3

3.1 Biharmonic curves in Sol_3

Biharmonic curves in a three-dimensional Riemannian manifold with constant sectional curvature $K \leq 0$ are geodesics [4]. In [2] the authors considered the case of positive curvature showing that biharmonic curves have constant geodesic curvature and geodesic torsion (helices).

Let $\psi : I \to Sol_3$ be a differentiable curve parametrized by arc length.
From (1.1) we have

$$\nabla_T^3 T = (-3kk')T + (k'' - k^3 - k\tau^2)N + (2k'\tau + k\tau')B, \tag{3.1}$$

where $k' = \frac{dk}{ds}$, $k'' = \frac{d^2k}{ds^2}$, $\tau' = \frac{d\tau}{ds}$.
Using (2.7) one obtains [10]

$$R(T, N, T, N) = 2B_3^2 - 1, \quad R(T, N, T, B) = -2N_3B_3, \tag{3.2}$$

where

$$\begin{cases} T = T_1e_1 + T_2e_2 + T_3e_3 \\ N = N_1e_1 + N_2e_2 + N_3e_3 \\ B = T \wedge N = B_1e_1 + B_2e_2 + B_3e_3. \end{cases} \tag{3.3}$$

Theorem 3.1 ([10]). *Let $\psi : I \to Sol_3$ be a differentiable curve parametrized by arc length. Then ψ is a proper non-geodesic biharmonic curve if and only if*

$$\begin{cases} k = \textit{constant} \neq 0 \\ k^2 + \tau^2 = 2B_3^2 - 1 \\ \tau' = 2N_3B_3. \end{cases} \tag{3.4}$$

Corollary 3.2. *If $\tau = 0$ and $k = \textit{constant} \neq 0$ for a curve ϕ. ϕ is a non-geodesic biharmonic curve then*

$$\begin{cases} k^2 = 2B_3^2 - 1 \\ N_3 = 0. \end{cases}$$

3.2 Triharmonic curves in Sol_3

To study the triharmonic curves in Sol_3, we shall use their Frenet vector fields and equations.

Let us denote by $\psi : I \to Sol_3$ an arclength parametrized curve in Sol_3. Assume that ψ is non-geodesic.

If $r = 2t,\ t \geq 1$, then (1.3) takes the form [7], [19]

$$E_{2t}(\psi) = \frac{1}{2}\int_M < \underbrace{(\mathrm{d}^*\mathrm{d})...(\mathrm{d}^*\mathrm{d})}_{t\ times}\psi,\ \underbrace{(\mathrm{d}^*\mathrm{d})...(\mathrm{d}^*\mathrm{d})}_{t\ times}\psi > v_g. \tag{3.5}$$

If $r = 2t + 1$, then (1.3) takes the form

$$E_{2t+1}(\psi) = \frac{1}{2}\int_M < \mathrm{d}\,\underbrace{(\mathrm{d}^*\mathrm{d})...(\mathrm{d}^*\mathrm{d})}_{t\ times}\psi,\ \mathrm{d}\,\underbrace{(\mathrm{d}^*\mathrm{d})...(\mathrm{d}^*\mathrm{d})}_{t\ times}\psi > v_g. \tag{3.6}$$

The Euler-Lagrange equations of (3.5) and (3.6), reduces to the equation

$$\tau_r(\psi) = \nabla_T^{2r-1}T + \sum_{s=0}^{r-1}(-1)^s R(\nabla_T^{2r-3-s}T,\ \nabla_T^s T)T,\ r \geq 1. \tag{3.7}$$

Solutions of $\tau_r(\psi) = 0$ are called r- harmonic curves.

Remark 3.3. We say that a $r-$ harmonic curve is proper if it is not harmonic. Any harmonic curve is a $r-$ harmonic curve, for any $r \geq 1$.

An arc-length parametrized curve $\psi : I \to M^n$ from $I \subset \mathbb{R}$ to a Riemannian manifold M^n of dimension n is called *triharmonic* if [9]

$$\nabla_T^5 T + R(\nabla_T^3 T,\ T)T - R(\nabla_T^2 T,\ \nabla_T T)T = 0. \tag{3.8}$$

Proposition 3.4. *Let $\psi : I \subset \mathbb{R} \to M^n$ be a differentiable curve parametrized by arc length. Then ψ is triharmonic curve if and only if*

$$\begin{cases} \xi_1(s) = 0 \\ \xi_2(s) - \xi_4(s)R(N,T,T,N) - \xi_5(s)R(B,T,T,N) + \xi_6(s)R(B,N,T,N) = 0 \\ \xi_3(s) - \xi_4(s)R(N,T,T,B) - \xi_5(s)R(B,T,T,B) + \xi_6(s)R(B,N,T,B) = 0, \end{cases} \tag{3.9}$$

where

$$\begin{aligned}
\xi_1(s) &= -10k'k'' - 5kk^{(3)} + 5kk'(2k^2+\tau^2) + 5k^2\tau\tau', \\
\xi_2(s) &= k^5 + k^{(4)} - 15kk'^2 - 10k^2k'' + 2k^3\tau^2 - 6\tau^2k'' \\
&\quad -12k'\tau\tau' - 3k\tau'^2 + k\tau^4 - 4k\tau\tau'', \\
\xi_3(s) &= 4\tau k^{(3)} + k\tau^{(3)} - 9k^2k'\tau - 4k'\tau^3 - 6k\tau^2\tau' + 6k''\tau' \\
&\quad -\tau'k^3 + 4k'\tau'',
\end{aligned}$$

$$\xi_4(s) = k'' - 2k^3 - k\tau^2, \quad \xi_5(s) = 2k'\tau + k\tau', \quad \xi_6(s) = k^2\tau.$$

Proof. From (1.1) we have

$$\nabla_T^2 T = (-k^2)T + (k')N + (k\tau)B, \tag{3.10}$$

$$\nabla_T^3 T = (-3kk')T + (k'' - k(k^2+\tau^2))N + (2k'\tau + k\tau')B, \tag{3.11}$$

$$\nabla_T^5 T = \xi_1(s)T + \xi_2(s)N + \xi_3(s)B, \tag{3.12}$$

By (3.8) we see that ψ is a triharmonic curve if and only if

$$\xi_1(s)T+\xi_2(s)N+\xi_3(s)B+\xi_4(s)R(N,T)T+\xi_5(s)R(B,T)T-\xi_6(s)R(B,N)T = 0. \tag{3.13}$$

Using (2.4), we have (3.9). This completes the proof. □

Theorem 3.5. *Let $\psi : I \to Sol_3$ be a differentiable curve parametrized by arc length. Then ψ is a proper non-geodesic triharmonic curve if and only if*

$$\begin{cases} \xi_1(s) = 0 \\ \xi_2(s) + \xi_4(s)(2B_3^2 - 1) - 2\xi_5(s)N_3B_3 - 2\xi_6(s)T_3B_3 = 0 \\ \xi_3(s) - 2\xi_4(s)N_3B_3 - \xi_5(s)(1-2N_3^2) + 2\xi_6(s)T_3N_3 = 0. \end{cases} \tag{3.14}$$

Proof. Using (2.7) we get

$$\begin{cases} R(B,N,T,N) = -2T_3B_3, \quad R(B,T,T,B) = 1 - 2N_3^2 \\ R(B,N,T,B) = 2T_3N_3, \quad R(T,N,T,N) = 2B_3^2 - 1 \\ R(T,N,T,B) = -2N_3B_3. \end{cases} \tag{3.15}$$

Combining (3.15) and (3.9), it is obtained (3.14). This completes the proof. □

Corollary 3.6. *If $\tau = 0$ and $N_3B_3 \neq 0$. Then, $k = 0$.*

4 Triharmonic helices in Sol_3

We shall call helix a curve in Sol_3 with constant geodesic curvature and torsion. Now, for any helix in Sol_3, the system (3.14) becomes

$$\begin{cases} (k^2+\tau^2)^2 - (2B_3^2 - 1)(2k^2+\tau^2) - 2k\tau T_3B_3 = 0 \\ N_3(B_3(2k^2+\tau^2) + \tau k T_3) = 0. \end{cases} \tag{4.1}$$

Theorem 4.1. *Let $\psi : I \to Sol_3$ be a non-geodesic triharmonic helix parametrized by arc length. Then $N_3 = 0$.*

Proof. If $N_3 \neq 0$, then from (4.1), we obtain

$$\begin{cases} (k^2+\tau^2)^2 - (2B_3^2-1)(2k^2+\tau^2) - 2k\tau T_3 B_3 = 0 \\ B_3(2k^2+\tau^2) + \tau k T_3 = 0. \end{cases} \tag{4.2}$$

Using second equation of (4.2), we have

$$(k^2+\tau^2)^2 + 2k^2 + \tau^2 = 0. \tag{4.3}$$

From the definition of helix, the curvature and torsion of ψ satisfy the following $k = constant \neq 0$ and $\tau = constant \neq 0$.

From (4.4) we have $k = 0 = \tau$, a contradiction. Thus, we must have $N_3 = 0$. □

Theorem 4.2. *Let $\psi : I \to Sol_3$ be a non-geodesic triharmonic helix parametrized by arc length. Then $B_3 \neq 0$.*

Proof. If $B_3 = 0$, from the first equation in (4.1), we obtain

$$(k^2+\tau^2)^2 + (2k^2+\tau^2) = 0. \tag{4.4}$$

Equation (4.4) implies that $k = 0 = \tau$, a contradiction. This completes the proof. □

5 General helix in Sol_3

In 1845, de Saint Venant first proved that a space curve is a general helix if and only if the ratio of curvature to torsion be constant (see [16] for details).

Definition 5.1. Let ψ be a curve in Sol_3 and $\{T, N, B\}$ be the Frenet frame on Sol_3 along ψ.
1) If both k and τ are constant along ψ, then is called circular helix with respect to Frenet frame. 2) A curve ψ such that

$$\frac{\tau}{k} = c, \; c \in \mathbb{R}, \tag{5.1}$$

is called a general helix with respect to Frenet frame.

If k =constant $\neq 0$ and $\tau = 0$, then the curve ϕ is a circle.

Theorem 5.2. *Let $\psi : I \to Sol_3$ be a non-geodesic triharmonic general helix parametrized by arc length. If $N_3 = 0$, then ψ is a circular helix.*

Proof. From (5.1), we have

$$\begin{cases} \xi_1(s) = -10k'k'' - 5kk^{(3)} + 10k^3k'(c^2+1) \\ \xi_2(s) = k^5(c^2+1)^2 + k^{(4)} - 15kk'^2(c^2+1) - 10k^2k''(c^2+1) \\ \xi_3(s) = -c\xi_1(s) \\ \xi_4(s) = k'' - k^3(c^2+1) \\ \xi_5(s) = 3ck'k \\ \xi_6(s) = ck^3. \end{cases} \tag{5.2}$$

By using equations (5.2) in (3.14), equation (3.14), we can obtain a system of three differential equations characterizing triharmonic general helix in Sol_3

$$\begin{cases} \xi_1(s) = -10k'k'' - 5kk^{(3)} + 10k^3k'(c^2+1) = 0 \\ \xi_2(s) + (2B_3^2-1)(k'' - k^3(1+c^2)) - 6ck'kN_3B_3 - 2ck^3T_3B_3 = 0 \\ 2N_3B_3(k'' - k^3(1+c^2)) + 3ck'k(1-2N_3^2) - 2ck^3T_3N_3 = 0. \end{cases} \tag{5.3}$$

Substituting $N_3 = 0$ into the third equation in (5.3) we have $k'k = 0$, which implies k =constant and hence τ =constant. Then ψ is a circular helix.

6 Bi -f- harmonic curves in Sol_3

In this section we derive the bi -f- harmonic curves in Sol_3.
The authors of [11] gave the Euler-Lagrange equation of bi -f- harmonic maps.

Bi -f- harmonic maps $\psi : (\mathcal{N},\ g) \to (\tilde{\mathcal{N}},\ \tilde{g})$ between two Riemannian manifolds are critical points of the bi -f- energy functional [11], [12]:

$$E_{f,2}(\psi) = \frac{1}{2}\int_{\Omega} |\tau_f(\psi)|^2\, v_g, \tag{6.1}$$

where $\Omega \subset \mathcal{N}$ is a compact domain, $\tau_f(\psi) = f\tau(\psi) + \mathrm{d}\psi(\mathrm{grad} f)$ is the f- tension field of ψ, $\tau(\psi) = \mathrm{trace}\nabla \mathrm{d}\psi$ is the tension field of ψ.
In [11], the authors used the name f- biharmonic maps for the critical points of the functional (6.1).

Proposition 6.1 ([12]). *Let $\alpha : I \to (\tilde{\mathcal{N}},\ \tilde{g})$ be a curve in a Riemannian manifold $(\tilde{\mathcal{N}},\ \tilde{g})$, parametrized by its arclength, and $\alpha' = T$. Then α is a bi -f- harmonic curve if and only if*

$$\begin{aligned} 0 &= (ff''' + f'f'')T + (3ff'' + 2f'^2)\nabla_T^{\tilde{\mathcal{N}}}T \\ &\quad +4(ff')\nabla_T^2 T + f^2\nabla_T^3 T + f^2 R^{\tilde{\mathcal{N}}}(\nabla_T^{\tilde{\mathcal{N}}}T,\ T)T, \end{aligned} \tag{6.2}$$

where $f : I \to (0,\ \infty)$ is a smooth map, $\nabla_T^2 T = \nabla_T^{\tilde{\mathcal{N}}}\nabla_T^{\tilde{\mathcal{N}}}T$ and $\nabla_T^3 T = \nabla_T^{\tilde{\mathcal{N}}}\nabla_T^{\tilde{\mathcal{N}}}\nabla_T^{\tilde{\mathcal{N}}}T$.

Using (1.1), (3.10) and (3.11) in (6.2), we have

Theorem 6.2. *Let $\alpha : I \to (\mathbb{R}^3,\ g_{Sol_3})$ be a curve parametrized by arc length in Sol_3 space $(\mathbb{R}^3,\ g_{Sol_3})$. Then α is a bi -f- harmonic curve if and only if*

$$\begin{aligned} 0 &= (ff''' + f'f'' - 4k^2ff' - 3kk'f^2)T \\ &\quad +(3kff'' + 2kf'^2 + 4k'ff' + (k'' - k^3 - k\tau^2)f^2)N \\ &\quad +(4k\tau ff' + (2k'\tau + k\tau')f^2)B + kf^2R(N,\ T)T. \end{aligned} \tag{6.3}$$

From (3.15), we obtain

Theorem 6.3. *Let $\alpha : I \to (\mathbb{R}^3,\ g_{Sol_3})$ be a curve parametrized by arc length in Sol_3 space $(\mathbb{R}^3,\ g_{Sol_3})$. Then α is a bi -f- harmonic curve if and only if the following equations hold:*

$$\begin{cases} ff''' + f'f'' - 4k^2ff' - 3kk'f^2 = 0 \\ 3kff'' + 2kf'^2 + 4k'ff' + (k'' - k^3 - k\tau^2)f^2 + kf^2(1 - 2B_3^2) = 0 \\ 4k\tau ff' + (2k'\tau + k\tau')f^2 + 2kf^2(N_3B_3) = 0. \end{cases} \tag{6.4}$$

In the following cases, we find necessary and sufficient conditions for curves of Sol_3 space to be bi -f- harmonic:

Case 6.1. If $k = 0$, namely α is a geodesic curve, then from (6.4) we obtain that it is bi -f- harmonic if and only if $ff''' + f'f'' = (ff'')' = 0$. Then we have the following corollary:

Corollary 6.4. *A geodesic curve is bi -f- harmonic if and only if $ff'' = constant$.*

Case 6.2. If $k = constant = c \neq 0$ and $\tau = 0$, then (6.4) reduces to

$$\begin{cases} (ff'')' = 4c^2 ff' \\ 3ff'' + 2f'^2 + (1 - c^2 - 2B_3^2)f^2 = 0 \\ N_3 B_3 = 0. \end{cases} \tag{6.5}$$

Case 6.2.1. If $B_3 = 0$, then (6.5) reduces to

$$\begin{cases} (ff'')' = 4c^2 ff' \\ 3ff'' + 2f'^2 + (1 - c^2)f^2 = 0. \end{cases} \tag{6.6}$$

From the second equation above we obtain

$$(ff'')' = \frac{2(c^2 - 1)}{3} ff' - \frac{4}{3} f' f'',$$

which implies

$$((5c^2 + 1)f + 2f'')f' = 0.$$

Then we have

Corollary 6.5. *Let $\alpha : I \to (\mathbb{R}^3,\ g_{Sol_3})$ be a curve parametrized by arc length in Sol_3 space $(\mathbb{R}^3,\ g_{Sol_3})$, with $k = constant = c \neq 0$, $\tau = 0$ and $B_3 = 0$. Then α is a bi -f- harmonic curve if and only if either f is a constant function or f is given by*

$$f(s) = c_1 \cos(\xi s) + c_2 \sin(\xi s),\ s \in I,$$

where $c_1, c_2 \in \mathbb{R}$ and $\xi = \sqrt{\frac{5c^2+1}{2}}$.

Case 6.2.2. If $N_3 = 0$, then we have

Corollary 6.6. *Let $\alpha : I \to (\mathbb{R}^3,\ g_{Sol_3})$ be a curve parametrized by arc length in Sol_3 space $(\mathbb{R}^3,\ g_{Sol_3})$, with $k = constant = c \neq 0$, $\tau = 0$ and $N_3 = 0$ $(B_3 \neq 0)$. Then α is a bi -f- harmonic curve if and only if the following equations are satisfied:*

$$\begin{cases} (ff'')' = 4c^2 ff' \\ 3ff'' + 2f'^2 + (1 - c^2 - 2B_3^2)f^2 = 0. \end{cases}$$

Case 6.3. If $k = constant = c \neq 0$ and $\tau = constant = b \neq 0$, then (6.4) reduces to

$$\begin{cases} (ff'')' = 4c^2 ff' \\ 3ff'' + 2f'^2 + (1 - c^2 - b^2 - 2B_3^2)f^2 = 0 \\ 2bf' + (N_3 B_3)f = 0. \end{cases} \tag{6.7}$$

Case 6.3.1. If $N_3 = 0$, then the third equation of (6.7) implies that f is constant and $B_3 = constant$.

Case 6.3.2. If $N_3 \neq 0$, then the first and the second equations of (6.7) give

$$2f'f'' - 2B_3 B_3' f^2 + (5c^2 + 1 - b^2 - 2B_3^2)ff' = 0. \tag{6.8}$$

Substituting the third equation of (6.7) in (6.8), we obtain

$$2N_3 f'' + ((5c^2 + 1 - b^2 - 2B_3^2)N_3 + 4B_3 B_3')f = 0.$$

Hence, we give the following result:

Corollary 6.7. *Let $\alpha : I \to (\mathbb{R}^3,\ g_{Sol_3})$ be a curve parametrized by arc length in Sol_3 space $(\mathbb{R}^3,\ g_{Sol_3})$, with $k = constant = c \neq 0$, $\tau = constant = b \neq 0$ and $N_3 \neq 0$. Then α is a bi -f- harmonic curve if and only if*

$$2N_3 f'' + ((5c^2 + 1 - b^2 - 2B_3^2)N_3 + 4B_3 B_3')f = 0.$$

Case 6.4. If $k = constant = c \neq 0$ and $\tau \neq constant$, then (6.4) reduces to

$$\begin{cases} (ff'')' = 4c^2 ff' \\ 3ff'' + 2f'^2 + (1 - c^2 - \tau^2 - 2B_3^2)f^2 = 0 \\ 4\tau f' + (\tau' + 2N_3 B_3)f = 0. \end{cases} \tag{6.9}$$

If $N_3 = 0$, then the third equation of (6.9) implies that

$$f(s) = a\tau^{-\frac{1}{4}},\ a \in \mathbb{R}.$$

Substituting the third equation into the second one in (6.9) we have

$$\varrho'' - \varrho'\varrho + \frac{1}{8}\varrho^3 - 4c^2\varrho = 0$$

$$-12\varrho' + 5\varrho^2 + 16\delta = 0,$$

where $\varrho = \frac{\tau'}{\tau}$ and $\delta = 1 - c^2 - \tau^2 - 2B_3^2$.
Therefore, we conclude that

Corollary 6.8. *Let $\alpha : I \to (\mathbb{R}^3,\ g_{Sol_3})$ be a curve parametrized by arc length in Sol_3 space $(\mathbb{R}^3,\ g_{Sol_3})$, with $k = constant = c \neq 0$, $\tau \neq constant$ and $N_3 = 0$. Then α is a bi -f- harmonic curve if and only if $f(s) = a\tau^{-\frac{1}{4}}$, $a \in \mathbb{R}$ and the torsion τ solves the following*

$$\varrho'' - \varrho'\varrho + \frac{1}{8}\varrho^3 - 4c^2\varrho = 0,$$

$$-12\varrho' + 5\varrho^2 + 16\delta = 0,$$

where $\varrho = \frac{\tau'}{\tau}$ and $\delta = 1 - c^2 - \tau^2 - 2B_3^2$.

Case 6.5. If $k \neq constant \neq 0$ and $\tau = 0$, then (6.4) reduces to

$$\begin{cases} (ff'')' - 4k^2 ff' - 3kk'f^2 = 0 \\ 3kff'' + 2kf'^2 + 4k'ff' + (k'' - k^3)f^2 + kf^2(1 - 2B_3^2) = 0 \\ N_3 B_3 = 0. \end{cases} \tag{6.10}$$

Then we have the following corollary

Corollary 6.9. *Let $\alpha : I \to (\mathbb{R}^3,\ g_{Sol_3})$ be a differentiable bi -f- harmonic curve parametrized by arc length in Sol_3 space. If $k \neq constant \neq 0$ and $\tau = 0$, then $N_3 B_3 = 0$.*

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