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4-REMAINDER CORDIAL LABELING OF PARACHUTE,TWIG,KAYAK PADDALE GRAPH

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Abstract. Let G be a (p, q) graph. Let f be a map from $V(G)$ to the set $\{1, 2, \ldots, k\}$ where k is an integer $2 < k \leq |V(G)|$. For each edge uv assign the label r where r is the remainder when $f(u)$ is divided by $f(v)$ (or) $f(v)$ is divided by $f(u)$ according as $f(u) \geq f(v)$ or $f(v) \geq f(u)$. The function f is called a k-remainder cordial labeling of G if $|v_f(i) - v_f(j)| \leq 1$, $i, j \in \{1, \ldots, k\}$ where $v_f(x)$ denote the number of vertices labelled with x and $|\eta_o - \eta_e| \leq 1$ where η_e and η_o respectively denote the number of edges labeled with even integers and number of edges labelled with odd integers. A graph with a k-remainder cordial labeling is called a k-remainder cordial graph. In this paper we investigate the $4-$remainder cordial labeling behavior of parachute,twig and kayak paddale graph.

1 Introduction

In this paper we consider only finite, undirected and simple graphs.The notion of $k-$Remainder cordial labeling of a graph was introduced and studied some properties of k-Remainder cordial labeling in [3].The 4-Remainder cordial labeling behavior of several graphs like path, cycle, star, complete graph, wheel etc have been investigated in [3].In this paper we investigate the 4- Remainder cordial labeling behavior of parachute, twig and kayak paddale graph.Terms not in here followed from [1,2]

2 4- Remainder cordial labeling

Definition 2.1. Let G be a (p, q) graph. Let f be a map from $V(G)$ to the set $\{1, 2, \ldots, k\}$ where k is an integer $2 < k \leq |V(G)|$. For each edge uv assign the label r where r is the remainder when $f(u)$ is divided by $f(v)$ (or) $f(v)$ is divided by $f(u)$ according as $f(u) \geq f(v)$ or $f(v) \geq f(u)$. The function f is called a k-remainder cordial labeling of G if $|v_f(i) - v_f(j)| \leq 1$, $i, j \in \{1, \ldots, k\}$ where $v_f(x)$ denote the number of vertices labelled with x and $|\eta_e - \eta_o| \leq 1$ where η_e and η_o respectively denote the number of edges labeled with even integers and number of edges labelled with odd integers. A graph with a k-remainder cordial labeling is called a k-remainder cordial graph.

3 Preliminaries

Definition 3.1. [1] A graph obtained from the wheel W_{m+n}, $m \geq 3$ by deleting n consecutive spokes is said to be parachute and it is denoted by $P_{m,n}$.

Definition 3.2. [1] A twig $TW(P_n), n \geq 3$ is a graph obtained from a path by attaching exactly two pendant edges to each internal vertex of the path.

Definition 3.3. [1] A Kayak Paddale $KP\,(m, n, l)$ is the graph obtained by joining C_m and C_n by a path of length l. Let C_m be the cycle $x_1x_2 \cdots x_mx_1$ and C_n be the cycle $z_1z_2 \cdots z_nz_1$ and let P_l be the path $y_1y_2 \cdots y_l$. $E(KP\,(m, n, l)) = E\,(P_l) \cup E\,(C_m) \cup E\,(C_n)$, identifying x_1 with y_1 and y_n with z_1.

4 Main results

Theorem 4.1. *The Parachute $P_{m,n}$ is $4-$remainder cordial for all $m \geq 3, n \geq 1$.*

Proof. Let $V(P_{m,n}) = \{x, x_i, y_j : 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(P_{m,n}) = \{xx_i : 1 \leq i \leq m\} \cup \{x_i x_{i+1} : 1 \leq i \leq m-1\} \cup \{y_j y_{j+1} : 1 \leq j \leq n-1\} \cup \{x_1 y_1, x_m y_n\}$.
First assign the label the label 3 to the vertex x.Next assign the labels to the remaining vertices.There are 4 cases arises.

Case(i). $m \equiv 0 \pmod 4$
There are four cases arises.

Subcase(i). $n \equiv 0 \pmod 4$
First assign the labels $1, 2, 3, 4$ to the vertices x_1, x_2, x_3, x_4 and then assign the labels $1, 2, 3, 4$ to the vertices x_4, x_6, x_7, x_8.Proceed this process until we reach x_m.Secondly assign the labels $1, 2, 3, 4$ to the vertices y_1, y_2, y_3, y_4 and then assign the labels $1, 2, 3, 4$ to the vertices y_4, y_6, y_7, y_8.Proceed this process until we reach y_n.

Subcase(ii). $n \equiv 1 \pmod 4$
Now we assign the labels $1, 2, 3, 4$ to the vertices x_1, x_2, x_3, x_4 and then assign the labels $1, 2, 3, 4$ to the vertices x_4, x_6, x_7, x_8.Proceeding like this until we reach x_m.Also assign the labels $1, 2, 3, 4$ to the vertices y_1, y_2, y_3, y_4 and then assign the labels $1, 2, 3, 4$ to the vertices y_4, y_6, y_7, y_8.Proceeding like this until we reach y_{n-1}.Lastly assign the label 1 to the vertex y_n.

Subcase(iii). $n \equiv 2 \pmod 4$
Assign the labels $1, 2, 3, 4$ to the vertices x_1, x_2, x_3, x_4 and then assign the labels $1, 2, 3, 4$ to the vertices x_4, x_6, x_7, x_8.continuing like this until we reach x_{m-4}.Next assign the labels $1, 2, 4, 2$ to the vertices $x_{m-3}, x_{m-2}, x_{m-1}, x_m$.Also assign the labels $1, 2, 3, 4$ to the vertices y_1, y_2, y_3, y_4 and then assign the labels $1, 2, 3, 4$ to the vertices y_4, y_6, y_7, y_8.continuing like this until we reach y_{n-2}.Lastly assign the label $4, 3$ to the vertices y_{n-1}, y_n.

Subcase(iv). $n \equiv 3 \pmod 4$
First assign the labels $1, 2, 3, 4$ to the vertices x_1, x_2, x_3, x_4 and then assign the labels $1, 2, 3, 4$ to the vertices x_4, x_6, x_7, x_8.countinuing this process until we reach x_{m-4}.Next assign the labels $1, 2, 4, 2$ to the vertices $x_{m-3}, x_{m-2}, x_{m-1}, x_m$. Secondly assign the labels $1, 2, 3, 4$ to the vertices y_1, y_2, y_3, y_4 and then assign the labels $1, 2, 3, 4$ to the vertices y_4, y_6, y_7, y_8. Proceeding like this until we reach y_{n-3}.Lastly assign the label $1, 4, 3$ to the vertices y_{n-2}, y_{n-1}, y_n.

Thus the table 1 given below shows that $P_{m,n}$ is 4-remainder cordial.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	η_e	η_o
$n \equiv 0 \pmod 4$	$\frac{m+n}{4}$	$\frac{m+n+4}{4}$	$\frac{m+n}{4}$	$\frac{m+n}{4}$	$\frac{2m+n}{2}$	$\frac{2m+n}{2}$
$n \equiv 1 \pmod 4$	$\frac{m+n+3}{4}$	$\frac{m+n-1}{4}$	$\frac{m+n+3}{4}$	$\frac{m+n-1}{4}$	$\frac{2m+n+1}{2}$	$\frac{2m+n-1}{2}$
$n \equiv 2 \pmod 4$	$\frac{m+n-2}{4}$	$\frac{m+n+2}{4}$	$\frac{m+n+2}{4}$	$\frac{m+n+2}{4}$	$\frac{2m+n}{2}$	$\frac{2m+n}{2}$
$n \equiv 3 \pmod 4$	$\frac{m+n+1}{4}$	$\frac{m+n+1}{4}$	$\frac{m+n+1}{4}$	$\frac{m+n+1}{4}$	$\frac{2m+n+1}{2}$	$\frac{2m+n-1}{2}$

Table 1.

Case(ii). $m \equiv 1 \pmod 4$
There are four cases arises.

Subcase(i). $n \equiv 0 \pmod 4$
First assign the labels $1, 2, 3, 4$ to the vertices x_1, x_2, x_3, x_4 and then assign the labels $1, 2, 3, 4$ to the vertices x_4, x_6, x_7, x_8.Proceed this process until we reach x_{m-1}.Next assign the label 2 to the vertex x_m. Also assign the labels $1, 2, 3, 4$ to the vertices y_1, y_2, y_3, y_4 and then assign the labels

$1, 2, 3, 4$ to the vertices y_4, y_6, y_7, y_8.Proceed this process until we reach y_n.

Subcase(ii). $n \equiv 1 \pmod 4$.
Assign the labels $1, 2, 3, 4$ to the vertices x_1, x_2, x_3, x_4 and then assign the labels $1, 2, 3, 4$ to the vertices x_4, x_6, x_7, x_8.Proceeding like this until we reach x_m.Next assign the label 2 to the vertex x_m. Secondly assign the labels $1, 2, 3, 4$ to the vertices y_1, y_2, y_3, y_4 and then assign the labels $1, 2, 3, 4$ to the vertices y_4, y_6, y_7, y_8.Proceeding like this until we reach y_{n-1}.Lastly assign the label 1 to the vertex y_n.

Subcase(iii). $n \equiv 2 \pmod 4$.
We assign the labels $1, 2, 3, 4$ to the vertices x_1, x_2, x_3, x_4 and then assign the labels $1, 2, 3, 4$ to the vertices x_4, x_6, x_7, x_8.In similar manner assign the labels until we reach x_{m-5}.Next assign the label 2 to the vertices $x_{m-4}, x_{m-3}, x_{m-2}$ then assign the label 4 to the vertices x_{m-1}, x_m. Secondly assign the labels $1, 2, 3, 4$ to the vertices y_1, y_2, y_3, y_4 and then assign the labels $1, 2, 3, 4$ to the vertices y_4, y_6, y_7, y_8.In similar way proceed this until we reach y_{n-6}.Lastly Next assign the label 1 to the vertices $y_{n-5}, y_{n-4}, y_{n-3}$, and assign the label 4 to the vertex y_{n-1} then assign the label 3 to the vertices y_{n-2}, y_n.

Subcase(iv). $n \equiv 3 \pmod 4$.
Now assign the labels $1, 2, 3, 4$ to the vertices x_1, x_2, x_3, x_4 and then assign the labels $1, 2, 3, 4$ to the vertices x_4, x_6, x_7, x_8.Proceed this process until we reach x_{m-5}.Next assign the labels $1, 2, 2, 4, 4$ to the vertices $x_{m-4}, x_{m-3}, x_{m-2}, x_{m-1}, x_m$. Secondly assign the labels $1, 2, 3, 4$ to the vertices y_1, y_2, y_3, y_4 and then assign the labels $1, 2, 3, 4$ to the vertices y_4, y_6, y_7, y_8.Proceeding this process until we reach y_{n-3}. Lastly assign the label $1, 2, 3$ to the vertices y_{n-2}, y_{n-1}, y_n.

Thus the table 2 given below shows that $P_{m,n}$ is 4−remainder cordial.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	η_e	η_o
$n \equiv 0 \pmod 4$	$\frac{m+n-1}{4}$	$\frac{m+n+3}{4}$	$\frac{m+n+3}{4}$	$\frac{m+n-1}{4}$	$\frac{2m+n}{2}$	$\frac{2m+n}{2}$
$n \equiv 1 \pmod 4$	$\frac{m+n+2}{4}$	$\frac{m+n+2}{4}$	$\frac{m+n+2}{4}$	$\frac{m+n-2}{4}$	$\frac{2m+n+1}{2}$	$\frac{2m+n-1}{2}$
$n \equiv 2 \pmod 4$	$\frac{m+n+1}{4}$	$\frac{m+n+1}{4}$	$\frac{m+n+1}{4}$	$\frac{m+n+1}{4}$	$\frac{2m+n}{2}$	$\frac{2m+n}{2}$
$n \equiv 3 \pmod 4$	$\frac{m+n}{4}$	$\frac{m+n+4}{4}$	$\frac{m+n}{4}$	$\frac{m+n}{4}$	$\frac{2m+n+1}{2}$	$\frac{2m+n-1}{2}$

Table 2.

Case(iii). $m \equiv 2 \pmod 4$.
There are four cases arises.

Subcase(i). $n \equiv 0 \pmod 4$.
First assign the labels $1, 2, 3, 4$ to the vertices x_1, x_2, x_3, x_4 and then assign the labels $1, 2, 3, 4$ to the vertices x_4, x_6, x_7, x_8.In similar manner assign the labels to the vertices upto x_{m-2}.Next assign the label $4, 2$ to the vertex x_{m-1}, x_m.Secondly assign the labels $1, 2, 3, 4$ to the vertices y_1, y_2, y_3, y_4 and then assign the labels $1, 2, 3, 4$ to the vertices y_4, y_6, y_7, y_8.continue like this until we reach y_n.

Subcase(ii). $n \equiv 1 \pmod 4$
Now assign the labels $1, 2, 3, 4$ to the vertices x_1, x_2, x_3, x_4 and then assign the labels $1, 2, 3, 4$ to the vertices x_4, x_6, x_7, x_8.Proceeding like this until we reach x_m.Next assign the label $4, 2$ to the vertex x_{m-1}, x_m. Secondly assign the labels $1, 2, 3, 4$ to the vertices y_1, y_2, y_3, y_4 and then assign the labels $1, 2, 3, 4$ to the vertices y_4, y_6, y_7, y_8.Proceeding like this until we reach y_{n-1}.Lastly assign the label 1 to the vertex y_n.

Subcase(iii). $n \equiv 2 \pmod 4$.
Assign the labels $1, 2, 3, 4$ to the vertices x_1, x_2, x_3, x_4 and then assign the labels $1, 2, 3, 4$ to the vertices x_4, x_6, x_7, x_8.Proceed this process until we reach x_{m-2}.Next assign the labels $1, 4$ to

the vertices x_{m-2}, x_{m-1}, x_m.Secondly assign the labels $1, 2, 3, 4$ to the vertices y_1, y_2, y_3, y_4 and then assign the labels $1, 2, 3, 4$ to the vertices y_4, y_6, y_7, y_8. continuing this process until we reach y_{n-2}.Lastly assign the label $2, 3$ to the vertices y_{n-1}, y_n.

Subcase(iv). $n \equiv 3 \pmod 4$
We assign the labels $1, 2, 3, 4$ to the vertices x_1, x_2, x_3, x_4 and then assign the labels $1, 2, 3, 4$ to the vertices x_4, x_6, x_7, x_8.Proceed like this until we reach x_{m-2}.Next assign the labels $4, 2$ to the vertices x_{m-1}, x_m. Secondly assign the labels $1, 2, 3, 4$ to the vertices y_1, y_2, y_3, y_4 and then assign the labels $1, 2, 3, 4$ to the vertices y_4, y_6, y_7, y_8.Proceeding this process until we reach y_{n-3}. Lastly assign the label $3, 2, 1$ to the vertices y_{n-2}, y_{n-1}, y_n.

Thus the table 3 given below shows that $P_{m,n}$ is $4-$remainder cordial.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	η_e	η_o
$n \equiv 0 \pmod 4$	$\frac{m+n-2}{4}$	$\frac{m+n+2}{4}$	$\frac{m+n+2}{4}$	$\frac{m+n+2}{4}$	$\frac{2m+n}{2}$	$\frac{2m+n}{2}$
$n \equiv 1 \pmod 4$	$\frac{m+n+1}{4}$	$\frac{m+n+1}{4}$	$\frac{m+n+1}{4}$	$\frac{m+n+1}{4}$	$\frac{2m+n+1}{2}$	$\frac{2m+n-1}{2}$
$n \equiv 2 \pmod 4$	$\frac{m+n}{4}$	$\frac{m+n}{4}$	$\frac{m+n+4}{4}$	$\frac{m+n}{4}$	$\frac{2m+n}{2}$	$\frac{2m+n}{2}$
$n \equiv 3 \pmod 4$	$\frac{m+n-1}{4}$	$\frac{m+n+3}{4}$	$\frac{m+n+3}{4}$	$\frac{m+n-1}{4}$	$\frac{2m+n-11}{2}$	$\frac{2m+n+1}{2}$

Table 3.

Case(iv). $m \equiv 3 \pmod 4$.
There are four cases arises.

Subcase(i). $n \equiv 0 \pmod 4$.
First assign the labels $1, 2, 3, 4$ to the vertices x_1, x_2, x_3, x_4 and then assign the labels $1, 2, 3, 4$ to the vertices x_4, x_6, x_7, x_8.Proceeding like this until we reach x_{m-3}.Next assign the labels $2, 2, 4$ to the vertices x_{m-2}, x_{m-1}, x_m.Secondly assign the labels $1, 2, 3, 4$ to the vertices y_1, y_2, y_3, y_4 and then assign the labels $1, 2, 3, 4$ to the vertices y_4, y_6, y_7, y_8. Proceeding like this until we reach y_{n-4}.Finally assign the labels to $3, 4, 1, 1$ to the vertices $y_{n-3}, y_{n-2}, y_{n-1}, y_n$.

Subcase(ii).$n \equiv 1 \pmod 4$.
Now assign the labels $1, 2, 3, 4$ to the vertices x_1, x_2, x_3, x_4 and then assign the labels $1, 2, 3, 4$ to the vertices x_4, x_6, x_7, x_8.Proceed this pattern until we reach x_{m-3}.Next assign the labels $2, 2, 4$ to the vertices x_{m-2}, x_{m-1}, x_m.Secondly assign the labels $1, 2, 3, 4$ to the vertices y_1, y_2, y_3, y_4 and then assign the labels $1, 2, 3, 4$ to the vertices y_4, y_6, y_7, y_8.Proceeding this until we reach y_{n-1}.Lastly assign the label 1 to the vertex y_n.

Subcase(iii). $n \equiv 2 \pmod 4$.
Assign the labels $1, 2, 3, 4$ to the vertices x_1, x_2, x_3, x_4 and then assign the labels $1, 2, 3, 4$ to the vertices x_4, x_6, x_7, x_8.In similar pattern assign the labels until we reach x_{m-3}..Next assign the labels $2, 2, 4$ to the vertices x_{m-2}, x_{m-1}, x_m. Secondly assign the labels $1, 2, 3, 4$ to the vertices y_1, y_2, y_3, y_4 and then assign the labels $1, 2, 3, 4$ to the vertices y_4, y_6, y_7, y_8.This similar pattern is repeated until we reach y_{n-2}. Lastly assign the label $3, 1$ to the vertices y_{n-1}, y_n.

Subcase(iv). $n \equiv 3 \pmod 4$.
We assign the labels $1, 2, 3, 4$ to the vertices x_1, x_2, x_3, x_4 and then assign the labels $1, 2, 3, 4$ to the vertices x_4, x_6, x_7, x_8.Continuing like this until we reach x_{m-3}.Next assign the labels $1, 2, 4$, to the vertices x_{m-2}, x_{m-1}, x_m Secondly assign the labels $1, 2, 3, 4$ to the vertices y_1, y_2, y_3, y_4 and then assign the labels $1, 2, 3, 4$ to the vertices y_4, y_6, y_7, y_8. Continue this process until we reach y_{n-3}.Lastly assign the label $1, 2, 3$ to the vertices y_{n-2}, y_{n-1}, y_n.

Thus the table 4 given below shows that $P_{m,n}$ is $4-$remainder cordial.

$\square$

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	η_e	η_o
$n \equiv 0 \pmod 4$	$\frac{m+n+1}{4}$	$\frac{m+n+1}{4}$	$\frac{m+n+1}{4}$	$\frac{m+n+1}{4}$	$\frac{2m+n}{2}$	$\frac{2m+n}{2}$
$n \equiv 1 \pmod 4$	$\frac{m+n}{4}$	$\frac{m+n+4}{4}$	$\frac{m+n}{4}$	$\frac{m+n}{4}$	$\frac{2m+n+1}{2}$	$\frac{2m+n-1}{2}$
$n \equiv 2 \pmod 4$	$\frac{m+n-1}{4}$	$\frac{m+n+3}{4}$	$\frac{m+n+3}{4}$	$\frac{m+n-1}{4}$	$\frac{2m+n}{2}$	$\frac{2m+n}{2}$
$n \equiv 3 \pmod 4$	$\frac{m+n+2}{4}$	$\frac{m+n+2}{4}$	$\frac{m+n+2}{4}$	$\frac{m+n-2}{4}$	$\frac{2m+n+1}{2}$	$\frac{2m+n-1}{2}$

Table 4.

Theorem 4.2. *The Twig $TW(P_n)$ is 4-remainder cordial for all $n \geq 3$.*

Proof. Let $V(TW(P_n)) = \{u_i, v_j, w_j : 1 \leq i \leq n, 1 \leq j \leq n-2\}$ and $E(TW(P_n)) = \{u_i ui+1, v_i u_{i+1}, w_i,$ There are four cases arises.
Case(i). $n \equiv 0 \pmod 4$
First assign the labels $1, 2, 3, 4$ to the vertices u_1, u_2, u_3, u_4 and assign the labels $1, 2, 3, 4$ to the vertices u_5, u_6, u_7, u_8.Proceeding this manner until we reach u_n.Secondly assign the labels $1, 2, 3, 4$ to the vertices v_1, v_2, v_3, v_4 and assign the labels $1, 2, 3, 4$ to the vertices v_5, v_6, v_7, v_8. Proceed this pattern until we reach v_{n-4} then assign the labels $1, 2$ to the vertices v_{n-3}, v_{n-2}.Lastly assign the labels $1, 2, 3, 4$ to the vertices w_1, w_2, w_3, w_4 and assign the labels $1, 2, 3, 4$ to the vertices w_5, w_6, w_7, w_8. Proceeding like this until we reach w_{n-4} then assign the labels $4, 3$ to the vertices w_{n-3}, w_{n-2}.

Case(ii). $n \equiv 1 \pmod 4$.
Now assign the labels $1, 2, 3, 4$ to the vertices u_1, u_2, u_3, u_4 and assign the labels $1, 2, 3, 4$ to the vertices u_5, u_6, u_7, u_8.In similar manner assign the labels until we reach u_{n-1} then assign the label 4 to the vertex u_n.Secondly assign the labels $1, 2, 3, 4$ to the vertices v_1, v_2, v_3, v_4 and assign the labels $1, 2, 3, 4$ to the vertices v_5, v_6, v_7, v_8.Continuing like this until we reach v_{n-5} then assign the labels $1, 2, 3$ to the vertices $v_{n-4}, v_{n-3}, v_{n-2}$.Lastly assign the labels $1, 2, 3, 4$ to the vertices w_1, w_2, w_3, w_4 and assign the labels $1, 2, 3, 4$ to the vertices w_5, w_6, w_7, w_8.Repeating like this until we reach w_{n-5} then assign the labels $2, 4, 1$ to the vertices $w_{n-4}, w_{n-3}, w_{n-2}$.

Case(iii). $n \equiv 2 \pmod 4$.
We assign the labels $1, 2, 3, 4$ to the vertices u_1, u_2, u_3, u_4 and assign the labels $1, 2, 3, 4$ to the vertices u_5, u_6, u_7, u_8.Proceed this pattern until we reach u_{n-2} then assign the label $4, 2$ to the vertex u_{n-1}, u_n.Secondly assign the labels $1, 2, 3, 4$ to the vertices v_1, v_2, v_3, v_4 and assign the labels $1, 2, 3, 4$ to the vertices v_5, v_6, v_7, v_8.Repeat this pattern until we reach v_{n-2}.Lastly assign the labels $1, 2, 3, 4$ to the vertices w_1, w_2, w_3, w_4 and assign the labels $1, 2, 3, 4$ to the vertices w_5, w_6, w_7, w_8.Proceed like this until we reach w_{n-6} then assign the labels $2, 4, 1, 3$ to the vertices $w_{n-5}, w_{n-4}, w_{n-3}, w_{n-2}$.

Case(iv). $n \equiv 3 \pmod 4$.
Assign the labels $1, 2, 3, 4$ to the vertices u_1, u_2, u_3, u_4 and assign the labels $1, 2, 3, 4$ to the vertices u_5, u_6, u_7, u_8.In similar manner assign the labels until we reach u_{n-3} then assign the label $4, 2, 3$ to the vertex u_{n-2}, u_{n-1}, u_n.Secondly assign the labels $1, 2, 3, 4$ to the vertices v_1, v_2, v_3, v_4 and assign the labels $1, 2, 3, 4$ to the vertices v_5, v_6, v_7, v_8. Continuing like this until we reach v_{n-3} then assign the label 1 to the vertex v_{n-2}.Lastly assign the labels $1, 2, 3, 4$ to the vertices w_1, w_2, w_3, w_4 and assign the labels $1, 2, 3, 4$ to the vertices w_5, w_6, w_7, w_8.Continue this proces until we reach w_{n-7} then assign the labels $2, 4, 3, 1, 3$ to the vertices $w_{n-6}, w_{n-5}, w_{n-4}, w_{n-3}, w_{n-2}$.

Thus the table 5 given below shows that $TW(P_n), n \geq 3$ is $4-$remainder cordial.

□

Theorem 4.3. *The Kayak Paddale $KP(n, n, n)$ is $4-$remainder cordial for all $n \geq 3$.*

Proof. Now describe the vertex labeling as follows.There are four cases arises.
Case(i). $n \equiv 0 \pmod 4$
First assign the labels $4, 3, 2, 1$ to the vertices x_1, x_2, x_3, x_4 and then assign the labels $4, 3, 2, 1$ to

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	η_e	η_o
$n \equiv 0 \pmod 4$	$\frac{3n-4}{4}$	$\frac{3n-4}{4}$	$\frac{3n-4}{4}$	$\frac{3n-4}{4}$	$\frac{3n-4}{2}$	$\frac{3n-6}{2}$
$n \equiv 1 \pmod 4$	$\frac{3n-3}{4}$	$\frac{3n-3}{4}$	$\frac{3n-7}{4}$	$\frac{3n-3}{4}$	$\frac{3n-5}{2}$	$\frac{3n-5}{2}$
$n \equiv 2 \pmod 4$	$\frac{3n-6}{4}$	$\frac{3n-2}{4}$	$\frac{3n-6}{4}$	$\frac{3n-2}{4}$	$\frac{3n-4}{2}$	$\frac{3n-6}{2}$
$n \equiv 3 \pmod 4$	$\frac{3n-5}{4}$	$\frac{3n-5}{4}$	$\frac{3n-1}{4}$	$\frac{3n-5}{4}$	$\frac{3n-5}{2}$	$\frac{3n-5}{2}$

Table 5.

the vertices x_5, x_6, x_7, x_8.In similar manner assign the labels $4, 3, 2, 1$ to the vertices $x_{n-3}, x_{n-2}, x_{n-1}, x_n$.Secondly assign the labels to the vertices $z_i(1 \le i \le n)$ as in $x_i(1 \le i \le n)$.Finally assign the labels $1, 2, 3, 4$ to the vertices y_2, y_3, y_4, y_5 and then assign the labels $1, 2, 3, 4$ to the vertices y_6, y_7, y_8, y_9.Repeat this process until we reach y_{n-4}, then assign the labels $2, 4, 3$ to the vertices $y_{n-3}, y_{n-2}, y_{n-1}$.

Case(ii). $n \equiv 1 \pmod 4$.
Now assign the labels $4, 3, 2, 1$ to the vertices x_1, x_2, x_3, x_4 and then assign the labels $4, 3, 2, 1$ to the vertices x_5, x_6, x_7, x_8.In similar pattern assign the labels $4, 3, 2, 1$ to the vertices $x_{n-4}, x_{n-3}, x_{n-2}, x_{n-1}$.Secondly assign the labels to the vertices $z_i(1 \le i \le n-1)$ as in $x_i(1 \le i \le n-1)$.Next assign the labels $1, 2, 3, 4$ to the vertices y_2, y_3, y_4, y_5 and then assign the labels $1, 2, 3, 4$ to the vertices y_6, y_7, y_8, y_9.Proceeding like this until we reach y_{n-5}, then assign the labels $2, 4, 3, 1$ to the vertices $y_{n-4}, y_{n-3}, y_{n-2}, y_{n-1}$.Finally assign the label 3 to the vertex x_n and assign the label 1 to the vertex z_n.

Case(iii). $n \equiv 2 \pmod 4$.
we assign the labels $4, 3, 2, 1$ to the vertices x_1, x_2, x_3, x_4 and then assign the labels $4, 3, 2, 1$ to the vertices x_5, x_6, x_7, x_8.In similar pattern way assign the labels $4, 3, 2, 1$ to the vertices $x_{n-5}, x_{n-4}, x_{n-3}, x_{n-2}$.Secondly assign the labels to the vertices $z_i(1 \le i \le n-2)$ as in $x_i(1 \le i \le n-2)$. Next assign the labels $1, 2, 3, 4$ to the vertices y_2, y_3, y_4, y_5 and then assign the labels $4, 3, 2, 1$ to the vertices y_6, y_7, y_8, y_9.Proceed this pattern until we reach y_{n-6}, then assign the labels $3, 2, 4, 1, 3$ to the vertices $y_{n-5}, y_{n-4}, y_{n-3}, y_{n-2}, y_{n-1}$.Finally assign the label $4, 3$ to the vertices x_{n-1}, x_n and assign the label $1, 2$ to the vertices z_{n-1}, z_n.

Case(iv). $n \equiv 3 \pmod 4$.
Assign the labels $4, 3, 2, 1$ to the vertices x_1, x_2, x_3, x_4 and then assign the labels $4, 3, 2, 1$ to the vertices x_5, x_6, x_7, x_8.In similar way assign the labels $4, 3, 2, 1$ to the vertices $x_{n-6}, x_{n-5}, x_{n-4}, x_{n-3}$.Secondly assign the labels to the vertices $z_i(1 \le i \le n-3)$ as in $x_i(1 \le i \le n-3)$.Next assign the labels $1, 2, 3, 4$ to the vertices y_2, y_3, y_4, y_5 and then assign the labels $1, 2, 3, 4$ to the vertices y_6, y_7, y_8, y_9.Proceeding like this until we reach y_{n-3}. Finally assign the label $1, 2, 3$ to the vertices x_{n-2}, x_{n-1}, x_n and assign the label $4, 1, 4$ to the vertices z_{n-2}, z_{n-1}, z_n, then assign the label $3, 2$ to the vertex y_{n-2}, y_{n-1}.

Thus the table 6 given below shows that $KP\,(n, n, n)$ is $4-$remainder cordial.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	η_e	η_o
$n \equiv 0 \pmod 4$	$\frac{3n-4}{4}$	$\frac{3n}{4}$	$\frac{3n}{4}$	$\frac{3n}{4}$	$\frac{3n}{2}$	$\frac{3n}{2}$
$n \equiv 1 \pmod 4$	$\frac{3n+1}{4}$	$\frac{3n-3}{4}$	$\frac{3n+1}{4}$	$\frac{3n-3}{4}$	$\frac{3n+1}{2}$	$\frac{3n-1}{2}$
$n \equiv 2 \pmod 4$	$\frac{3n-2}{4}$	$\frac{3n-2}{4}$	$\frac{3n+2}{4}$	$\frac{3n-2}{4}$	$\frac{3n}{2}$	$\frac{3n}{2}$
$n \equiv 3 \pmod 4$	$\frac{3n-1}{4}$	$\frac{3n-1}{4}$	$\frac{3n-1}{4}$	$\frac{3n-1}{4}$	$\frac{3n+1}{2}$	$\frac{3n-1}{2}$

Table 6.

□

Theorem 4.4. *The Kayak Paddale $KP\,(m, n, l)$ is $4-$ remainder cordial if $m \ge 3, n \ge 3, l \ge 1$ and among any of them are congruent modulo 4 while the other may assume any values other* *than those of the other two.*

Proof. Now we describe the vertex labeling as follows.There are 12 cases arises.

Case(i). $m \equiv 0 \pmod 4, n \equiv 0 \pmod 4$.
First assign the labels $4, 3, 2, 1$ to the vertices x_1, x_2, x_3, x_4 and assign the labels $4, 3, 2, 1$ to the vertices x_5, x_6, x_7, x_8.Proceed this process until we reach x_m.Next assign the labels to the vertices $y_i (2 \leq i \leq l-1)$ and $z_i (1 \leq i \leq n)$.There are three cases arises.

Subcase(i). $l \equiv 1 \pmod 4$.
Now we assign the labels $1, 2, 3, 4$ to the vertices y_2, y_3, y_4, y_5 and assign the labels $1, 2, 3, 4$ to the vertices y_6, y_7, y_8, y_9.Proceeding like this until we reach y_{l-1}.Finally assign the labels to the vertices $z_i (1 \leq i \leq n)$ as in $x_i (1 \leq i \leq m)$.

Subcase(ii). $l \equiv 2 \pmod 4$.
Now assign the labels $1, 2, 3, 4$ to the vertices y_2, y_3, y_4, y_5 and assign the labels $1, 2, 3, 4$ to the vertices y_6, y_7, y_8, y_9.Continuing this process until we reach y_{l-2} and assign the label 3 to the vertex y_{l-1}.Finally assign the labels to the vertices $z_i (5 \leq i \leq n)$ as in $x_i (5 \leq i \leq m)$ and assign the labels $2, 4, 3, 1$ to the vertices z_1, z_2, z_3, z_4.

Subcase(iii). $l \equiv 3 \pmod 4$.
We assign the labels $1, 2, 3, 4$ to the vertices y_2, y_3, y_4, y_5 and assign the labels $1, 2, 3, 4$ to the vertices y_6, y_7, y_8, y_9.Continue this pattern until we reach y_{l-3} and assign the label $2, 3$ to the vertex y_{l-2}, y_{l-1}.Finally assign the labels to the vertices $z_i (1 \leq i \leq n)$ as in $x_i (1 \leq i \leq m)$.

Thus the table 7 given below shows that $KP\,(m, n, l)$ is $4-$remainder cordial.

Nature of l	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	η_e	η_o
$l \equiv 1 \pmod 4$	$\frac{m+n+l-1}{4}$	$\frac{m+n+l-1}{4}$	$\frac{m+n+l-1}{4}$	$\frac{m+n+l-1}{4}$	$\frac{m+n+l-1}{2}$	$\frac{m+n+l+1}{2}$
$l \equiv 2 \pmod 4$	$\frac{m+n+l-2}{4}$	$\frac{m+n+l-2}{4}$	$\frac{m+n+l+2}{4}$	$\frac{m+n+l-2}{4}$	$\frac{m+n+l}{2}$	$\frac{m+n+l}{2}$
$l \equiv 3 \pmod 4$	$\frac{m+n+l-3}{4}$	$\frac{m+n+l+1}{4}$	$\frac{m+n+l+1}{4}$	$\frac{m+n+l-3}{4}$	$\frac{m+n+l-1}{2}$	$\frac{m+n+l+1}{2}$

Table 7.

Case(ii). $m \equiv 0 \pmod 4, l \equiv 0 \pmod 4$.
First assign the labels $4, 3, 2, 1$ to the vertices x_1, x_2, x_3, x_4 and assign the labels $4, 3, 2, 1$ to the vertices x_5, x_6, x_7, x_8.Proceed this pattern until we reach x_mAlso assign the labels $1, 2, 3, 4$ to the vertices y_2, y_3, y_4, y_5 and assign the labels $1, 2, 3, 4$ to the vertices y_6, y_7, y_8, y_9.Proceed this manner until we reach y_{l-4} and assign the label $1, 2, 3$ to the vertex $y_{l-3}, y_{l-2}, y_{l-1}$.Next assign the labels to the vertices $z_i (1 \leq i \leq n)$.There are three cases arises.

Subcase(i). $n \equiv 1 \pmod 4$.
Finally assign the labels to the vertices $z_i (1 \leq i \leq n-1)$ as in $x_i (1 \leq i \leq m)$ and assign the label 4 to the vertex z_n.

Subcase(ii). $n \equiv 2 \pmod 4$.
Next assign the labels to the vertices $z_i (1 \leq i \leq n-2)$ as in $x_i (1 \leq i \leq m)$ and assign the label $3, 4$ to the vertex z_{n-1}, z_n.

Subcase(iii). $n \equiv 3 \pmod 4$.
Next assign the labels to the vertices $z_i (1 \leq i \leq n-3)$ as in $x_i (1 \leq i \leq m)$ and assign the label $1, 4, 3$ to the vertex z_{n-2}, z_{n-1}, z_n.

Thus the table 8 given below shows that $KP\,(m, n, l)$ is $4-$remainder cordial.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	η_e	η_o
$n \equiv 1 \pmod 4$	$\frac{m+n+l-1}{4}$	$\frac{m+n+l-1}{4}$	$\frac{m+n+l-1}{4}$	$\frac{m+n+l-1}{4}$	$\frac{m+n+l+1}{2}$	$\frac{m+n+l-1}{2}$
$n \equiv 2 \pmod 4$	$\frac{m+n+l-2}{4}$	$\frac{m+n+l-2}{4}$	$\frac{m+n+l+2}{4}$	$\frac{m+n+l-2}{4}$	$\frac{m+n+l}{2}$	$\frac{m+n+l}{2}$
$n \equiv 3 \pmod 4$	$\frac{m+n+l+1}{4}$	$\frac{m+n+l-3}{4}$	$\frac{m+n+l+1}{4}$	$\frac{m+n+l-3}{4}$	$\frac{m+n+l-1}{2}$	$\frac{m+n+l+1}{2}$

Table 8.

Case(iii). $m \equiv 1 \pmod 4, n \equiv 0 \pmod 4, l \equiv 0 \pmod 4$.
First as in case(ii), assign the labels to the vertices $x_i(1 \leq i \leq m)$ like as $z_i(1 \leq i \leq n)$ and $z_i(1 \leq i \leq n)$ like as $x_i(1 \leq i \leq m)$. Finally assign the labels to the vertices $y_i(1 \leq i \leq l-1)$ as in case(ii).

Thus the table 8 given below shows that $KP\,(m, n, l)$ is $4-$remainder cordial.

Case(iv). $m \equiv 1 \pmod 4, n \equiv 1 \pmod 4$.
First assign the labels $4, 3, 2, 1$ to the vertices x_1, x_2, x_3, x_4 and assign the labels $4, 3, 2, 1$ to the vertices x_5, x_6, x_7, x_8.Proceed this process until we reach x_{m-1}.Secondly assign the labels to the vertices $z_i(1 \leq i \leq n-1)$ as in $x_i(1 \leq i \leq m-1)$.Next assign the labels to the vertices $y_i(2 \leq i \leq l-1)$. There are three cases arises.

Subcase(i). $l \equiv 0 \pmod 4$.
Now we assign the labels $1, 2, 3, 4$ to the vertices y_2, y_3, y_4, y_5 and assign the labels $1, 2, 3, 4$ to the vertices y_6, y_7, y_8, y_9.Proceeding like this until we reach y_{l-4}.Finally assign the labels $1, 2, 3$ to the vertices $y_{l-3}, y_{l-2}, y_{l-1}$ and then assign the label 3 to the vertex x_m and 4 to the vertex z_n.

Subcase(ii). $l \equiv 2 \pmod 4$.
We assign the labels $1, 2, 3, 4$ to the vertices y_2, y_3, y_4, y_5 and assign the labels $1, 2, 3, 4$ to the vertices y_6, y_7, y_8, y_9.continuously assign this labels until we reach y_{l-2} and assign the label 3 to the vertex y_{l-1}.Finally assign the label 2 to the vertex x_m and 4 to the vertex z_n.

Subcase(iii). $l \equiv 3 \pmod 4$.
Now assign the labels $1, 2, 3, 4$ to the vertices y_2, y_3, y_4, y_5 and assign the labels $1, 2, 3, 4$ to the vertices y_6, y_7, y_8, y_9.Continue this process until we reach y_{l-3} and assign the label $4, 3$ to the vertex y_{l-2}, y_{l-1}.Finally and then assign the label 2 to the vertex x_m and 1 to the vertex z_n.

Thus the table 9 given below shows that $KP\,(m, n, l)$ is $4-$remainder cordial.

Nature of l	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	η_e	η_o
$l \equiv 0 \pmod 4$	$\frac{m+n+l-2}{4}$	$\frac{m+n+l-2}{4}$	$\frac{m+n+l+2}{4}$	$\frac{m+n+l-2}{4}$	$\frac{m+n+l}{2}$	$\frac{m+n+l}{2}$
$l \equiv 2 \pmod 4$	$\frac{m+n+l-4}{4}$	$\frac{m+n+l}{4}$	$\frac{m+n+l}{4}$	$\frac{m+n+l}{4}$	$\frac{m+n+l}{2}$	$\frac{m+n+l}{2}$
$l \equiv 3 \pmod 4$	$\frac{m+n+l-1}{4}$	$\frac{m+n+l-1}{4}$	$\frac{m+n+l-1}{4}$	$\frac{m+n+l-1}{4}$	$\frac{m+n+l-1}{2}$	$\frac{m+n+l+1}{2}$

Table 9.

Case(v). $m \equiv 1 \pmod 4, l \equiv 1 \pmod 4$.
First assign the labels $4, 3, 2, 1$ to the vertices x_1, x_2, x_3, x_4 and assign the labels $4, 3, 2, 1$ to the vertices x_5, x_6, x_7, x_8.This pattern is repeated until we reach x_{m-1}.Also assign the labels $1, 2, 3, 4$ to the vertices y_2, y_3, y_4, y_5 and assign the labels $1, 2, 3, 4$ to the vertices y_6, y_7, y_8, y_9.In similar manner assign the labels to the vetices upto y_{l-1} then assign the label 4 to the vertex x_m.Next assign the labels to the vertices $z_i(1 \leq i \leq n)$.There are three cases arises.

Subcase(i). $n \equiv 0 \pmod 4$.
Finally assign the labels to the vertices $z_i(1 \leq i \leq n)$ as in $x_i(1 \leq i \leq m-1)$.

Subcase(ii). $n \equiv 2 \pmod 4$.
Next assign the labels to the vertices $z_i(1 \leq i \leq n-2)$ as in $x_i(1 \leq i \leq m-1)$ and assign the label $2, 3$ to the vertex z_{n-1}, z_n.

Subcase(iii). $n \equiv 3 \pmod 4$.
Next assign the labels to the vertices $z_i(1 \leq i \leq n-3)$ as in $x_i(1 \leq i \leq m-1)$ and assign the label $1, 2, 3$ to the vertex z_{n-2}, z_{n-1}, z_n.

Thus the table 10 given below shows that $KP(m, n, l)$ is $4-$remainder cordial.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	η_e	η_o
$n \equiv 0 \pmod 4$	$\frac{m+n+l-2}{4}$	$\frac{m+n+l-2}{4}$	$\frac{m+n+l-2}{4}$	$\frac{m+n+l+2}{4}$	$\frac{m+n+l}{2}$	$\frac{m+n+l}{2}$
$n \equiv 2 \pmod 4$	$\frac{m+n+l-4}{4}$	$\frac{m+n+l}{4}$	$\frac{m+n+l}{4}$	$\frac{m+n+l}{4}$	$\frac{m+n+l}{2}$	$\frac{m+n+l}{2}$
$n \equiv 3 \pmod 4$	$\frac{m+n+l-1}{4}$	$\frac{m+n+l-1}{4}$	$\frac{m+n+l-1}{4}$	$\frac{m+n+l-1}{4}$	$\frac{m+n+l+1}{2}$	$\frac{m+n+l-1}{2}$

Table 10.

Case(vi). $m \equiv 1 \pmod 4, n \equiv 1 \pmod 4, l \equiv 1 \pmod 4$.
First as in case(v), assign the labels to the vertices $x_i(1 \leq i \leq m)$ like as $z_i(1 \leq i \leq n)$ and $z_i(1 \leq i \leq n)$ like as $x_i(1 \leq i \leq m)$.Finally assign the labels to the vertices $y_i(1 \leq i \leq l-1)$ as in case(v).

Thus the table 10 given below shows that $KP(m, n, l)$ is $4-$remainder cordial.

Case(vii). $m \equiv 2 \pmod 4, n \equiv 2 \pmod 4$.
First assign the labels $4, 3, 2, 1$ to the vertices x_1, x_2, x_3, x_4 and assign the labels $4, 3, 2, 1$ to the vertices x_5, x_6, x_7, x_8.Proceed this manner until we reach x_{m-2}.Secondly assign the labels to the vertices $z_i(1 \leq i \leq n-2)$ as in $x_i(1 \leq i \leq m-2)$.And assign the label $4, 3$ to the vertices x_{m-1}, x_m then assign the label $1, 2$ to the vertices z_{n-1}, z_n.Next assign the labels to the vertices $y_i(2 \leq i \leq l-1)$. There are three cases arises.

Subcase(i). $l \equiv 0 \pmod 4$.
Now assign the labels $1, 2, 3, 4$ to the vertices y_2, y_3, y_4, y_5 and assign the labels $1, 2, 3, 4$ to the vertices y_6, y_7, y_8, y_9.This pattern is repeated until we reach y_{l-4}.Finally assign the labels $1, 2, 3$ to the vertices $y_{l-3}, y_{l-2}, y_{l-1}$.

Subcase(ii). $l \equiv 1 \pmod 4$.
Assign the labels $1, 2, 3, 4$ to the vertices y_2, y_3, y_4, y_5 and assign the labels $1, 2, 3, 4$ to the vertices y_6, y_7, y_8, y_9.Proceed this pattern until we reach y_{l-1}.

Subcase(iii). $l \equiv 3 \pmod 4$.
Finally assign the labels $1, 2, 3, 4$ to the vertices y_2, y_3, y_4, y_5 and assign the labels $1, 2, 3, 4$ to the vertices y_6, y_7, y_8, y_9.Proceeding like this until we reach y_{l-3} and assign the label $1, 2$ to the vertex y_{l-2}, y_{l-1}.

Thus the table 11 given below shows that $KP(m, n, l)$ is $4-$remainder cordial.

Case(viii). $m \equiv 2 \pmod 4, l \equiv 2 \pmod 4$.
First assign the labels $4, 3, 2, 1$ to the vertices x_1, x_2, x_3, x_4 and assign the labels $4, 3, 2, 1$ to the vertices x_5, x_6, x_7, x_8.In similar manner assign the labels until we reach x_{m-2} then assign the labels $4, 3$ to the vertices x_{m-1}, x_m.Also assign the labels $1, 2, 3, 4$ to the vertices y_2, y_3, y_4, y_5 and assign the labels $1, 2, 3, 4$ to the vertices y_6, y_7, y_8, y_9.This pattern is repeated until we reach y_{l-2} then assign the label 2 to the vertex y_{l-1}.Next assign the labels to the vertices $z_i(1 \leq i \leq n)$.There are three cases arises.

Nature of l	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	η_e	η_o
$l \equiv 0 \pmod 4$	$\frac{m+n+l}{4}$	$\frac{m+n+l}{4}$	$\frac{m+n+l}{4}$	$\frac{m+n+l-4}{4}$	$\frac{m+n+l}{2}$	$\frac{m+n+l}{2}$
$l \equiv 1 \pmod 4$	$\frac{m+n+l-1}{4}$	$\frac{m+n+l-1}{4}$	$\frac{m+n+l-1}{4}$	$\frac{m+n+l-1}{4}$	$\frac{m+n+l+1}{2}$	$\frac{m+n+l-1}{2}$
$l \equiv 3 \pmod 4$	$\frac{m+n+l-3}{4}$	$\frac{m+n+l+1}{4}$	$\frac{m+n+l+1}{4}$	$\frac{m+n+l-3}{4}$	$\frac{m+n+l-1}{2}$	$\frac{m+n+l+1}{2}$

Table 11.

Subcase(i). $n \equiv 0 \pmod 4$.
Finally assign the labels to the vertices $z_i(1 \le i \le n)$ as in $x_i(1 \le i \le m-2)$.

Subcase(ii). $n \equiv 1 \pmod 4$.
Next assign the labels to the vertices $z_i(1 \le i \le n-2)$ as in $x_i(1 \le i \le m-2)$ and assign the label $1, 2$ to the vertex z_{n-1}, z_n.

Subcase(iii). $n \equiv 3 \pmod 4$.
Next assign the labels to the vertices $z_i(1 \le i \le n-3)$ as in $x_i(1 \le i \le m-2)$ and assign the label $1, 2, 3$ to the vertex z_{n-2}, z_{n-1}, z_n.

Thus the table 12 given below shows that $KP\,(m, n, l)$ is $4-$remainder cordial.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	η_e	η_o
$n \equiv 0 \pmod 4$	$\frac{m+n+l-4}{4}$	$\frac{m+n+l}{4}$	$\frac{m+n+l}{4}$	$\frac{m+n+l}{4}$	$\frac{m+n+l}{2}$	$\frac{m+n+l}{2}$
$n \equiv 1 \pmod 4$	$\frac{m+n+l-1}{4}$	$\frac{m+n+l-1}{4}$	$\frac{m+n+l-1}{4}$	$\frac{m+n+l-1}{4}$	$\frac{m+n+l+1}{2}$	$\frac{m+n+l-1}{2}$
$n \equiv 3 \pmod 4$	$\frac{m+n+l-3}{4}$	$\frac{m+n+l+1}{4}$	$\frac{m+n+l+1}{4}$	$\frac{m+n+l-3}{4}$	$\frac{m+n+l-1}{2}$	$\frac{m+n+l+1}{2}$

Table 12.

Case(ix). $m \equiv 2 \pmod 4, n \equiv 2 \pmod 4, l \equiv 2 \pmod 4$.
First as in case(viii), assign the labels to the vertices $x_i(1 \le i \le m)$ like as $z_i(1 \le i \le n)$ and $z_i(1 \le i \le n)$ like as $x_i(1 \le i \le m)$.Finally assign the labels to the vertices $y_i(1 \le i \le l-1)$ as in case(viii).

Thus the table 12 given below shows that $KP\,(m, n, l)$ is $4-$remainder cordial.

Case(x). $m \equiv 3 \pmod 4, n \equiv 3 \pmod 4$.
First assign the labels $4, 3, 2, 1$ to the vertices x_1, x_2, x_3, x_4 and assign the labels $4, 3, 2, 1$ to the vertices x_5, x_6, x_7, x_8.Proceeding this pattern until we reach x_{m-3}.Secondly assign the labels to the vertices $z_i(1 \le i \le n-3)$ as in $x_i(1 \le i \le m-3)$.Next assign the labels to the vertices $y_i(2 \le i \le l-1)$. There are three cases arises.

Subcase(i). $l \equiv 0 \pmod 4$.
Now assign the labels $1, 2, 3, 4$ to the vertices y_2, y_3, y_4, y_5 and assign the labels $1, 2, 3, 4$ to the vertices y_6, y_7, y_8, y_9.continuing this manner until we reach y_{l-4}.Finally assign the labels $2, 3, 4$ to the vertices $y_{l-3}, y_{l-2}, y_{l-1}$.And assign the label $1, 4, 3$ to the vertices x_{m-3}, x_{m-1}, x_m then assign the label $2, 3, 1$ to the vertices z_{n-2}, z_{n-1}, z_n.

Subcase(ii). $l \equiv 1 \pmod 4$.
Now we assign the labels $1, 2, 3, 4$ to the vertices y_2, y_3, y_4, y_5 and assign the labels $1, 2, 3, 4$ to the vertices y_6, y_7, y_8, y_9.Repeat this pattern until we reach y_{l-1}.And assign the label $1, 2, 3$ to the vertices x_{m-3}, x_{m-1}, x_m then assign the label $2, 3, 4$ to the vertices z_{n-2}, z_{n-1}, z_n.

Subcase(iii). $l \equiv 2 \pmod 4$.
Finally assign the labels $1, 2, 3, 4$ to the vertices y_2, y_3, y_4, y_5 and assign the labels $1, 2, 3, 4$ to

the vertices y_6, y_7, y_8, y_9.This pattern is repeated until we reach y_{l-2} and assign the label 1 to the vertex y_{l-1}.And assign the label $1, 2, 3$ to the vertices x_{m-3}, x_{m-1}, x_m then assign the label $2, 3, 4$ to the vertices z_{n-2}, z_{n-1}, z_n.

Thus the table 13 given below shows that $KP\,(m, n, l)$ is $4-$remainder cordial.

Nature of l	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	η_e	η_o
$l \equiv 0 \pmod 4$	$\frac{m+n+l-2}{4}$	$\frac{m+n+l-2}{4}$	$\frac{m+n+l+2}{4}$	$\frac{m+n+l-2}{4}$	$\frac{m+n+l}{2}$	$\frac{m+n+l}{2}$
$l \equiv 1 \pmod 4$	$\frac{m+n+l-3}{4}$	$\frac{m+n+l+1}{4}$	$\frac{m+n+l+1}{4}$	$\frac{m+n+l-3}{4}$	$\frac{m+n+l-1}{2}$	$\frac{m+n+l+1}{2}$
$l \equiv 2 \pmod 4$	$\frac{m+n+l}{4}$	$\frac{m+n+l}{4}$	$\frac{m+n+l}{4}$	$\frac{m+n+l-4}{4}$	$\frac{m+n+l}{2}$	$\frac{m+n+l}{2}$

Table 13.

Case(xi). $m \equiv 3 \pmod 4, l \equiv 3 \pmod 4$.
First assign the labels $4, 3, 2, 1$ to the vertices x_1, x_2, x_3, x_4 and assign the labels $4, 3, 2, 1$ to the vertices x_5, x_6, x_7, x_8.Proceed this pattern until we reach x_{m-3}.Also assign the labels $1, 2, 3, 4$ to the vertices y_2, y_3, y_4, y_5 and assign the labels $1, 2, 3, 4$ to the vertices y_6, y_7, y_8, y_9.Proceed this pattern until we reach y_{l-3}. Next assign the labels to the vertices $z_i (1 \le i \le n)$.There are three cases arises.

Subcase(i). $n \equiv 0 \pmod 4$
Now assign the labels to the vertices $z_i (1 \le i \le n)$ as in $x_i (1 \le i \le m-3)$ and assign the label $2, 3, 4$ to the vertices x_{m-3}, x_{m-1}, x_m. Finally assign the labels $1, 3$ to the vertices y_{l-2}, y_{l-1}.

Subcase(ii). $n \equiv 1 \pmod 4$
Next assign the labels to the vertices $z_i (1 \le i \le n-1)$ as in $x_i (1 \le i \le m-3)$ and assign the label 3 to the vertex z_n.Finally assign the label $1, 2, 3$ to the vertices x_{m-3}, x_{m-1}, x_m and then assign the labels $2, 4$ to the vertices y_{l-2}, y_{l-1}.

Subcase(iii). $n \equiv 2 \pmod 4$.
Next assign the labels to the vertices $z_i (1 \le i \le n-2)$ as in $x_i (1 \le i \le m-3)$ and assign the label $4, 3$ to the vertex z_{n-1}, z_n.Finally assign the label $1, 2, 3$ to the vertices x_{m-3}, x_{m-1}, x_m and then assign the labels $2, 4$ to the vertices y_{l-2}, y_{l-1}

Thus the table 14 given below shows that $KP\,(m, n, l)$ is $4-$remainder cordial.

Nature of n	$v_f(1)$	$v_f(2)$	$v_f(3)$	$v_f(4)$	η_e	η_o
$n \equiv 0 \pmod 4$	$\frac{m+n+l-2}{4}$	$\frac{m+n+l-2}{4}$	$\frac{m+n+l+2}{4}$	$\frac{m+n+l-2}{4}$	$\frac{m+n+l}{2}$	$\frac{m+n+l}{2}$
$n \equiv 1 \pmod 4$	$\frac{m+n+l-3}{4}$	$\frac{m+n+l+1}{4}$	$\frac{m+n+l+1}{4}$	$\frac{m+n+l-3}{4}$	$\frac{m+n+l+1}{2}$	$\frac{m+n+l-1}{2}$
$n \equiv 2 \pmod 4$	$\frac{m+n+l-4}{4}$	$\frac{m+n+l}{4}$	$\frac{m+n+l}{4}$	$\frac{m+n+l}{4}$	$\frac{m+n+l}{2}$	$\frac{m+n+l}{2}$

Table 14.

Case(xii). $m \equiv 3 \pmod 4, n \equiv 3 \pmod 4, l \equiv 3 \pmod 4$.
First as in case(xi), assign the labels to the vertices $x_i (1 \le i \le m)$ like as $z_i (1 \le i \le n)$ and $z_i (1 \le i \le n)$ like as $x_i (1 \le i \le m)$. Finally assign the labels to the vertices $y_i (1 \le i \le l-1)$ as in case(xi).

Thus the table 14 given below shows that $KP\,(m, n, l)$ is $4-$remainder cordial.

□

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