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Several inequalities involving the generalized multi-index Mittag-Leffler functions

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Abstract In the present article, we choose the generalized multi-index Mittag-Leffler function to establish some presumably new and potentially useful inequalities . Also, we point out that the results presented here can be reduced to those corresponding to some relatively simple Mittag-Leffler functions including certain known ones.

1 Introduction

In 1903, Gösta Mittag-Leffler [7] introduced and investigated the Mittag-Leffler function defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)} \qquad (\Re(\alpha) > 0;\ z \in \mathbb{C}), \tag{1.1}$$

where Γ denotes the familiar Gamma function (see, e.g., [8, 9, Section 1.1]). Here and in the following, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}$ be the sets of complex numbers, real numbers, positive real numbers and positive integers, respectively, and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Since then, various extensions (or generalizations) of this Mittag-Leffler function have been presented. The generalized Mittag-Leffler functions have been connected and applied to diverse research fields such as mathematics itself, engineering, statistics, biology, chemistry, and physics (see, e.g., [2, 19, 11, 10, 18, 17]).

Among numerous extensions of the Mittag-Leffler function (1.1), we choose to recall some of them. Wiman [3, 4] generalized the Mittag-Leffler function (1.1)

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)} \qquad (\Re(\alpha) > 0;\ \beta,\, z \in \mathbb{C}). \tag{1.2}$$

Prabhakar [21] extended the $E_{\alpha,\beta}(z)$ (1.2)

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!} \qquad (\Re(\alpha) > 0;\ \beta,\, \gamma,\, z \in \mathbb{C}). \tag{1.3}$$

Srivastava and Tomovski [11] gave a further extension of (1.3)

$$E^{\gamma,\delta}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n}}{\Gamma(\alpha n + \beta)} \frac{z^n}{n!} \tag{1.4}$$

$$(\Re(\alpha) > \max\{0, \Re(\kappa) - 1\},\ \Re(\kappa) > 0;\ \beta,\, \gamma,\, z \in \mathbb{C})\,.$$

The special case of (1.4) when

$$\kappa = q \in (0, 1) \cup \mathbb{N} \quad \text{and} \quad \min\{\Re(\beta),\, \Re(\gamma)\} > 0$$

was already considered by Shukla and Prajapati [1]. Here $(\lambda)_\nu$ denotes the Pochhammer symbol which is defined (for $\lambda,\ \nu \in \mathbb{C}$), in terms of the familiar Gamma function Γ, by

$$(\lambda)_\nu := \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} = \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C}\setminus\{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases} \tag{1.5}$$

it being understood conventionally that $(0)_0 := 1$ (see, e.g., [5, 8, 9]).

Saxena and Nishimoto [15, 16] introduced the following generalized multi-index Mittag-Leffler function

$$E^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}[z] = E_{\gamma,\kappa}\left[(\alpha_j,\beta_j)^m_{j=1}; z\right] = \sum_{n=0}^{\infty} \frac{(\gamma)_{\kappa n}}{\prod_{j=1}^{m}\Gamma(\alpha_j n+\beta_j)} \frac{z^n}{n!} \tag{1.6}$$

$$\left(\Re(\beta_j) > 0\ \ (j = 1,\ldots,m),\ \Re\Big(\sum_{j=1}^{m}\alpha_j\Big) > \max\{0, \Re(\kappa)-1\};\ \gamma,\ z \in \mathbb{C}\right).$$

The generalized multi-index Mittag-Leffler function (1.6) is normalized as

$$\mathbf{E}^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}[z] = \Big(\prod_{j=1}^{m}\Gamma(\beta_j)\Big)\, E^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}[z], \tag{1.7}$$

which satisfies the following differential formula

$$\left(\mathbf{E}^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}[z]\right)' = \frac{d}{dz}\mathbf{E}^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}[z] = \frac{(\gamma)_\kappa \prod_{j=1}^{m}\Gamma(\beta_j)}{\prod_{j=1}^{m}\Gamma(\alpha_j+\beta_j)}\mathbf{E}^{\gamma+\kappa,\kappa}_{(\alpha_j,\alpha_j+\beta_j)_m}[z]. \tag{1.8}$$

Or, equivalently,

$$\left(E^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}[z]\right)' = \frac{d}{dz}E^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}[z] = (\gamma)_\kappa\, E^{\gamma+\kappa,\kappa}_{(\alpha_j,\alpha_j+\beta_j)_m}[z]. \tag{1.9}$$

In this paper, we aim to establish several inequalities involving the normalized general multi-index Mittag-Leffler function (1.7). Also we present three inequalities associated with the normalized general multi-index Mittag-Leffler function (1.7). Also we point out that the results presented here can be reduced to those corresponding to some relatively simple Mittag-Leffler functions including certain known ones.

For our purpose, first we recall the definition of log-convexity (log-concavity). A function $h : [a,b] \subseteq \mathbb{R} \longrightarrow \mathbb{R}^+$ is said to be *log-convex* if $\log h$ is convex on the interval $[a,b]$, that is,

$$h(\alpha x + (1-\alpha)y) \leq [h(x)]^\alpha [h(y)]^{1-\alpha} \tag{1.10}$$

holds for all $x,\ y \in [a,b]$ and $\alpha \in [0,1]$. If the inequality in (1.10) is reversed, then h is said to be *log-concave* on $[a,b]$.

The following statement obviously holds: If a function $g : [a,b] \subseteq \mathbb{R} \longrightarrow \mathbb{R}^+$ is differentiable, then g is log-convex (log-concave) if and only if g'/g is increasing (decreasing).

Next we give a theorem which states monotonicity of ratio of two functions depends on that of the sequence of ratios of the coefficients of the two respective functions, which is asserted in Theorem A (see, e.g., [20]).

Theorem 1.1. *Let* $F(x) = \sum_{n=0}^{\infty} f_n\, x^n$ *and* $G(x) = \sum_{n=0}^{\infty} g_n\, x^n$ *be two series with real coefficients* f_n *and* g_n $(n \in \mathbb{N}_0)$*, which are convergent for* $|x| < r$ *for some* $r \in \mathbb{R}^+$*. If* $g_n \in \mathbb{R}^+$ $(n \in \mathbb{N}_0)$ *and the sequence* $\{f_n/g_n\}_{n=0}^{\infty}$ *is (strictly) increasing (decreasing), then the function* $F(x)/G(x)$ *is also (strictly) increasing (decreasing) on* $[0,r)$.

Further we recall another theorem which states monotonicity of ratio of certain differences of two functions depends on that of the ratio of derivatives of two respective functions, which is asserted in Theorem 1.2 (see, e.g., [6]).

Theorem 1.2. *Let two functions f, $g : [a, b] \longrightarrow \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b). Also let $g'(x) \neq 0$ for all $x \in (a, b)$. If f'/g' is increasing (decreasing) on the interval (a, b), then the functions*

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)} \tag{1.11}$$

are increasing (decreasing) on the interval (a, b).

2 Inequalities

Turán [14] showed that the Legendre polynomials $P_n(x)$ satisfy the following determinant inequality

$$\begin{vmatrix} P_n(x) & P_{n+1}(x) \\ P_{n+1}(x) & P_{n+2}(x) \end{vmatrix} = P_n(x)\, P_{n+2}(x) - \{P_{n+1}(x)\}^2 \leq 0 \tag{2.1}$$
$$(-1 \leq x \leq 1;\ n \in \mathbb{N}_0)\,,$$

where the equality occurs only when $x = \pm 1$ (see also [13]). Recently, many researchers have applied the above classical inequality (2.1) in various polynomials and functions such as ultraspherical polynomials, Laguerre polynomials, Hermite polynomials, Bessel functions of the first kind, modified Bessel functions, and polygamma functions.

We present a Turán-type inequality for the normalized general multi-index Mittag-Leffler function (1.7), asserted in Theorem 2.1.

Theorem 2.1. *Let γ, κ, α_j, $\beta_j \in \mathbb{R}^+$ $(j = 1, \ldots, m;\ m \in \mathbb{N})$. Then*

$$\left(\mathbf{E}^{\gamma,\kappa}_{(\alpha_j,\beta_j+1)_m}[z]\right)^2 \leq \mathbf{E}^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}[z]\ \mathbf{E}^{\gamma,\kappa}_{(\alpha_j,\beta_j+2)_m}[z] \tag{2.2}$$

holds for all $z \in \mathbb{R}^+$.

Proof. We begin by recalling the Psi (or Digamma) function $\psi(z)$ defined by (see, e.g., [9, Section 1.3])

$$\psi(z) := \frac{d}{dz}\{\log \Gamma(z)\} = \frac{\Gamma'(z)}{\Gamma(z)} \qquad \text{or} \qquad \log \Gamma(z) = \int_1^z \psi(t)\, dt. \tag{2.3}$$

The $\psi(z)$ has the following property

$$\psi(z) = -\gamma - \frac{1}{z} + \sum_{n=1}^{\infty} \frac{z}{n(z+n)}, \tag{2.4}$$

where γ is the Euler-Mascheroni constant (see, e.g., [9, Section 1.2]).

From (1.6) and (1.7), we write

$$\mathbf{E}^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}[z] = \sum_{n=0}^{\infty} a^{\kappa}_n(\alpha_j, \beta_j, m,\ \gamma)\, z^n,$$

where

$$a^{\kappa}_n(\alpha_j, \beta_j, m,\ \gamma) := \frac{(\gamma)_{\kappa n} \prod\limits_{j=1}^{m} \Gamma(\beta_j)}{n! \prod\limits_{j=1}^{m} \Gamma(\alpha_j n + \beta_j)}. \tag{2.5}$$

We prove the function $\beta_j \longmapsto \mathbf{E}^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}[z]$ is log-convex for $\beta_j \in \mathbb{R}^+$ and $j = 1, \ldots, m$. Since sum of log-convex functions is log-convex, it suffices to show that $\beta_j \longmapsto a^{\kappa}_n(\alpha_j, \beta_j, m, \gamma)$ is log-convex for $\beta_j \in \mathbb{R}^+$ and $j = 1, \ldots, m$, that is,

$$\frac{\partial^2}{\partial{\beta_j}^2}\log(a^{\kappa}_n(\alpha_j,\beta_j,m,\gamma)) = \psi'(\beta_j) - \psi'(\beta_j+\alpha_j\, n) \geq 0 \tag{2.6}$$

$$\left(\alpha_j,\, \beta_j \in \mathbb{R}^+;\; n \in \mathbb{N}_0\right).$$

Indeed, using (2.4), we have

$$\psi^{(2)}(z) = -2\sum_{n=0}^{\infty}\frac{1}{(z+n)^3} < 0 \quad \left(z \in \mathbb{R}^+\right), \tag{2.7}$$

which implies that $\psi'(z)$ is a decreasing function on $z \in \mathbb{R}^+$. Therefore the inequality (2.6) holds true. Hence the function $\beta_j \longmapsto \mathbf{E}^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}[z]$ is log-convex for $\beta_j \in \mathbb{R}^+$ and $j = 1, \ldots, m$. In view of (1.10), we find

$$\mathbf{E}^{\gamma,\kappa}_{(\alpha_j,\, t\mu_j+(1-t)\nu_j)_m}[z] \leq \left\{\mathbf{E}^{\gamma,\kappa}_{(\alpha_j,\mu_j)_m}[z]\right\}^t \left\{\mathbf{E}^{\gamma,\kappa}_{(\alpha_j,\nu_j)_m}[z]\right\}^{1-t} \tag{2.8}$$

$$\left(\alpha_j,\, \gamma,\, \kappa,\, \mu,\, \nu,\, z \in \mathbb{R}^+,\, m \in \mathbb{N};\; 0 \leq t \leq 1\right).$$

Setting $\mu_j = \beta_j$, $\nu_j = \beta_j + 2$ and choosing $t = 1/2$ in (2.8), we obtain the desired inequality (2.2). □

Lemma 2.2. *The function* $z \mapsto \Gamma(z+a)/\Gamma(z)$ *is increasing for* $z,\, a \in \mathbb{R}^+$. *Also, the function* $z \mapsto \Gamma(z)/\Gamma(z+a)$ *is decreasing for* $z,\, a \in \mathbb{R}^+$. *Further, the function*

$$z \longmapsto \frac{\Gamma^2(z(n+1)+a)}{\Gamma(z(n+2)+a)\Gamma(zn+a)} \tag{2.9}$$

is decreasing for $z,\, a \in \mathbb{R}^+$ *and* $n \in \mathbb{N}_0$.

Proof. Let $f(z) := \Gamma(z+a)/\Gamma(z)$. Using (2.4) and taking logarithmic derivative, we get

$$f'(z) = \frac{a\,\Gamma(z+a)}{\Gamma(z)}\sum_{n=0}^{\infty}\frac{1}{(z+n)(z+a+n)} > 0 \quad \left(z,\, a \in \mathbb{R}^+\right).$$

So the function $z \mapsto \Gamma(z+a)/\Gamma(z)$ is increasing for $z,\, a \in \mathbb{R}^+$.

Similarly, we can prove the second statement. We omit the details.

Indeed, let

$$g(z) := \frac{\Gamma^2(z(n+1)+a)}{\Gamma(z(n+2)+a)\Gamma(zn+a)} \quad \left(z \in \mathbb{R}^+\right).$$

Taking logarithmic derivative, we have

$$g'(z) = g(z)\left\{2(n+1)\,\psi(z(n+1)+a) - (n+2)\,\psi(z(n+2)+a) - n\,\psi(zn+a)\right\}. \tag{2.10}$$

Using (2.4) in (2.10), we get

$$g'(z) = -2z\, g(z)\sum_{k=0}^{\infty}\frac{a+k}{(zn+a+k)\{z(n+1)+a+k\}\{z(n+2)+a+k\}} < 0$$

for $z,\, a \in \mathbb{R}^+$ and $n \in \mathbb{N}_0$. This proves the third statement. □

Theorem 2.3. *Let* $\gamma,\, \kappa,\, \alpha_j,\, \mu_j,\, \nu_j\ (j = 1, \ldots, m) \in \mathbb{R}^+$. *Then the following statements hold.*

(i) If $\mu_j < \nu_j\ (\nu_j < \mu_j)\ (j = 1, \ldots, m)$, *the function*

$$z \longmapsto E^{\gamma,\kappa}_{(\alpha_j,\mu_j)_m}[z]\Big/E^{\gamma,\kappa}_{(\alpha_j,\nu_j)_m}[z] \tag{2.11}$$

is increasing (decreasing) for $z \in \mathbb{R}^+$.

(ii) If $\mu_j < \nu_j$ $(j = 1, \ldots, m)$,

$$E^{\gamma,\kappa}_{(\alpha_j,\nu_j)_m}[z]\, E^{\gamma+\kappa,\kappa}_{(\alpha_j,\alpha_j+\mu_j)_m}[z] - E^{\gamma,\kappa}_{(\alpha_j,\mu_j)_m}[z]\, E^{\gamma+\kappa,\kappa}_{(\alpha_j,\alpha_j+\nu_j)_m}[z] \geq 0 \tag{2.12}$$

for $z \in \mathbb{R}^+$.

(iii) If β_j $(j = 1, \ldots, m) \in \mathbb{R}^+$,

$$E^{\gamma,\kappa}_{(\alpha_j,\alpha_j+\beta_j)_m}[z]\, E^{\gamma+\kappa,\kappa}_{(\alpha_j,\alpha_j+\beta_j)_m}[z] - E^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}[z]\, E^{\gamma+\kappa,\kappa}_{(\alpha_j,2\alpha_j+\beta_j)_m}[z] \geq 0 \tag{2.13}$$

for $z \in \mathbb{R}^+$.

Proof. To prove the statement in (i), using (1.6), in view of Theorem 1.1, we need to show monotonicity of the following sequence $\{a_n\}_{n=0}^{\infty}$ given by

$$a_n = \prod_{j=1}^{m} \Gamma(\alpha_j n + \nu_j) \Big/ \prod_{j=1}^{m} \Gamma(\alpha_j n + \mu_j). \tag{2.14}$$

Then we have

$$\frac{a_{n+1}}{a_n} = \prod_{j=1}^{m} \frac{\Gamma(\alpha_j(n+1)+\nu_j)}{\Gamma(\alpha_j n + \nu_j)} \Big/ \prod_{j=1}^{m} \frac{\Gamma(\alpha_j(n+1)+\mu_j)}{\Gamma(\alpha_j n + \mu_j)}. \tag{2.15}$$

If $\mu_j < \nu_j$ $(j = 1, \ldots, m)$, in view of the first statement in Lemma 2.2, we have

$$\frac{\Gamma(\alpha_j(n+1)+\nu_j)}{\Gamma(\alpha_j n + \nu_j)} > \frac{\Gamma(\alpha_j(n+1)+\mu_j)}{\Gamma(\alpha_j n + \mu_j)}. \tag{2.16}$$

Using (2.16) in (2.15), we find $\frac{a_{n+1}}{a_n} \geq 1$. If $\mu_j > \nu_j$ $(j = 1, \ldots, m)$, we can have $\frac{a_{n+1}}{a_n} \leq 1$. This proves the statement in (i).

For (ii), in view of (i), using (1.9), we have

$$\begin{aligned} &\frac{d}{dz}\left\{E^{\gamma,\kappa}_{(\alpha_j,\mu_j)_m}[z] \Big/ E^{\gamma,\kappa}_{(\alpha_j,\nu_j)_m}[z]\right\} \\ &= \frac{(\gamma)_\kappa\left[E^{\gamma,\kappa}_{(\alpha_j,\nu_j)_m}[z]E^{\gamma+\kappa,\kappa}_{(\alpha_j,\alpha_j+\mu_j)_m}[z] - E^{\gamma,\kappa}_{(\alpha_j,\mu_j)_m}[z]E^{\gamma+\kappa,\kappa}_{(\alpha_j,\alpha_j+\nu_j)_m}[z]\right]}{\left(E^{\gamma,\kappa}_{(\alpha_j,\nu_j)_m}[z]\right)^2} \geq 0, \end{aligned}$$

which implies the statement in (ii).

Setting $\mu_j = \beta_j$ and $\nu_j = \alpha_j + \beta_j$ in the result in (ii), we prove the statement in (iii). □

Corollary 2.4. *Let γ, κ, α_j, μ_j, ν_j $(j = 1, \ldots, m) \in \mathbb{R}^+$ such that $\mu_j < \nu_j$ $(j = 1, \ldots, m)$. Then*

$$\mathbf{E}^{\gamma,\kappa}_{(\alpha_j,\mu_j)_m}[z] \geq \mathbf{E}^{\gamma,\kappa}_{(\alpha_j,\nu_j)_m}[z] \tag{2.17}$$

for $z \in \mathbb{R}^+$.

Proof. If $\mu_j < \nu_j$ $(j = 1, \ldots, m)$, then, from (i) of Theorem 2.3, we have

$$E^{\gamma,\kappa}_{(\alpha_j,\mu_j)_m}[z] \Big/ E^{\gamma,\kappa}_{(\alpha_j,\nu_j)_m}[z] \geq E^{\gamma,\kappa}_{(\alpha_j,\mu_j)_m}[0] \Big/ E^{\gamma,\kappa}_{(\alpha_j,\nu_j)_m}[0] = \prod_{j=1}^{m} \Gamma(\nu_j) \Big/ \prod_{j=1}^{m} \Gamma(\mu_j),$$

from which we get

$$\prod_{j=1}^{m} \Gamma(\mu_j)\, E^{\gamma,\kappa}_{(\alpha_j,\mu_j)_m}[z] \geq \prod_{j=1}^{m} \Gamma(\nu_j)\, E^{\gamma,\kappa}_{(\alpha_j,\nu_j)_m}[z].$$

In view of (1.7), the last inequality is the same as that in (2.17). □

3 Lazarević and Wilker-type inequalities

We present Lazarević and Wilker-type inequalities for the normalized general multi-index Mittag-Leffler function (1.7), asserted in Theorem 3.1.

Theorem 3.1. *Let* $\gamma,\ \kappa,\ \alpha_j,\ \mu_j,\ \nu_j\ \in \mathbb{R}^+$ *with* $\mu_j \le \nu_j$ $(j=1,\ldots,m)$*. Then*

$$\left\{\mathbf{E}^{\gamma,\kappa}_{(\alpha_j,\mu_j)_m}[z]\right\}^{\prod_{j=1}^{m}\frac{\Gamma(\mu_j)}{\Gamma(\alpha_j+\mu_j)}} \ge \left\{\mathbf{E}^{\gamma,\kappa}_{(\alpha_j,\nu_j)_m}[z]\right\}^{\prod_{j=1}^{m}\frac{\Gamma(\nu_j)}{\Gamma(\alpha_j+\nu_j)}} \tag{3.1}$$

for $z \in \mathbb{R}^+$.

Proof. Using the results in Corollary 2.4 and Lemma 2.2, we can prove the result here. □

4 Further inequalities

Here we present several inequalities product of generalized multi-index Mittag-Leffler functions (1.7). To do this, we begin by proving log-concavity of the function (1.7), asserted by the following lemma.

Lemma 4.1. *Let* $\gamma,\ \kappa,\ \alpha_j,\ \beta_j \in \mathbb{R}^+$ *such that* $\kappa \le \alpha_j$ *and* $\beta_j = \gamma$ *for some* $j \in \{1,\ldots,m\}$*. Then the function* $z \mapsto \mathbf{E}^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}[z]$ *is log-concave for* $z \in \mathbb{R}^+$.

Proof. We find that $f(z) := \mathbf{E}^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}[z]$ is log-concave for $z \in \mathbb{R}^+$ if and only if $\log f(z)$ is concave for $z \in \mathbb{R}^+$ if and only if $\frac{d}{dz}\log f(z) = f'(z)/f(z)$ is decreasing for $z \in \mathbb{R}^+$. From (1.6), (1.7), and (1.8), we have

$$\begin{aligned}\frac{f'(z)}{f(z)} &= \frac{(\gamma)_\kappa\, E^{\gamma+\kappa,\kappa}_{(\alpha_j,\alpha_j+\beta_j)_m}[z]}{E^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}[z]}\\ &= (\gamma)_\kappa \sum_{n=0}^{\infty}\frac{(\gamma+\kappa)_{\kappa n}}{\prod_{j=1}^{m}\Gamma(\alpha_j n+\alpha_j+\beta_j)}\frac{z^n}{n!}\Bigg/\sum_{n=0}^{\infty}\frac{(\gamma)_{\kappa n}}{\prod_{j=1}^{m}\Gamma(\alpha_j n+\beta_j)}\frac{z^n}{n!}.\end{aligned} \tag{4.1}$$

In order to prove that $f'(z)/f(z)$ is decreasing for $z \in \mathbb{R}^+$, in view of Theorem A, it suffices to show that the following sequence, which is obtained by dividing the coefficient of z^n in the numerator series by that in the denominator series in (4.1),

$$a_n := \frac{\Gamma(\gamma+\kappa(n+1))\prod_{j=1}^{m}\Gamma(\alpha_j n+\beta_j)}{\Gamma(\gamma+\kappa n)\prod_{j=1}^{m}\Gamma(\alpha_j(n+1)+\beta_j)} \quad (n \in \mathbb{N}_0) \tag{4.2}$$

is decreasing if and only if $a_{n+1}/a_n \le 1$ $(n \in \mathbb{N}_0)$. Now we have to show that, for $n \in \mathbb{N}_0$,

$$\frac{a_{n+1}}{a_n} = \frac{\Gamma(\gamma+\kappa(n+2))\Gamma(\gamma+\kappa n)}{\Gamma^2(\gamma+\kappa(n+1))}\,\frac{\prod_{j=1}^{m}\Gamma^2(\alpha_j(n+1)+\beta_j)}{\prod_{j=1}^{m}\Gamma(\alpha_j(n+2)+\beta_j)\,\Gamma(\alpha_j n+\beta_j)} \le 1,$$

or, equivalently,

$$\prod_{j=1}^{m}\frac{\Gamma^2(\alpha_j(n+1)+\beta_j)}{\Gamma(\alpha_j(n+2)+\beta_j)\,\Gamma(\alpha_j n+\beta_j)} \le \frac{\Gamma^2(\kappa(n+1)+\gamma)}{\Gamma(\kappa(n+2)+\gamma)\Gamma(\kappa n+\gamma)}. \tag{4.3}$$

By assumption, say, $\alpha_{j_0} \ge \kappa$ and $\beta_{j_0} = \gamma$ for some $j_0 \in \{1,\ \ldots,\ m\}$. Then, in view of the third statement in Lemma 2.2, we obtain

$$\frac{\Gamma^2(\kappa(n+1)+\gamma)}{\Gamma(\kappa(n+2)+\gamma)\Gamma(\kappa n+\gamma)} \ge \frac{\Gamma^2(\alpha_{j_0}(n+1)+\gamma)}{\Gamma(\alpha_{j_0}(n+2)+\gamma)\Gamma(\alpha_{j_0}n+\gamma)}. \tag{4.4}$$

Also, we find from the first or second statement in Lemma 2.2 that

$$\frac{\Gamma^2(\alpha_j(n+1)+\beta_j)}{\Gamma(\alpha_j n+\beta_j)\Gamma(\alpha_j(n+2)+\beta_j)} \leq 1 \quad (j=1,\,\ldots,\,m). \tag{4.5}$$

Then, using (4.5) in the right side of (4.4), we prove (4.3).

□

Theorem 4.2. *Let* $\gamma,\ \kappa,\ \alpha_j,\ \beta_j \in \mathbb{R}^+$ *such that* $\kappa \leq \alpha_j$ *and* $\beta_j = \gamma$ *for some* $j \in \{1,\ldots,m\}$. *Then*

(i)

$$\left\{\mathbf{E}^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}[z_1]\right\}^t \left\{\mathbf{E}^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}[z_2]\right\}^{1-t} \leq \mathbf{E}^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}(tz_1+(1-t)z_2), \tag{4.6}$$

where $z_1,\ z_2 \in \mathbb{R}^+$ *and* $t \in [0,1]$.

(ii)

$$\mathbf{E}^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}[z]\mathbf{E}^{\gamma+2\kappa,\kappa}_{(\alpha_j,2\alpha_j+\beta_j)_m}[z] \leq C(\alpha_j,\beta_j;m)\left\{\mathbf{E}^{\gamma+\kappa,\kappa}_{(\alpha_j,\alpha_j+\beta_j)_m}[z]\right\}^2 \tag{4.7}$$

for $z \in \mathbb{R}^+$, *where*

$$C(\alpha_j,\beta_j;m) := \prod_{j=1}^{m} \frac{\Gamma(\beta_j)\,\Gamma(2\alpha_j+\beta_j)}{\Gamma^2(\alpha_j+\beta_j)} \geq 1.$$

(iii)

$$\mathbf{E}^{\gamma+\kappa,\kappa}_{(\alpha_j,\alpha_j+\beta_j)_m}[z] \leq \mathbf{E}^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}[z] \quad \left(z \in \mathbb{R}^+\right). \tag{4.8}$$

Proof. The inequality in (i) is a restatement of the log-concavity of the function $z \mapsto \mathbf{E}^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}[z]$ for $z \in \mathbb{R}^+$ in Lemma 4.1.

By Lemma 4.1, $\mathbf{E}[z] := \mathbf{E}^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}[z]$ is log-concave for $z \in \mathbb{R}^+$ if and only if

$$\frac{d}{dz}\frac{\mathbf{E}'[z]}{\mathbf{E}[z]} = \frac{\mathbf{E}''[z]\,\mathbf{E}[z]-\{\mathbf{E}'[z]\}^2}{\{\mathbf{E}[z]\}^2} \leq 0$$

for $z \in \mathbb{R}^+$ if and only if

$$\mathbf{E}''[z]\,\mathbf{E}[z] \leq \{\mathbf{E}'[z]\}^2 \quad \left(z \in \mathbb{R}^+\right). \tag{4.9}$$

Using (1.8) in (4.9), we obtain

$$\mathbf{E}^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}[z]\mathbf{E}^{\gamma+2\kappa,\kappa}_{(\alpha_j,2\alpha_j+\beta_j)_m}[z] \leq \frac{(\gamma)_\kappa}{(\gamma+\kappa)_\kappa}\,C(\alpha_j,\beta_j;m)\left\{\mathbf{E}^{\gamma+\kappa,\kappa}_{(\alpha_j,\alpha_j+\beta_j)_m}[z]\right\}^2 \tag{4.10}$$

for $z \in \mathbb{R}^+$. In view of the first or second statement in Lemma 2.2, we have

$$\frac{(\gamma)_\kappa}{(\gamma+\kappa)_\kappa} = \frac{\Gamma^2(\gamma+\kappa)}{\Gamma(\gamma)\,\Gamma(\gamma+2\kappa)} \leq 1, \tag{4.11}$$

which is used in (4.10) to prove (4.7). Here, $C(\alpha_j,\beta_j;m) \geq 1$ can also be shown by the first or second statement in Lemma 2.2.

As in the proof of (ii), $\mathbf{E}[z] := \mathbf{E}^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}[z]$ is log-concave for $z \in \mathbb{R}^+$ if and only if $\mathbf{E}'[z]/\mathbf{E}[z]$ is decreasing for $z \in \mathbb{R}^+$. Thus we find

$$\mathbf{E}'[z]/\mathbf{E}[z] \leq \lim_{z\to 0+} \mathbf{E}'[z]/\mathbf{E}[z] = \mathbf{E}'[0],$$

from which we have

$$\mathbf{E}'[z] \leq \mathbf{E}'[0]\,\mathbf{E}[z]. \tag{4.12}$$

Applying (1.8) to (4.12), we obtain the inequality (4.8). □

5 Special cases

Since the generalized multi-index Mittag-Leffler function $E^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}[z]$ (1.6) contain relatively simple Mittag-Leffler functions such as (1.2), (1.3) and (1.4) as its special cases, the results presented here may be reduced to yield those corresponding to the functions (1.2), (1.3) and (1.4).

Setting $m = \gamma = \kappa = 1$ in the result in Theorem 2.1 yields the corresponding known inequality [12, Theorem 1]. Taking $m = 1$ and $\kappa \in (0, 1) \cup \mathbb{N}$ in Theorem 2.1 gives the corresponding known inequality [12, Theorem 5]. The known result [12, Theorem 4] is a special case of the inequality in Theorem 3.1.

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