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EULER-ZAGIER MULTIPLE L-FUNCTIONS INVOLVING BALANCING-LIKE POLYNOMIALS ASSOCIATED TO DIRICHLET CHARACTERS

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Abstract In the article, we study the analytic continuation of Euler-Zagier multiple shifted zeta functions involving balancing-like polynomials and Euler-Zagier multiple L-functions involving balancing-like polynomials associated to Dirichlet characters. We also compute a complete list of exact singularities and residues of these functions at poles. We further examine the values of these functions at negative integer arguments.

1 Introduction

The Euler-Zagier multiple zeta function $\zeta_{EZ,k}$ is defined by

$$\zeta_{EZ,k}(s_1, s_2, \ldots, s_k) = \sum_{0<m_1<m_2\cdots<m_k<\infty} \frac{1}{{m_1}^{s_1}{m_2}^{s_2}\cdots{m_k}^{s_k}}, \tag{1.1}$$

where $s_1, s_2, \ldots, s_k$ are complex variables [19]. This series is absolutely convergent in the region

$$\{(s_1, s_2, \ldots, s_k) \in \mathbb{C}^k | \mathrm{Re}(s_{k-r+1} + s_{k-r+2} + \cdots + s_k) > r,\ r = 1, 2, \ldots, k\}.$$

Arakawa and Kaneko [3] demonstrated the analytic continuation of (1.1) as a function of single variable s_k, where $s_1, s_2, \ldots, s_{k-1}$ are positive integers. The analytic continuation of (1.1) has been studied by several researchers [1, 11, 20]. Hurwitz [8] defined the shifted zeta function $\zeta(s, x)$ as:

$$\zeta(s, x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s},\ Re(s) > 1$$

where $0 < x \leq 1$. The Dirichlet L-function written in terms of Hurwitz zeta functions as

$$\mathcal{L}(s, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n^s} = q^{-s} \sum_{a \pmod q} \chi(a)\zeta(s; a/q),$$

where χ is the Dirichlet character modulo q. Various generalizations of the multiple zeta functions were introduced and their analytic properties have been studied. One of the most valuable generalization is the multiple series associated to Dirichlet characters which are called multiple Dirichlet L-functions. The multiple Dirichlet L-function is defined as

$$\mathcal{L}(s_1, \ldots, s_k \mid \chi_1, \ldots, \chi_k) = \sum_{0<m_1<m_2\cdots<m_k<\infty} \frac{\chi_1(m_1)}{{m_1}^{s_1}} \cdots \frac{\chi_k(m_k)}{{m_k}^{s_k}},$$

where $\chi_1, \ldots, \chi_k$ are Dirichlet characters of same modulo $q \in \mathbb{N}_{\geq 2}$. The analytic continuation of this multiple L-function has been studied by Akiyama and Ishikawa [2].

In [12], Navas introduced Fibonacci Dirichlet series $\zeta_F(s) = \sum_{n=1}^{\infty} F_n^{-s},\ Re(s) > 0$ for $s \in \mathbb{C}$, where F_n denotes the n-th Fibonacci number and proved that $\zeta_F(s)$ is analytically continued to a meromorphic function on the complex plane $\mathbb{C}$. In [9], Kamano considered the Lucas zeta function $\Phi^{(P,\,Q)}(s)=\sum_{n=1}^{\infty}\frac{1}{U_n^s},\ Re(s)>0,\ s\in\mathbb{C}$, which is a generalization of Fibonacci zeta function and derived its analytic continuation, where U_n is the n-th Lucas number of first kind. The Lucas L-function is defined as

$$\mathcal{L}_U(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{U_n^s},\ Re(s)>0,$$

where χ is the Dirichlet character modulo q and can be analytical continued to the whole complex plane [9]. Rout and Meher [16] defined the multiple Fibonacci zeta function

$$\zeta_F(s_1,\dots,s_d)=\sum_{0<n_1<n_2\cdots<n_d}\frac{1}{F_{n_1}^{s_1}\cdots F_{n_d}^{s_d}},$$

where d is the depth and and $s_1+\cdots+s_d$ is the weight of $\zeta_F(s_1,s_2,\dots,s_d)$. They studied the analytic continuation of $\zeta_F(s_1,s_2,\dots,s_d)$ of depth $d=2$ and found a complete list of poles and their corresponding residues. In [16], they also examined the arithmetic nature of $\zeta_F(s_1,s_2,\dots,s_d)$ at negative integer arguments. Recently, Meher and Rout [10] proved the meromorphic continuation of multiple Lucas zeta functions of depth d :

$$\zeta_U(s_1,\dots,s_d)=\sum_{0<n_1<n_2\cdots<n_d}\frac{1}{U_{n_1}^{s_1}\cdots U_{n_d}^{s_d}},$$

where (U_n) is the Lucas sequence of first kind. They calculated a complete list of poles and their residues and proved that the multiple Lucas zeta values are rational at negative integers.

Now, our premier task is to talk about the theory of balancing numbers and balancing-like numbers. A natural number m is said to be a balancing number if it is the solution of a simple Diophantine equation $1+2+\cdots+(m-1)=(m+1)+(m+2)+\cdots+(m+r)$, where r is a balancer corresponding to m [5]. Let $\{B_m\}_{m\geq 0}$ be the balancing sequence and is recursively defined as $B_0=0$, $B_1=1$ and $B_m=6B_{m-1}-B_{m-2}$ for $m\geq 2$. The closed form expression for balancing sequence is $B_m=\dfrac{\lambda_1^m-\lambda_2^m}{\lambda_1-\lambda_2}$, where $\lambda_1=3+2\sqrt{2}$ and $\lambda_2={\lambda_1}^{-1}=3-2\sqrt{2}$ are the roots of the balancing characteristic equation $\lambda^2-6\lambda+1=0$ [14]. The balancing-like sequence is recursively defined as $x_0=0$, $x_1=1$ and $x_{m+1}=Ax_m-x_{m-1}$, $m\geq 1$ where $A\in\mathbb{N}_{>2}$ [18]. For $A=6$, balancing-like sequence gives sequence of balancing numbers. The closed form expression for balancing-like sequence is $x_m=\dfrac{\alpha^m-\beta^m}{\alpha-\beta}$, where $\alpha=\frac{A+\sqrt{D}}{2}$ and $\beta=\alpha^{-1}=\frac{A-\sqrt{D}}{2}$ are the roots of $x^2-Ax+1=0$, where $D=A^2-4>0$ [18]. The sequence of balancing polynomials is recursively defined as $B_0(x)=0$, $B_1(x)=1$ and $B_{m+1}(x)=6xB_m(x)-B_{m-1}(x)$ for $m\geq 1$ and extensively studied in [13, 15]. Rout and Panda [17] considered balancing zeta function $\zeta_B(s)=\sum_{m=1}^{\infty}B_m^{-s},\ Re(s)>0$ for $s\in\mathbb{C}$, where B_m denotes the m-th balancing number and derived that $\zeta_B(s)$ can be meromorphically continued to the whole complex plane. They also shown that $\zeta_B(-m)$ is an irrational number when m is an odd natural number. In [17], they also studied the analytic continuation of the balancing L-function $L_B(s,\chi)$. In a subsequent paper, Behera et al. [4] proved the analytical continuation of $\zeta_C(s)=\sum_{n=1}^{\infty}C_n^{-s},\ Re(s)>0$ for $s\in\mathbb{C}$, where C_n denotes the n-th Lucas-balancing number and $\zeta_C(-m)$ is a rational number for any odd natural number m. They also discuss the meromorphic continuation of Lucas-balancing L-function $L_C(s,\chi)$. Recently, Dutta and Ray [7] defined Euler-Zagier multiple Lucas-balancing zeta functions of depth k :

$$\zeta_{EZC}(s_1,\dots,s_k)\quad=\sum_{1\leq m_1<m_2<\cdots<m_k<\infty}\frac{1}{C_{m_1}^{s_1}\cdots C_{m_k}^{s_k}},$$

and studied its meromorphic continuation. The authors have calculated a complete list of poles and their residues and discussed the Euler-Zagier multiple Lucas-balancing zeta values at negative integer arguments.

In this note, we introduce Euler-Zagier multiple shifted zeta function involving balancing-like polynomials, Euler-Zagier multiple L-function involving balancing-like polynomials associated to Dirichlet characters and investigate the analytic continuation along with poles and their corresponding residues. Further, we prove that the values of Euler-Zagier multiple L-functions involving balancing-like polynomials associated to Dirichlet characters are rational at negative integers.

2 Preliminaries

Definition 2.1. For any positive integer y, the sequence of balancing-like polynomials is recursively defined as $x_0(y) = 0$, $x_1(y) = 1$ and $x_{m+1}(y) = Ayx_m(y) - x_{m-1}(y)$, $m \geq 1$ where $A \in \mathbb{N}_{>2}$.

For $A = 6$, the sequence of balancing-like polynomials gives the sequence of balancing polynomials studied in [15]. The closed form expression for balancing-like polynomials is $x_m(y) = \dfrac{\alpha^m(y) - \beta^m(y)}{\alpha(y) - \beta(y)}$, where $\alpha(y) = \frac{Ay+\sqrt{D(y)}}{2}$ and $\beta(y) = \alpha(y)^{-1} = \frac{Ay-\sqrt{D(y)}}{2}$ are the roots of $x^2 - Ayx + 1 = 0$, where $D(y) = (Ay)^2 - 4 > 0$.

Definition 2.2. Let $\chi_1, \ldots, \chi_k$ be the Dirichlet characters of same modulus $t \in \mathbb{N}_{\geq 2}$ and χ_0 be the principal character. The Euler-Zagier multiple shifted zeta function involving balancing-like polynomials and Euler-Zagier multiple L-function involving balancing-like polynomials associated to Dirichlet characters are defined as:

$$\zeta^k_{EZB}(s_1, \ldots, s_k \mid h_1, \ldots, h_k) = \sum_{m_1,\ldots,m_k=0}^{\infty} \frac{1}{x^{s_1}_{tm_1+h_1}(y)} \cdots \frac{1}{x^{s_k}_{t(m_1+\cdots+m_k)+(h_1+\cdots+h_k)}(y)} \tag{2.1}$$

and

$$\mathcal{L}^k_{EZB}(s_1, \ldots, s_k \mid \chi_1, \ldots, \chi_k) = \sum_{1\leq m_1<m_2<\cdots<m_k} \frac{\chi_1(m_1)}{x^{s_1}_{m_1}(y)} \cdots \frac{\chi_k(m_k)}{x^{s_k}_{m_k}(y)}, \tag{2.2}$$

where $m_i, h_i \in \mathbb{N}$ for $1 \leq i \leq k$.

For the sake of convenience, we denote $\zeta^k_{EZB}(s|\mathbf{h})$ and $\mathcal{L}^k_{EZB}(s|\chi)$ for $\zeta^k_{EZB}(s_1, \ldots, s_k \mid h_1, \ldots, h_k)$ and $\mathcal{L}^k_{EZB}(s_1, \ldots, s_k \mid \chi_1, \ldots, \chi_k)$ respectively.

For any integer $k \geq 1$, we consider the open subset of $\mathcal{D}_k$ of $\mathbb{C}^k$, i.e.

$$\mathcal{D}_k = \{(s_1, \ldots, s_k) \in \mathbb{C}^k \big| \sum_{i=d}^{k} Re(s_i) > 0, 1 \leq d \leq k\}.$$

Proposition 2.3. *The Euler-Zagier multiple shifted zeta function involving balancing-like polynomials $\zeta^k_{EZB}(s|\mathbf{h})$, is absolutely convergent in the domain $\mathcal{D}_k$.*

Proof. Let $s_j = \sigma_j + it_j \in \mathbb{C}$ and $\sigma_j = Re(s_j) > 0$ for $j = 1, \ldots, k$. Now,

$$\begin{aligned}
\sum_{m_1=1}^{\infty} \left| \frac{1}{x^{s_1}_{m_1}(y)} \right| &= (\alpha(y) - \beta(y))^{\sigma_1} \sum_{m_1=1}^{\infty} \left| \frac{1}{\left(\alpha^{m_1}(y) - \beta^{m_1}(y)\right)^{s_1}} \right| \\
&\leq (\alpha(y) - \beta(y))^{\sigma_1} \sum_{m_1=1}^{\infty} \frac{1}{\alpha^{m_1\sigma_1}(y)\left(1 - (|\frac{\beta(y)}{\alpha(y)}|)^{m_1}\right)^{s_1}} \\
&\leq \left(\frac{\alpha(y) - \beta(y)}{1 - |\beta(y)/\alpha(y)|} \right)^{\sigma_1} \sum_{m_1=1}^{\infty} \frac{1}{\alpha^{\sigma_1 m_1}(y)} \\
&= \Lambda_{\sigma_1} (\alpha(y) - \beta(y))^{\sigma_1} \frac{1}{\alpha^{\sigma_1}(y) - 1},
\end{aligned}$$

and for $2 \le d \le k$, we have the following estimate

$$\left|\frac{1}{x^{s_d}_{m_1+\cdots+m_d}(y)}\right| \le \Lambda_{\sigma_d}\frac{(\alpha(y)-\beta(y))^{\sigma_d}}{\alpha^{\sigma_d(m_1+\cdots+m_d)}(y)},$$

where $\Lambda_{\sigma_j} = \frac{1}{(1-|\beta(y)/\alpha(y)|)^{\sigma_j}}$.
Now,

$$\begin{aligned}
&\sum_{0<m_1<m_2<\cdots<m_k}\left|\frac{1}{x^{s_1}_{m_1}(y)x^{s_2}_{m_2}(y)\cdots x^{s_k}_{m_k}(y)}\right| \\
\le\;& \sum_{m_1=1}^{\infty}\left|\frac{1}{x^{s_1}_{m_1}(y)}\right|\sum_{m_2=1}^{\infty}\left|\frac{1}{x^{s_2}_{m_1+m_2}(y)}\right|\cdots\sum_{m_k=1}^{\infty}\left|\frac{1}{x^{s_k}_{m_1+\cdots+m_k}(y)}\right| \\
\le\;& \Lambda_{\sigma_1}\cdots\Lambda_{\sigma_k}(\alpha(y)-\beta(y))^{\sigma_1+\cdots+\sigma_k}\sum_{m_1=1}^{\infty}\frac{1}{\alpha^{\sigma_1 m_1}(y)}\sum_{m_2=1}^{\infty}\frac{1}{\alpha^{\sigma_2(m_1+m_2)}(y)} \\
&\times\cdots\times\sum_{m_k=1}^{\infty}\frac{1}{\alpha^{\sigma_k(m_1+\cdots+m_k)}(y)} \\
=\;& \Lambda(\alpha(y)-\beta(y))^{\sigma_1+\cdots+\sigma_k}\sum_{m_1=1}^{\infty}\frac{1}{\alpha^{(\sigma_1+\cdots+\sigma_k)m_1}(y)}\sum_{m_2=1}^{\infty}\frac{1}{\alpha^{(\sigma_2+\cdots+\sigma_k)m_2}(y)} \\
&\times\cdots\times\sum_{m_k=1}^{\infty}\frac{1}{\alpha^{\sigma_k m_k}(y)} \\
=\;& \Lambda(\alpha(y)-\beta(y))^{\sigma_1+\cdots+\sigma_k}\frac{1}{(\alpha^{\sigma_1+\cdots+\sigma_k}(y)-1)}\frac{1}{(\alpha^{\sigma_2+\cdots+\sigma_k}(y)-1)} \\
&\times\cdots\times\frac{1}{(\alpha^{\sigma_k}(y)-1)} \\
<\;& \infty,
\end{aligned}$$

since $\alpha(y) > 1$ where $\Lambda = \Lambda_{\sigma_1}\cdots\Lambda_{\sigma_k}$. Now,

$$\begin{aligned}
&\sum_{m_1=0,\dots,m_k=0}\left|\frac{1}{x^{s_1}_{tm_1+h_1}(y)}\cdots\frac{1}{x^{s_k}_{t(m_1+\cdots+m_k)+(h_1+\cdots+h_k)}(y)}\right| \\
\le\;& \sum_{m_1=0,\dots,m_k=0}\left|\frac{1}{x^{s_1}_{tm_1+h_1}(y)}\right|\cdots\left|\frac{1}{x^{s_k}_{t(m_1+\cdots+m_k)+(h_1+\cdots+h_k)}(y)}\right| \\
\le\;& \sum_{m_1=1}^{\infty}\left|\frac{1}{x^{s_1}_{m_1}(y)}\right|\sum_{m_2=1}^{\infty}\left|\frac{1}{x^{s_2}_{m_1+m_2}(y)}\right|\cdots\sum_{m_k=1}^{\infty}\left|\frac{1}{x^{s_k}_{m_1+\cdots+m_k}(y)}\right| < \infty. \qquad (2.3)
\end{aligned}$$

Therefore, the series (2.1) converges absolutely in the domain $\mathcal{D}_k$. This completes the proof. □

Proposition 2.4. *For any positive integer $k \ge 1$, let $\chi_1, \dots, \chi_k$ be the Dirichlet characters of same modulus $t \in \mathbb{N}_{\ge 2}$. The infinite sum $\sum_{1\le m_1<m_2<\cdots<m_k}\frac{\chi_1(m_1)}{x^{s_1}_{m_1}(y)}\cdots\frac{\chi_k(m_k)}{x^{s_k}_{m_k}(y)}$ is absolutely convergent in $\mathcal{D}_k$.*

Proof. It is observed that

$$\begin{aligned}
&\sum_{1\le m_1<m_2<\cdots<m_k}\frac{\chi_1(m_1)}{x^{s_1}_{m_1}(y)}\cdots\frac{\chi_k(m_k)}{x^{s_k}_{m_k}(y)} \\
=\;& \sum_{m_1=1}^{\infty}\frac{\chi_1(m_1)}{x^{s_1}_{m_1}(y)}\sum_{m_2=1}^{\infty}\frac{\chi_2(m_1+m_2)}{x^{s_2}_{m_1+m_2}(y)}\cdots\sum_{m_k=1}^{\infty}\frac{\chi_k(m_1+\cdots+m_k)}{x^{s_k}_{m_1+\cdots+m_k}(y)}.
\end{aligned}$$

For any $m \in \mathbb{N}$, $|\chi(m)| \le 1$. Therefore, by Proposition 2.3, the series $\mathcal{L}^k_{EZB}(s|\chi)$ converges absolutely in $\mathcal{D}_k$. □

3 Analytic continuation of Euler-Zagier multiple L-functions involving balancing-like polynomials associated to Dirichlet characters

In this section, we demonstrate the analytic continuation of Euler-Zagier multiple L-functions involving balancing-like polynomials associated to Dirichlet characters.

Theorem 3.1. *The Euler-Zagier multiple shifted zeta function involving balancing-like polynoials $\zeta^k_{EZB}(s|\mathbf{h})$ can be analytically continued to a meromorphic function on all of $\mathbb{C}^k$ with exact list of poles on the hyperplanes*

$$s_d + \cdots + s_k = -2(r_d + \cdots + r_k) + \frac{2\pi i a}{t \log \alpha(y)},\ 1 \leq d \leq k,$$

where $r_1, \ldots, r_k \in \mathbb{Z}_{\geq 0},$ and $a \in \mathbb{Z}$.

Proof. As $\alpha(y) - \beta(y) = \sqrt{D(y)}$ and $\alpha(y)\beta(y) = 1$, then for any complex number $s \in \mathbb{C}$, we have

$$\begin{aligned} x^s_m(y) = \left(\frac{\alpha^m(y) - \beta^m(y)}{\alpha(y) - \beta(y)}\right)^s &= D^{-s/2}(y)\alpha^{ms}(y)\left(1 - \left(\frac{\beta(y)}{\alpha(y)}\right)^m\right)^s \\ &= D^{-s/2}(y)\alpha^{ms}(y)\left(1 - \frac{1}{\alpha^{2m}(y)}\right)^s \\ &= D^{-s/2}(y)\sum_{r=0}^{\infty}\binom{s}{r}(-1)^r\alpha^{m(s-2r)}(y). \end{aligned} \tag{3.1}$$

Now,

$$x^s_{tm+h}(y) = D^{-\frac{s}{2}}(y)\sum_{r=0}^{\infty}\binom{s}{r}(-1)^r\alpha^{(tm+h)(s-2r)}(y). \tag{3.2}$$

Since the series $\zeta^k_{EZB}(s|\mathbf{h})$ is absolutely convergent. Then by interchanging the order of summation, we get

$$\begin{aligned} &\zeta^k_{EZB}(s \mid \mathbf{h}) \\ = &\sum_{m_1=0,\ldots,m_k=0}^{\infty} \frac{1}{x^{s_1}_{tm_1+h_1}(y)} \cdots \frac{1}{x^{s_k}_{t(m_1+\cdots+m_k)+(h_1+\cdots+h_k)}(y)} \\ = &\sum_{m_1=0,\ldots,m_k=0}^{\infty}\left(D^{\frac{s_1}{2}}(y)\sum_{r_1=0}^{\infty}\binom{-s_1}{r_1}(-1)^{r_1}\alpha^{-(tm_1+h_1)(s_1+2r_1)}(y)\right) \\ &\times\cdots\times\left(D^{\frac{s_k}{2}}(y)\sum_{r_k=0}^{\infty}\binom{-s_k}{r_k}(-1)^{r_k}\alpha^{-\left(t(m_1+\cdots+m_k)+(h_1+\cdots+h_k)\right)(s_k+2r_k)}(y)\right) \\ = &D^{\frac{s_1+\cdots+s_k}{2}}(y)\sum_{r_1=0}^{\infty}\binom{-s_1}{r_1}(-1)^{r_1}\cdots\sum_{r_k=0}^{\infty}\binom{-s_k}{r_k}(-1)^{r_k} \\ &\times\alpha^{-\left(s_1+\cdots+s_k+2(r_1+\cdots+r_k)\right)h_1}(y)\times\ \cdots\times\alpha^{-(s_k+2r_k)h_k}(y) \\ &\times\sum_{m_1,\ldots,m_k=0}^{\infty}\left(\alpha^{-t(s_1+\cdots+s_k+2(r_1+\cdots+r_k))}(y)\right)^{m_1}\times\ \cdots\times\left(\alpha^{-t(s_k+2r_k)}(y)\right)^{m_k} \end{aligned}$$

$$= D^{\frac{s_1+\cdots+s_k}{2}}(y)\sum_{r_1=0}^{\infty}\binom{-s_1}{r_1}(-1)^{r_1}\cdots\sum_{r_k=0}^{\infty}\binom{-s_k}{r_k}(-1)^{r_k}$$
$$\times\frac{\alpha^{-h_1(s_1+\cdots+s_k+2(r_1+\cdots+r_k))}(y)}{1-\alpha^{-t(s_1+\cdots+s_k+2(r_1+\cdots+r_k))}(y)}\times\cdots\times\frac{\alpha^{-h_k(s_k+2r_k)}(y)}{1-\alpha^{-t(s_k+2r_k)}(y)}$$
$$= D^{\frac{s_1+\cdots+s_k}{2}}(y)\sum_{r_1=0}^{\infty}\binom{-s_1}{r_1}(-1)^{r_1}\cdots\sum_{r_k=0}^{\infty}\binom{-s_k}{r_k}(-1)^{r_k}\frac{\alpha^{-h_1(s_1+2r_1)}(y)}{1-\alpha^{-t(s_1+\cdots+s_k+2(r_1+\cdots+r_k))}(y)}$$
$$\times\cdots\times\frac{\alpha^{-(h_1+\cdots+h_k)(s_k+2r_k)}(y)}{1-\alpha^{-t(s_k+2r_k)}(y)}. \tag{3.3}$$

The infinite series (3.3) is holomorphic function on $\mathbb{C}^k$ except for the poles derived from the functions

$$F_{r_d,\ldots,r_k}(s_d,\ldots,s_k)=\frac{\alpha^{-h_d(s_d+\cdots+s_k+2(r_d+\cdots+r_k))}(y)}{1-\alpha^{-t(s_d+\cdots+s_k+2(r_d+\cdots+r_k))}(y)} \text{ for } 1\leq d\leq k.$$

Therefore, $\zeta^k_{EZB}(s|\mathbf{h})$ is meromorphically continued on $\mathbb{C}^k$ with poles on the hyperplanes

$$s_d+\cdots+s_k = -2(r_d+\cdots+r_k)+\tfrac{2\pi i a}{t\log\alpha(y)},\ 1\leq d\leq k, \tag{3.4}$$

where $r_1,\ldots,r_k\in\mathbb{Z}_{\geq 0},$ *and* $a\in\mathbb{Z}$. This completes the proof. □

Now,

$$\begin{aligned}\mathcal{L}^k_{EZB}(s\mid\chi) &= \sum_{1\leq m_1<m_2<\cdots<m_k}\frac{\chi_1(m_1)}{x^{s_1}_{m_1}(y)}\cdots\frac{\chi_k(m_k)}{x^{s_k}_{m_k}(y)}\\
&= \sum_{m_1=1}^{\infty}\frac{\chi_1(m_1)}{x^{s_1}_{m_1}(y)}\sum_{m_2=1}^{\infty}\frac{\chi_2(m_1+m_2)}{x^{s_2}_{m_1+m_2}(y)}\cdots\sum_{m_k=1}^{\infty}\frac{\chi_k(m_1+\cdots+m_k)}{x^{s_k}_{m_1+\cdots+m_k}(y)}\\
&= \sum_{h_1=1}^{t}\sum_{h_2=1}^{t}\cdots\sum_{h_k=1}^{t}\sum_{m_1=1}^{\infty}\frac{\chi_1(h_1)}{x^{s_1}_{tm_1+h_1}(y)}\cdots\sum_{m_k=1}^{\infty}\frac{\chi_k(h_1+\cdots+h_k)}{x^{s_k}_{t(m_1+\cdots+m_k)+h_1+\cdots+h_k}(y)}\\
&= \sum_{h_1=1}^{t}\sum_{h_2=1}^{t}\cdots\sum_{h_k=1}^{t}\chi_1(h_1)\cdots\chi_k(h_1+\cdots+h_k)\zeta^k_{EZB}(s|\mathbf{h}).\end{aligned} \tag{3.5}$$

From the above expression, it is obtained that the function $\mathcal{L}^k_{EZB}(s\mid\chi)$ is a linear combination of $\zeta^k_{EZB}(s|\mathbf{h})$. Thus, we get the following result.

Theorem 3.2. *For any positive integer $k\geq 1$, let $\chi_1,\ldots,\chi_k$ be the Dirichlet characters of same modulus $t\in\mathbb{N}_{\geq 2}$, then the Euler-Zagier multiple L-function involving balancing-like polynomials $\mathcal{L}^k_{EZB}(s\mid\chi)$ of depth k can be continued to a meromorphic function on all of $\mathbb{C}^k$ with all possible simple poles on the hyperplanes (3.4).*

4 Poles and residues of Euler-Zagier multiple L-functions involving balancing-like polynomials associated to Dirichlet characters

For $1\leq d\leq k$, let

$$s_k(d)=s_d+\cdots+s_k,\ r_k(d)=r_d+\cdots+r_k,\ r'_k(d)=r'_d+\cdots+r'_k,$$
$$h_k(d)=h_d+\cdots+h_k \text{ and } \zeta_t=e^{\frac{2\pi i}{t}}$$

with $s_k(k+1)=0,\ r_k(k+1)=0,\ r'_k(k+1)=0$. The residues of the Euler-Zagier multiple L-functions involving balancing-like polynomials $\mathcal{L}^k_{EZB}(s\mid\chi)$ along the hyperplanes (3.4) to be the restriction of the meromorphic function

$$\left(s_k(d)+2r_k(d)-\frac{2\pi i a}{t\log\alpha(y)}\right)\mathcal{L}^k_{EZB}(s\mid\chi)$$

to the hyperplanes (3.4).

Theorem 4.1. *Let r'_k be a non-negative integer. Let $l_k = -2r'_k + \frac{2\pi i a}{t \log \alpha(y)}$ and set $\mathcal{L}^{k-1}_{EZB}(s \mid \mathbf{h}) = 1$ for $k = 1$. Then*

$$\underset{s_k=l_k}{Res}\, \mathcal{L}^k_{EZB}(s \mid \chi) = \sum_{h_1=1}^{t} \chi_1(h_1) \sum_{h_2=1}^{t} \chi_2(h_2(1)) \cdots \sum_{h_k=1}^{t} \chi_k(h_k(1))$$

$$\zeta^{k-1}_{EZB}(s_1, \ldots, s_{k-1} \mid h_1, \ldots, h_{k-1}) \frac{D^{l_k/2}(y)(-1)^{r'_k}}{t \log \alpha(y)} \binom{-l_k}{r'_k} \zeta_t^{-ah_k(1)}.$$

Proof. Let us assume that $k \geq 1$ and $s_k(k) = s_k = l_k$. Now, $s_k + 2r'_k = \frac{2\pi i a}{t \log \alpha(y)}$ which implies that $\alpha^{s_k+2r'_k}(y) = \zeta_t^a$. Thus, $\alpha^{-h_k(1)(s_k+2r'_k)}(y) - \zeta_t^{-h_k(1)a}$ is an analytic function with simple zeros at l_k. Then

$$\begin{aligned} \lim_{s_k \to l_k} \frac{s_k - l_k}{(1 - \alpha^{-t(s_k+2r_k)}(y))} &= \underset{s_k=l_k}{\text{Res}} \frac{1}{(1 - \alpha^{-t(s_k+2r_k)}(y))} = \left. \frac{1}{\frac{d}{ds_k}(1 - \alpha^{-t(s_k+2r_k)}(y))} \right|_{s_k=l_k} \\ &= \frac{1}{t \log \alpha(y)}. \end{aligned}$$

The residue of $\zeta^k_{EZB}(s|\mathbf{h})$ along the hyper plane $s_k = l_k$ is given by

$$\begin{aligned} \underset{s_k=l_k}{\text{Res}}\, \zeta^k_{EZB}(s|\mathbf{h}) &= \lim_{s_k \to l_k} (s_k - l_k)\, D^{s_k(1)/2}(y) \sum_{r_1=0}^{\infty} \binom{-s_1}{r_1} (-1)^{r_1} \cdots \sum_{r_k=0}^{\infty} \binom{-s_k}{r_k} (-1)^{r_k} \\ &\quad \frac{\alpha^{-h_1(s_1+2r_1)}(y)}{1 - \alpha^{-t(s_k(1)+2r_k(1))}(y)} \times \cdots \times \frac{\alpha^{-(h_k(1)(s_k+2r_k)}(y)}{1 - \alpha^{-t(s_k+2r_k)}(y)} \\ &= D^{s_{k-1}(1)/2}(y) \sum_{r_1=0}^{\infty} \binom{-s_1}{r_1} (-1)^{r_1} \cdots \sum_{r_k=0}^{\infty} \binom{-s_{k-1}}{r_{k-1}} (-1)^{r_{k-1}} \\ &\quad \left. \frac{\alpha^{-h_1(s_1+2r_1)}(y)}{1 - \alpha^{-t(s_k(1)+2r_k(1))}(y)} \right|_{s_k=l_k} \times \cdots \times \left. \frac{\alpha^{-(h_{k-1}(1)(s_{k-1}+2r_{k-1})}(y)}{1 - \alpha^{-t(s_k(k-1)+2r_k(k-1))}(y)} \right|_{s_k=l_k} \\ &\quad \times \lim_{s_k \to l_k} D^{s_k/2}(y)(-1)^{r_k} \binom{-s_k}{r_k} (s_k - l_k) \frac{\alpha^{-h_k(1)(s_k+2r_k)}(y)}{1 - \alpha^{-t(s_k+2r_k)}(y)}. \end{aligned}$$

From the above expression, after applying limit, the terms containing only r_k but not $r_1, \cdots, r_{k-1}$ survives for $r_k = r'_k$ and rest of the terms vanish.
Therefore, from the above calculation, we have

$$\begin{aligned} \underset{s_k=l_k}{\text{Res}}\, \zeta^k_{EZB}(s|\mathbf{h}) &= D^{l_k/2}(y) D^{s_{k-1}(1)/2}(y) \sum_{r_1=0}^{\infty} \binom{-s_1}{r_1} (-1)^{r_1} \cdots \sum_{r_{k-1}=0}^{\infty} \binom{-s_{k-1}}{r_{k-1}} (-1)^{r_{k-1}} \\ &\quad \times \frac{\alpha^{-h_1(s_1+2r_1)}(y)}{1 - \alpha^{-t(s_{k-1}(1)+2r_{k-1}(1))}(y)} \times \cdots \times \frac{\alpha^{-(h_{k-1}(1)(s_{k-1}+2r_{k-1})}(y)}{1 - \alpha^{-t(s_{k-1}(k-1)+2r_{k-1}(k-1))}(y)} \\ &\quad \times \frac{(-1)^{r'_k}}{t \log \alpha(y)} \binom{-l_k}{r'_k} \zeta_t^{-h_k(1)a} \\ &= \zeta^{k-1}_{EZB}(s_1, \ldots, s_{k-1}|h_1, \ldots, h_{k-1}) D^{l_k/2}(y)(-1)^{r'_k} \binom{-l_k}{r'_k} \frac{\zeta_t^{-h_k(1)a}}{t \log \alpha(y)}. \quad (4.1) \end{aligned}$$

Therefore, using (3.5) and (4.1), we have

$$\begin{aligned}\operatorname*{Res}_{s_k=l_k} \mathcal{L}^k_{EZB}(s \mid \chi) &= \sum_{h_1=1}^{t} \chi_1(h_1) \sum_{h_2=1}^{t} \chi_2(h_2(1)) \cdots \sum_{h_k=1}^{t} \chi_k(h_k(1)) \operatorname*{Res}_{s_k=l_k} \zeta^k_{EZB}(s|\mathbf{h}) \\ &= \sum_{h_1=1}^{t} \chi_1(h_1) \sum_{h_2=1}^{t} \chi_2(h_2(1)) \cdots \sum_{h_k=1}^{t} \chi_k(h_k(1)) \\ &\quad \times \zeta^{k-1}_{EZB}(s_1, \ldots, s_{k-1} \mid h_1, \ldots, h_{k-1}) \frac{D^{l_k/2}(y)(-1)^{r'_k}}{t \log \alpha(y)} \binom{-l_k}{r'_k} \zeta_t^{-ah_k(1)}.\end{aligned}$$

This finishes the proof. □

For $a \in \mathbb{Z}$, let us define the Gauss sum

$$\mathcal{G}(\chi, a) = \sum_{x \pmod t} \chi(x) \zeta_t^{ax}.$$

For fixed $(h_1, \ldots, h_{k-1}) \in \mathbb{Z}^{k-1}_{>0}$, we have

$$\begin{aligned}\sum_{h_k=1}^{t} \chi_k(h_k(1)) \zeta_t^{-ah_k(1)} &= \sum_{h_k=1}^{t} \chi_k(-1) \chi_k(-ah_k(1)) \zeta_t^{-ah_k(1)} \\ &= \chi_k(-1) \mathcal{G}(\chi_k, a)\end{aligned}$$

as χ_k is periodic modulo t. The following result related to Gauss sum is found in [6].

Lemma 4.2. *For $d = gcd(a, t)$, if we assume that χ can not be defined modulo $\frac{t}{d}$, then $\mathcal{G}(\chi, a) = 0$.*

It is known that the Dirichlet L-function $L(s, \chi)$ is holomorphic on the whole complex plane where χ is non-principal and the function $L(s, \chi)$ has trivial zeros at non-positive integers. For the function $\mathcal{L}^k_{EZB}(s \mid \chi)$, we have the following result.

Lemma 4.3. *Let χ_k be non-principal character modulo t. Then the function $\mathcal{L}^k_{EZB}(s \mid \chi)$ is holomorphic on the real axis of s_k.*

Proof. From Theorem 3.2, the series $\mathcal{L}^k_{EZB}(s \mid \chi)$ is holomorphic except the singularities given by (3.4). If $s_k(k)$ is a real number, then $2a/t = 0$. Since $\frac{2a}{t} \in \mathbb{Z}$, then $t|a$ or $\frac{t}{2}|a$ with respect to t is odd or even that implies that $\gcd(a, t) = t$ or $\frac{t}{2}$. Therefore, $\frac{t}{\gcd(a,t)} = 1$ or 2. As χ_k is non-principal, the character χ_k cannot be defined for modulo 1 or 2. Using Lemma 4.2, we have $\mathcal{G}(\chi_k, a) = 0$, which desires the result. □

In the following theorem, we calculate the residues of $\mathcal{L}^k_{EZB}(s \mid \chi)$ along the hyperplanes (3.4) for $1 \leq d \leq k-1$.

Theorem 4.4. *Let $k > 1$ and d be positive integers such that $1 \leq d \leq k-1$. Let $r'_d \cdots r'_k$ be non-negative integers. Let $l_k(d) = -2r'_k(d) + \frac{2\pi i a}{t \log \alpha(y)}$. Then*

$$\begin{aligned}\operatorname*{\mathit{Res}}_{s_k(d)=l_k(d)} \mathcal{L}^k_{EZB}(s \mid \chi) &= D^{l_k(d)/2}(y) \zeta^{d-1}_{EZB}(s|\chi) \sum_{\substack{r_d \geq 0, \ldots, r_k \geq 0 \\ r_k(d) = r'_k(d)}} \binom{-s_d}{r_d} (-1)^{r_d} \cdots \binom{-s_k}{r_k} (-1)^{r_k} \\ &\quad \sum_{h_1=1}^{t} \chi_1(h_1) \times \cdots \times \sum_{h_k=1}^{t} \chi_k(h_k(1)) \frac{\alpha^{-h_{d+1}(s_k(d+1)+2r_k(d+1))}(y)}{1 - \alpha^{-t(s_k(d+1)+2r_k(d+1))}(y)} \\ &\quad \times \cdots \times \frac{\alpha^{-h_k(s_k+2r_k)}(y)}{1 - \alpha^{-t(s_k+2r_k)}(y)} \times \frac{\zeta_t^{-ah_d(1)}}{t \log \alpha(y)}.\end{aligned}$$

Proof. Let $s_k(d) = l_k(d) = -2r'_k(d) + \frac{2\pi ia}{t\log\alpha(y)}$. Now $(s_k(d) + 2r'_k(d))\log\alpha(y) = \frac{2\pi ia}{t}$ that implies $\alpha^{s_k(d)+2r'_k(d)}(y) = \zeta_t^a$. Thus, for $1 \le j \le d$, we obtain

$$\alpha^{-h_j(s_k(d)+2r'_k(d))}(y) = \zeta_t^{ah_j}, \alpha^{-h_j(1)(s_k(d)+2r'_k(d))}(y) = \zeta_t^{ah_j(1)}, \text{ and } \alpha^{-t(s_k(d)+2r'_k(d))}(y) = 1.$$

Hence, $\alpha^{-t(s_k(d)+2r'_k(d))}(y) - 1$ is an analytic function with simple zeros at $l_k(d)$ for $1 \le d \le k$. By proceeding as in the proof of Theorem 4.1, we have

$$\lim_{s_k(d)\to l_k(d)} \frac{s_k(d) - l_k(d)}{1 - \alpha^{-t(s_k(d)+2r_k(d))}(y)} = \operatorname*{Res}_{s_k(d)=l_k(d)} \frac{1}{1 - \alpha^{-t(s_k(d)+2r_k(d))}(y)} = \frac{1}{t\log\alpha(y)}.$$

Now we evaluate the limit as follows:

$$\lim_{s_k(d)\to l_k(d)} D^{s_k(d)/2}(y) \sum_{r_d,\ldots,r_k=0}^{\infty} \binom{-s_d}{r_d}(-1)^{r_d} \cdots \binom{-s_k}{r_k}(-1)^{r_k}$$
$$\frac{\alpha^{-h_d(s_k(d)+2r_k(d))}(y)(s_k(d) - l_k(d))}{1 - \alpha^{-t(s_k(d)+2r_k(d))}(y)} \times \cdots \times \frac{\alpha^{-h_k(s_k+2r_k)}(y)}{1 - \alpha^{-t(s_k+2r_k)}(y)}.$$

In the above calculation after applying the limit, only those terns containing $r_d, \ldots, r_k$ will survive when $r_k(d) = r'_k(d)$ and rest of the terms will vanish.
Therefore, the above limit reduces to

$$D^{l_k(d)/2}(y) \sum_{\substack{r_d\ge 0,\ldots,r_k\ge 0\\ r_k(d)=r'_k(d)}} \binom{-s_d}{r_d}(-1)^{r_d} \cdots \binom{-s_k}{r_k}(-1)^{r_k} \frac{\alpha^{-h_{d+1}(s_k(d+1)+2r_k(d+1))}(y)}{1 - \alpha^{-t(s_k(d+1)+2r_k(d+1))}(y)}$$
$$\times \cdots \times \frac{\alpha^{-h_k(s_k+2r_k)}(y)}{1 - \alpha^{-t(s_k+2r_k)}(y)} \times \frac{\zeta_t^{-ah_d}}{t\log\alpha(y)}.$$

Therefore, the residue of $\zeta^k_{EZB}(s|\mathbf{h})$ along the hyper plane $s_k(d) = l_k(d)$ is given by

$$\begin{aligned}
&\operatorname*{Res}_{s_k(d)=l_k(d)} \zeta^k_{EZB}(s|\mathbf{h}) \\
= &\lim_{s_k(d)\to l_k(d)} (s_k(d) - l_k(d))\zeta^k_{EZB}(s|\mathbf{h}) \\
= &D^{s_k(d)/2}(y) D^{s_{d-1}(1)/2}(y) \sum_{r_1,\ldots,r_{d-1}=0}^{\infty} \binom{-s_1}{r_1}(-1)^{r_1} \cdots \binom{-s_{d-1}}{r_{d-1}}(-1)^{r_{d-1}} \\
&\times \sum_{r_d,\ldots,r_k=0}^{\infty} \binom{-s_d}{r_d}(-1)^{r_d} \cdots \binom{-s_k}{r_k}(-1)^{r_k} \frac{\alpha^{-h_1(s_k(1)+2r_k(1))}(y)}{1 - \alpha^{-t(s_k(1)+2r_k(1))}(y)}\Bigg|_{s_k(d)=l_k(d)} \\
&\times \cdots \times \frac{\alpha^{-h_{d-1}(s_k(d-1)+2r_k(d-1))}(y)}{1 - \alpha^{-t(s_k(d-1)+2r_k(d-1))}(y)}\Bigg|_{s_k(d)=l_k(d)} \\
&\times \lim_{s_k(d)\to l_k(d)} (s_k(d) - l_k(d)) \frac{\alpha^{-h_d(s_k(d)+2r_k(d))}(y)}{1 - \alpha^{-t(s_k(d)+2r_k(d))}(y)} \cdots \frac{\alpha^{-h_k(s_k+2r_k)}(y)}{1 - \alpha^{-t(s_k+2r_k)}(y)} \\
= &D^{l_k(d)/2}(y)\zeta^{d-1}_{EZB}(s_1, \ldots, s_{d-1}|h_1, \ldots, h_{d-1}) \sum_{\substack{r_d\ge 0,\ldots,r_k\ge 0\\ r_k(d)=r'_k(d)}} \binom{-s_d}{r_d}(-1)^{r_d} \cdots \binom{-s_k}{r_k}(-1)^{r_k} \\
&\times \frac{\alpha^{-h_{d+1}(s_k(d+1)+2r_k(d+1))}(y)}{1 - \alpha^{-t(s_k(d+1)+2r_k(d+1))}(y)} \cdots \frac{\alpha^{-h_k(s_k+2r_k)}(y)}{1 - \alpha^{-t(s_k+2r_k)}(y)} \times \frac{\zeta_t^{-ah_d(1)}}{t\log\alpha(y)}.
\end{aligned} \tag{4.2}$$

By virtue of (3.5) and (4.2), we have

$$\begin{aligned}
&\operatorname*{Res}_{s_k(d)=l_k(d)} \mathcal{L}^k_{EZB}(s \mid \chi) \\
&= \sum_{h_1=1}^{t} \chi_1(h_1) \sum_{h_2=1}^{t} \chi_2(h_2(1)) \cdots \sum_{h_k=1}^{t} \chi_k(h_k(1)) \operatorname*{Res}_{s_k=l_k} \zeta^k_{EZB}(s|\mathbf{h}) \\
&= \sum_{h_1=1}^{t} \chi_1(h_1) \sum_{h_2=1}^{t} \chi_2(h_2(1)) \cdots \sum_{h_k=1}^{t} \chi_k(h_k(1)) D^{l_k(d)/2}(y) \\
&\quad \times \zeta^{d-1}_{EZB}(s_1, \ldots, s_{d-1}|h_1, \ldots, h_{d-1}) \sum_{\substack{r_d \geq 0, \ldots, r_k \geq 0 \\ r_k(d)=r'_k(d)}} \binom{-s_d}{r_d}(-1)^{r_d} \cdots \binom{-s_k}{r_k}(-1)^{r_k} \\
&\quad \times \frac{\alpha^{-h_{d+1}(s_k(d+1)+2r_k(d+1))}(y)}{1-\alpha^{-t(s_k(d+1)+2r_k(d+1))}(y)} \cdots \frac{\alpha^{-h_k(s_k+2r_k)}(y)}{1-\alpha^{-t(s_k+2r_k)}(y)} \times \frac{\zeta_t^{-ah_d(1)}}{t \log \alpha(y)} \\
&= D^{l_k(d)/2}(y) \zeta^{d-1}_{EZB}(s|\chi) \sum_{\substack{r_d \geq 0, \ldots, r_k \geq 0 \\ r_k(d)=r'_k(d)}} \binom{-s_d}{r_d}(-1)^{r_d} \cdots \binom{-s_k}{r_k}(-1)^{r_k} \sum_{h_1=1}^{t} \chi_1(h_1) \\
&\quad \times \cdots \times \sum_{h_k=1}^{t} \chi_k(h_k(1)) \frac{\alpha^{-h_{d+1}(s_k(d+1)+2r_k(d+1))}(y)}{1-\alpha^{-t(s_k(d+1)+2r_k(d+1))}(y)} \cdots \frac{\alpha^{-h_k(s_k+2r_k)}(y)}{1-\alpha^{-t(s_k+2r_k)}(y)} \times \frac{\zeta_t^{-ah_d(1)}}{t \log \alpha(y)}.
\end{aligned}$$

This completes the proof. □

5 Values of Euler-Zagier multiple L-functions involving balancing-like polynomials associated to Dirichlet characters at negative integers

In this section, we discuss the values of $\mathcal{L}^k_{EZB}(s \mid \chi)$ at negative integers. First we give a sufficient condition for $\mathcal{L}^k_{EZB}(s \mid \chi)$ to be holomorphic at $(s_1, \ldots, s_k) = (-n_1, \ldots, -n_k)$, where $n_i \in \mathbb{N}$ for $i = 1, \ldots, k$. Let us denote $n_k(d) = n_d + \cdots + n_k$, $1 \leq d \leq k$.

Lemma 5.1. *Let $(n_1, \ldots, n_k) \in \mathbb{N}^k$ and χ be a Dirichlet character of modulus t, where t is a positive integer. Then the function $\mathcal{L}^k_{EZB}(s \mid \chi)$ is holomorphic at $(s_1, \ldots, s_k) = (-n_1, \ldots, -n_k)$ if and only if*

$$n_k(1) \not\equiv 0 \pmod 2,\ n_k(2) \not\equiv 0 \pmod 2, \ldots, n_k(k) \not\equiv 0 \pmod 2.$$

Proof. The infinite series (3.3) is holomorphic except the poles derived from

$$\left(\alpha^{t(s_k(1)+2r_k(1))}(y) - 1\right) \times \cdots \times \left(\alpha^{t(s_k+2r_k)}(y) - 1\right) = 0.$$

This is true if and only if, one of the following equations holds:

$$ts_k(1) = -2tr_k(1), ts_k(2) = -2tr_k(2), \cdots, ts_k = -2tr_k,$$

with $tr_k(1) \equiv 0 \pmod 2$, $tr_k(2) \equiv 0 \pmod 2, \ldots, tr_k \equiv 0 \pmod 2$, which desires the result. □

For $1 \leq d \leq k$, let us denote

$$\delta_d(r_d, \ldots r_k; h_d) = (-1)^{r_d} \frac{\alpha^{-h_d(-n_k(d)+2r_k(d))}(y)}{1-\alpha^{-t(-n_k(d)+2r_k(d))}(y)}. \tag{5.1}$$

In particular,

$$\delta_1(r_1, \ldots, r_k; h_1) = \frac{(-1)^{r_1}\alpha^{-h_1(-n_k(1)+2r_k(1))}(y)}{1-\alpha^{-t(-n_k(1)+2r_k(1))}(y)}, \cdots,$$

$$\delta_k(r_k; h_k) = \frac{(-1)^{r_k}\alpha^{-h_k(-n_k(k)+2r_k(k))}(y)}{1-\alpha^{-t(-n_k(k)+2r_k(k))}(y)}.$$

By replacing r_d by $n_d - r_d$ in the above notation (5.1), we have

$$\delta_d(\hat{r}_d,\dots,r_k;h_d) = (-1)^{n_d-r_d}\frac{\alpha^{-h_d\left(-n_k(d)+2(n_d-r_d+r_k(d+1))\right)}(y)}{1-\alpha^{-t\left(-n_k(d)+2(n_d-r_d+r_k(d+1))\right)}(y)}. \tag{5.2}$$

Further, we denote

$$\begin{aligned}
&\sigma_0(r_1,\dots,r_k) = \delta_1(r_1,\dots,r_k;h_1)\times\delta_2(r_2,\dots,r_k;h_2)\times\cdots\times\delta_k(r_k;h_k),\\
&\sigma_p(r_1,\dots,\hat{r}_{c_1},\dots,\hat{r}_{cP},\dots,r_k) = \delta_1(r_1,\dots,\hat{r}_{c_1},\dots,\hat{r}_{cP},\dots,r_k;h_1)\\
&\times\cdots\times\delta_{c_1}(\hat{r}_{c_1},\dots,\hat{r}_{c_p},\dots,r_k;h_{c_1})\delta_{c_2}(\hat{r}_{c_2},\dots,\hat{r}_{c_p},\dots,r_k;h_{c_2})\\
&\times\cdots\times\ \delta_{c_p}(\hat{r}_{c_p},\dots,r_k;h_{c_p})\times\cdots\times\delta_k(r_k;h_k).
\end{aligned} \tag{5.3}$$

The expression $\sigma_p(r_1,\dots,\hat{r}_{c_1},\dots,\hat{r}_{c_p},\dots,r_k)$ precisely represents that p number of integers in the tuple $(r_1,\dots,r_k)$ are replaced from r_t to $n_t - r_t$ for $1\le t\le p$, whenever these terms appears in that corresponding $\delta_i(r_i,\dots r_k;h_i)$ as in the above expression.

Using the above notations, we prove the following proposition which is very essential to prove our main result.

Proposition 5.2. *Let* Ψ *be the non-trivial automorphism of Gal*$(\mathbb{Q}\sqrt{D(y)}/\mathbb{Q})$ *and* σ_p *as in the expression (5.3). Then for any* $0\le p\le k$*, we have*

$$\Psi\big(\sigma_p(r_1,\dots,\hat{r}_{c_1},\dots,\hat{r}_{c_p},\dots,r_k)\big) = (-1)^{n_k(1)}\big(\sigma_{k-p}(\hat{r}_1,\dots,\hat{r}_{c_1-1},r_{c_1},\dots,r_{c_p},\hat{r}_{c_{p+1}},\dots,\hat{r}_k)\big). \tag{5.4}$$

Proof. Now,

$$\begin{aligned}
\sigma_p(r_1,\dots,\hat{r}_{c_1},\dots,\hat{r}_{cP},\dots,r_k) &= \delta_1(r_1,\dots,\hat{r}_{c_1},\dots,\hat{r}_{cP},\dots,r_k;h_1)\\
&\times\cdots\times\delta_{c_1}(\hat{r}_{c_1},\dots,\hat{r}_{c_p},\dots,r_k;h_{c_1})\\
&\times\cdots\times\delta_{c_p}(\hat{r}_{c_p},\dots,r_k;h_{c_p})\times\cdots\times\delta_k(r_k;h_k)
\end{aligned}$$

$$\begin{aligned}
&= (-1)^{r_1}\frac{\alpha^{-h_1\left(-n_k(1)+2(r_{c_1-1}(1)+r_k(c_p+1)+\sum_{t=1}^{p}(n_{c_t}-r_{c_t}))\right)}(y)}{1-\alpha^{-t\left(-n_k(1)+2(r_{c_1-1}(1)+r_k(c_p+1)+\sum_{t=1}^{p}(n_{c_t}-r_{c_t}))\right)}(y)}\\
&\times\cdots\times(-1)^{n_{c_1}-r_{c_1}}\frac{\alpha^{-h_{c_1}\left(-n_k(c_1)+2(r_k(c_p+1)+\sum_{t=1}^{p}(n_{c_t}-r_{c_t}))\right)}(y)}{1-\alpha^{-t\left(-n_k(c_1)+2(r_k(c_p+1)+\sum_{t=1}^{p}(n_{c_t}-r_{c_t}))\right)}(y)}\\
&\times\cdots\times(-1)^{n_{c_p}-r_{c_p}}\frac{\alpha^{-h_{c_p}\left(-n_k(c_p)+2(r_k(c_p+1)+n_{c_p}-r_{c_p})\right)}(y)}{1-\alpha^{-t\left(-n_k(c_p)+2(r_k(c_p+1)+n_{c_p}-r_{c_p})\right)}(y)}\\
&\times\cdots\times(-1)^{r_k}\frac{\alpha^{-h_k(-n_k(k)+2r_k(k))}(y)}{1-\alpha^{-t(-n_k(k)+2r_k(k))}(y)}.
\end{aligned} \tag{5.5}$$

Further simplification gives

$$\begin{aligned}
&-n_k(1)+2\big(r_{c_1-1}(1)+r_k(c_p+1)+\sum_{t=1}^{p}(n_{c_t}-r_{c_t})\big)\\
=\ &-\big[n_1+\cdots+n_{c_1-1}+n_{c_1}+\cdots+n_{c_p}+\cdots+n_k\big]+2\big[r_1+\cdots+r_{c_1-1}\\
&+r_{c_p+1}+\cdots+r_{c_k}+\sum_{t=1}^{p}(n_{c_t}-r_{c_t})\big]\\
=\ &n_k(1)-2\bigg(\sum_{t=1}^{c_1-1}(n_t-r_t)+\sum_{t=c_p+1}^{k}(n_t-r_t)+\sum_{t=1}^{p}r_{c_t}\bigg).
\end{aligned} \tag{5.6}$$

Similarly,

$$-n_k(c_1)+2\big(r_k(c_p+1)+\sum_{t=1}^{p}(n_{c_t}-r_{c_t})\big)=n_k(c_1)-2\Big(\sum_{t=c_p+1}^{k}(n_t-r_t)+\sum_{t=1}^{p}r_{c_t}\Big) \quad (5.7)$$

and

$$-n_k(c_p)+2\big(r_k(c_p+1)+n_{c_p}-r_{c_p}\big)=n_k(c_p)-2\Big(\sum_{t=c_p+1}^{k}(n_t-r_t)+r_{c_p}\Big). \quad (5.8)$$

Using (5.6), (5.7) and (5.8) in (5.5), we have

$$\begin{aligned}
&\sigma_p(r_1,\ldots,\hat{r}_{c_1},\ldots,\hat{r}_{c_P},\ldots,r_k)\\
=\;&(-1)^{r_1}\frac{\alpha^{-h_1\left(n_k(1)-2\left(\sum_{t=1}^{c_1-1}(n_t-r_t)+\sum_{t=c_p+1}^{k}(n_t-r_t)+\sum_{t=1}^{p}r_{c_t}\right)\right)}(y)}{1-\alpha^{-t\left(n_k(1)-2\left(\sum_{t=1}^{c_1-1}(n_t-r_t)+\sum_{t=c_p+1}^{k}(n_t-r_t)+\sum_{t=1}^{p}r_{c_t}\right)\right)}(y)}\\
&\times\cdots\times(-1)^{n_{c_1}-r_{c_1}}\frac{\alpha^{-h_{c_1}\left(n_k(c_1)-2\left(\sum_{t=c_p+1}^{k}(n_t-r_t)+\sum_{t=1}^{p}r_{c_t}\right)\right)}(y)}{1-\alpha^{-t\left(n_k(c_1)-2\left(\sum_{t=c_p+1}^{k}(n_t-r_t)+\sum_{t=1}^{p}r_{c_t}\right)\right)}(y)}\\
&\times\cdots\times(-1)^{n_{c_p}-r_{c_p}}\frac{\alpha^{-h_{c_p}\left(n_k(c_p)-2\left(\sum_{t=c_p+1}^{k}(n_t-r_t)+r_{c_p}\right)\right)}(y)}{1-\alpha^{-t\left(n_k(c_p)-2\left(\sum_{t=c_p+1}^{k}(n_t-r_t)+r_{c_p}\right)\right)}(y)}\\
&\times\cdots\times(-1)^{r_k}\frac{\alpha^{-h_k\left(-n_k(k)+2r_k(k)\right)}(y)}{1-\alpha^{-t\left(-n_k(k)+2r_k(k)\right)}(y)}.
\end{aligned}$$

Similarly, we can deduce

$$\begin{aligned}
&\sigma_{k-p}(\hat{r}_1,\ldots,\hat{r}_{c_1-1},r_{c_1},\ldots,r_{c_p},\hat{r}_{c_{p+1}},\ldots,\hat{r}_k)\\
=\;&(-1)^{n_1-r_1}\frac{\alpha^{-h_1\left(-n_k(1)+2\left(\sum_{t=1}^{c_1-1}(n_t-r_t)+\sum_{t=c_p+1}^{k}(n_t-r_t)+\sum_{t=1}^{p}r_{c_t}\right)\right)}(y)}{1-\alpha^{-t\left(-n_k(1)+2\left(\sum_{t=1}^{c_1-1}(n_t-r_t)+\sum_{t=c_p+1}^{k}(n_t-r_t)+\sum_{t=1}^{p}r_{c_t}\right)\right)}(y)}\\
&\times\cdots\times(-1)^{r_{c_1}}\frac{\alpha^{-h_{c_1}\left(-n_k(c_1)+2\left(\sum_{t=c_p+1}^{k}(n_t-r_t)+\sum_{t=1}^{p}r_{c_t}\right)\right)}(y)}{1-\alpha^{-t\left(-n_k(c_1)+2\left(\sum_{t=c_p+1}^{k}(n_t-r_t)+\sum_{t=1}^{p}r_{c_t}\right)\right)}(y)}\\
&\times\cdots\times(-1)^{r_{c_p}}\frac{\alpha^{-h_{c_p}\left(-n_k(c_p)+2\left(\sum_{t=c_p+1}^{k}(n_t-r_t)+r_{c_p}\right)\right)}(y)}{1-\alpha^{-t\left(-n_k(c_p)+2\left(\sum_{t=c_p+1}^{k}(n_t-r_t)+r_{c_p}\right)\right)}(y)}\\
&\times\cdots\times(-1)^{n_k-r_k}\frac{\alpha^{-h_k\left(-n_k(k)+2(n_k(k)-r_k(k))\right)}(y)}{1-\alpha^{-t\left(-n_k(k)+2(n_k(k)-r_k(k))\right)}(y)}.
\end{aligned}$$

Since $\alpha(y)\beta(y)=1$, then

$$\begin{aligned}
&\sigma_{k-p}(\hat{r}_1,\ldots,\hat{r}_{c_1-1},r_{c_1},\ldots,r_{c_p},\hat{r}_{c_{p+1}},\ldots,\hat{r}_k)\\
=\;&(-1)^{n_1-r_1}\frac{\beta^{-h_1\left(n_k(1)-2\left(\sum_{t=1}^{c_1-1}(n_t-r_t)+\sum_{t=c_p+1}^{k}(n_t-r_t)+\sum_{t=1}^{p}r_{c_t}\right)\right)}(y)}{1-\beta^{-t\left(n_k(1)-2\left(\sum_{t=1}^{c_1-1}(n_t-r_t)+\sum_{t=c_p+1}^{k}(n_t-r_t)+\sum_{t=1}^{p}r_{c_t}\right)\right)}(y)}\\
&\times\cdots\times(-1)^{r_{c_1}}\frac{\beta^{-h_{c_1}\left(n_k(c_1)-2\left(\sum_{t=c_p+1}^{k}(n_t-r_t)+\sum_{t=1}^{p}r_{c_t}\right)\right)}(y)}{1-\beta^{-t\left(n_k(c_1)-2\left(\sum_{t=c_p+1}^{k}(n_t-r_t)+\sum_{t=1}^{p}r_{c_t}\right)\right)}(y)}\\
&\times\cdots\times(-1)^{r_{c_p}}\frac{\beta^{-h_{c_p}\left(n_k(c_p)-2\left(\sum_{t=c_p+1}^{k}(n_t-r_t)+r_{c_p}\right)\right)}(y)}{1-\beta^{-t\left(n_k(c_p)-2\left(\sum_{t=c_p+1}^{k}(n_t-r_t)+r_{c_p}\right)\right)}(y)}\\
&\times\cdots\times(-1)^{n_k-r_k}\frac{\beta^{-h_k\left(-n_k(k)+2r_k(k)\right)}(y)}{1-\beta^{-t\left(-n_k(k)+2r_k(k)\right)}(y)}\\
=\;&(-1)^{n_k(1)}\Psi\big(\sigma_p(r_1,\ldots,\hat{r}_{c_1},\ldots,\hat{r}_{c_p},\ldots,r_k)\big).
\end{aligned}$$

This ends the proof. □

Proposition 5.3. *Let k be a positive integer and $\boldsymbol{n}=(n_1,\ldots n_k)\in\mathbb{N}^k$. Let $A\in\mathbb{N}_{>2}$ and $\sqrt{D(y)}$ be an irrational number. Then $\zeta^k_{EZB}(-\boldsymbol{n}|\boldsymbol{h})\in\mathbb{Q}$ except for singularities.*

Proof. Consider $\mathbf{n}=(n_1,\ldots n_k)\in\mathbb{N}^k$, for some positive integer k. The binomial coefficient

$$\binom{n_i}{r_i}=0 \text{ for } r_i>n_i,\ 1\le i\le k.$$

Then using the notations $n_k(d), r_k(d)$ in (3.3), we have

$$\begin{aligned}
&\zeta^k_{EZB}(-\mathbf{n}|\mathbf{h})\\
=\;&D^{-n_k(1)/2}(y)\sum_{r_1=0}^{n_1}\binom{n_1}{r_1}(-1)^{r_1}\cdots\sum_{r_k=0}^{n_k}\binom{n_k}{r_k}(-1)^{r_k}\frac{\alpha^{-h_1(-n_k(1)+2r_k(1))}(y)}{1-\alpha^{-t(-n_k(1)+2r_k(1))}(y)}\\
&\times\cdots\times\frac{\alpha^{-h_k(-n_k(k)+2r_k(k))}(y)}{1-\alpha^{-t(-n_k(k)+2r_k(k))}(y)}.
\end{aligned}\tag{5.9}$$

It is clear that $\sum_{i=0}^{n}z_i=\sum_{i=0}^{n}z_{n-i}$ for any finite sequence of complex numbers $\{z_i\}$ and $\binom{n}{r}=\binom{n}{n-r}$. Therefore, we can write (5.9) as

$$\begin{aligned}
&\zeta^k_{EZB}(-\mathbf{n}|\mathbf{h})\\
=\;&D^{-n_k(1)/2}(y)\sum_{r_1=0}^{n_1}\binom{n_1}{r_1}(-1)^{r_1}\cdots\sum_{r_{k-1}=0}^{n_{k-1}}\binom{n_{k-1}}{r_{k-1}}(-1)^{r_{k-1}}\\
&\times\frac{1}{2}\Bigg[\sum_{r_k=0}^{n_k}\binom{n_k}{r_k}\frac{(-1)^{r_k}\alpha^{-h_1(-n_k(1)+2r_k(1))}(y)}{1-\alpha^{-t(-n_k(1)+2r_k(1))}(y)}\cdots\frac{\alpha^{-h_k(-n_k(k)+2r_k(k))}(y)}{1-\alpha^{-t(-n_k(k)+2r_k(k))}(y)}\\
&+\sum_{r_k=0}^{n_k}\binom{n_k}{n_k-r_k}\frac{(-1)^{n_k-r_k}\alpha^{-h_1(-n_k(1)+2r_{k-1}(1)+2(n_k-r_k))}(y)}{1-\alpha^{-t(-n_k(1)+2r_{k-1}(1)+2(n_k-r_k))}(y)}\cdots\frac{\alpha^{-h_k(n_k-2r_k)}(y)}{1-\alpha^{-t(n_k-2r_k)}(y)}\Bigg].
\end{aligned}\tag{5.10}$$

By continuing in this process for each index r_i, where $i=k-1,k-2,\ldots,1$ and using the notations in (5.1), we have

$$\begin{aligned}\zeta^k_{EZB}(-\mathbf{n}|\mathbf{h}) &= \frac{D^{-n_k(1)/2}(y)}{2^k}\sum_{r_1=0}^{n_1}\binom{n_1}{r_1}\cdots\sum_{r_k=0}^{n_k}\binom{n_k}{r_k}\Bigg[\prod_{i=1}^{k}\delta_i(r_i,\ldots,r_k;h_i)\\ &\quad+\sum_{c=1}^{k}\Big(\prod_{i=1}^{k}\delta_i(r_i,\ldots,\hat{r}_c,\ldots,r_k;h_i)\Big)\\ &\quad+\sum_{1\le c\le d\le k}\Big(\prod_{i=1}^{k}\delta_i(r_i,\ldots,\hat{r}_c,\ldots,\hat{r}_d,\ldots,r_k;h_i)\Big)\\ &\quad+\cdots+\prod_{i=1}^{k}\delta_i(\hat{r}_i,\ldots,\hat{r}_k;h_i)\Big)\Bigg].\end{aligned}$$

Using notations in (5.3), the above equation reduces

$$\begin{aligned}\zeta^k_{EZB}(-\mathbf{n}|\mathbf{h}) &= \frac{D^{-n_k(1)/2}(y)}{2^k}\Bigg[\sigma_0(r_1,\ldots,r_k)\\ &\quad+\sum_{p=1}^{k}\sum_{1\le c_1<c_2<\cdots<c_p\le k}^{k}\sigma_p(r_1,\ldots,\hat{r}_{c_1},,\ldots,\hat{r}_{c_p},\ldots,r_k)\Bigg].\end{aligned}$$

From Proposition 5.2, for any $0\le p\le k$,

$$\Psi\big(\sigma_p(r_1,\ldots,\hat{r}_{c_1},\ldots,\hat{r}_{c_p},\ldots,r_k)\big)=(-1)^{n_k(1)}\big(\sigma_{k-p}(\hat{r}_1,\ldots,\hat{r}_{c_1-1},r_{c_1},\ldots,r_{c_p},\hat{r}_{c_{p+1}},\ldots,\hat{r}_k)\big).$$

It is clear that, if even number of n_i's are odd in the tuple $(n_1,\ldots,n_k)$, then $(-1)^{n_k(1)}=1$ and hence

$$\Psi\big(\sigma_p(r_1,\ldots,\hat{r}_{c_1},\ldots,\hat{r}_{c_p},\ldots,r_k)\big)=\big(\sigma_{k-p}(\hat{r}_1,\ldots,\hat{r}_{c_1-1},r_{c_1},\ldots,r_{c_p},\hat{r}_{c_{p+1}},\ldots,\hat{r}_k)\big).$$

If odd number of n_i's are odd in the tuple $(n_1,\ldots,n_k)$, then $(-1)^{n_k(1)}=-1$ and therefore

$$\Psi\big(\sigma_p(r_1,\ldots,\hat{r}_{c_1},\ldots,\hat{r}_{c_p},\ldots,r_k)\big)=-\sigma_{k-p}(\hat{r}_1,\ldots,\hat{r}_{c_1-1},r_{c_1},\ldots,r_{c_p},\hat{r}_{c_{p+1}},\ldots,\hat{r}_k).$$

As Ψ is the non-trivial automorphism of Gal$(\mathbb{Q}\sqrt{D(y)}/\mathbb{Q})$, then

$$\sigma_p+\psi(\sigma_p)\in\mathbb{Q}\text{ and }\sigma_p-\psi(\sigma_p)\in\sqrt{D(y)}\mathbb{Q}.$$

Let $X(r_1,\ldots,r_k)=\sigma_0(r_1,\ldots,r_k)+\sum_{p=1}^{k}\sum_{1\le c_1<c_2<\cdots<c_p\le k}\sigma_p(r_1,\ldots,\hat{r}_{c_1},,\ldots,\hat{r}_{c_p},\ldots,r_k)$, then

$$\zeta^k_{EZB}(-\mathbf{n}|\mathbf{h})=\frac{D^{-n_k(1)/2}(y)}{2^k}X(r_1,\ldots,r_k).$$

The following two cases arise.

Case-I : (Even number of n_i's are odd in the tuple $(n_1,\ldots,n_k)$.)

In this case, $\frac{D^{-n_k(1)/2}(y)}{2^k}$ is a rational number. Now

$$\begin{aligned}\Psi\big(X(r_1,\ldots,r_k)\big) &= \Psi\big(\sigma_0(r_1,\ldots,r_k)\big)\\ &\quad+\Psi\Big(\sum_{p=1}^{k-1}\sum_{1\le c_1<c_2<\cdots<c_p\le k-1}\sigma_p(r_1,\ldots,\hat{r}_{c_1},,\ldots,\hat{r}_{c_p},\ldots,r_k)\Big)\\ &\quad+\Psi\big(\sigma_k(\hat{r}_1,\ldots,\hat{r}_k)\big)\\ &= \sigma_k(\hat{r}_1,\ldots,\hat{r}_k)+\sum_{p=1}^{k-1}\sum_{1\le c_1<c_2<\cdots<c_p\le k-1}\sigma_p(r_1,\ldots,\hat{r}_{c_1},,\ldots,\hat{r}_{c_p},\ldots,r_k)\\ &\quad+\sigma_0(r_1,\ldots,r_k)\\ &= X(r_1,\ldots,r_k),\end{aligned}$$

which implies that $X(r_1,\dots,r_k)\in\mathbb{Q}$. Therefore, $\zeta^k_{EZB}(-\mathbf{n}|\mathbf{h})\in\mathbb{Q}$.

Case-II : (Odd number of n_i's are odd in the tuple $(n_1,\dots,n_k)$.)

In this case, $\frac{D^{-n_k(1)/2}(y)}{2^k}\in\sqrt{D(y)}\mathbb{Q}$. Now

$$\begin{aligned}X(r_1,\dots,r_k) &= \sigma_0(r_1,\dots,r_k)+\sigma_1(\hat{r}_1,\dots,r_k)+\cdots+\sigma_{k-1}(\hat{r}_1,\hat{r}_2,\dots,\hat{r}_{k-1},r_k)\\ &\quad+\sigma_k(\hat{r}_1,\dots,\hat{r}_k)\\ &= \sigma_0(r_1,\dots,r_k)-\Psi\big(\sigma_0(r_1,\dots,r_k)\big)\\ &\quad+\sigma_1(\hat{r}_1,\dots,r_k)-\Psi\big(\sigma_1(\hat{r}_1,\dots,r_k)\big)+\cdots\\ &\quad+\sigma_{k-1}(\hat{r}_1,\hat{r}_2,\dots,r_k)-\Psi\big(\sigma_{k-1}(\hat{r}_1,\hat{r}_2,\dots,r_k)\big)\in\sqrt{D(y)}\mathbb{Q}.\end{aligned}$$

Therefore, in this case $\zeta^k_{EZB}(-\mathbf{n}|\mathbf{h})\in\mathbb{Q}$. This completes the proof. □

The following result deals with the values of Euler-Zagier multiple L-functions involving balancing-like polynomials associated to Dirichlet character.

Theorem 5.4. *Let k be a positive integer and $\boldsymbol{n}=(n_1,\dots n_k)\in\mathbb{N}^k$. For $A\in\mathbb{N}_{>2}$, $\sqrt{D(y)}$ irrational number and $\chi_1,\dots,\chi_k$ Dirichlet characters, then $\mathcal{L}^k_{EZB}(-\boldsymbol{n}|\chi)$ is rational except for the singularities.*

Proof. Note that

$$\mathcal{L}^k_{EZB}(-\mathbf{n}\mid\chi)=\sum_{h_1=1}^{t}\chi_1(h_1)\sum_{h_2=1}^{t}\chi_2(h_2(1))\cdots\sum_{h_k=1}^{t}\chi_k(h_k(1))\zeta^k_{EZB}(-\mathbf{n}|\mathbf{h}).$$

As χ_i's are Dirichlet characters and using Proposition 5.3, we have $\mathcal{L}^k_{EZB}(-\mathbf{n}|\chi)$ is rational except for the singularities. □

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