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Bounds for D-Radius and D-diameter of a graphs

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Abstract. In this article, we give bounds for D-radius and D-diameter of a graph. Also, as D-radius, D-diameter is greater than radius and diameter we give bounds for their difference.

1 Introduction

The concept of distance is one of the important parameters in a graph. In a graph, geodesic distance is well known. In an article [2], the first two authors introduced the concept of D-distance between vertices by considering degree of vertices present in the path also. Using this distance we can define the eccentricity of a vertex and radius, diameter of any graph.

In this article we obtained bounds for D-radius and D-diameter in terms of minimal degree, maximal degree of vertices and number of vertices . Also we proved examples to show that the above bounds are attained.

Further we give an upper bound for the difference of D-radius and radius, D-diameter and diameter and show that in case of cyclic graph this bounds attained.

For any connected graph we use Δ to denote the maximal degree of vertices and δ to denote minimal degree of vertex. Further, we denote radius and diameter by $r(G)$ and $d(G)$ of a graph and $r^D(G)$, $r^D(G)$ to denote the D-radius and D-diameter of a graph.

2 Main results

Theorem 2.1. *If any graph G with n vertices and $\Delta(G) \geq 2$, then $2\delta + 1 \leq r^D(G) \leq d^D(G) \leq n(\Delta + 1) - 4$.*

Proof. Let u and v be any two vertices of G such that $d^D(u, v) = d^D(G)$ and $(u = u_1, u_2, u_3, \cdots, u_m = v)$ be the corresponding path, where $m \leq n$. Then $d^D(u, v) = \sum deg(u) + m - 1 = m\Delta + (m-1) = m(\Delta+1) - 1 \leq n(\Delta+1) - 1 \Rightarrow d^D(u, v) \leq n(\Delta+1) - 1$. Now we claim that there is no graph G having $d^D(G) = n(\Delta+1) - 1$. Suppose G is a graph having $d^D(G) = n(\Delta+1) - 1$. This equality is obtained by taking the length of the D-path m as n and all vertices in the corresponding path are of degree Δ. Hence G must be a Δ- regular graph and $(u_1, u_2, u_3, \cdots, u_n)$ is longest D-path in G. Now as $deg(u_i) = \Delta$, $\Delta \geq 2$, the vertex u_1 must adjacent to a vertex u_k, where $k \geq 2$. Then $(u_1, u_k, u_{k+1}, \cdots, u_n)$ is a D-path between u_1 and u_n, of D-length less than $d^D(u, v)$. This is contradiction and so no graph can have the D-diameter $n(\Delta+1) - 1$. Therefore $d^D(G) \leq n(\Delta+1)$ Now let $x, y \in V(G)$ such that $d^D(x, y) = r^D(G)$ and $(x = v_1, v_2, v_3, \cdots, v_l = y)$ be the corresponding path with $l \geq 2$. Then $d^D(x, y) = \sum deg(v_i) + l - 1 \geq l\delta + l - 1 \geq r(\delta + 1) - 1 = 2(\delta + 1) - 1 = 2\delta + 1$.

Therefore $2\delta + 1 \leq r^D(G) \leq d^D(G) \leq n(\Delta + 1) - 4$. □

Theorem 2.2. *For any graph G with $\Delta(G) \geq 2$, $d^D(G) = n(\Delta + 1) - 4$, if and only if $G = C_3$ or C_4.*

Proof. Let $d^D(G) = n(\Delta + 1) - 4$ and p be a D- path in G of length $d^D(G)$.
Claim: the path p can exclude one vertex of G not belongs to p. Then $d^D(G) \leq (n-2)(\Delta+1) - (n-3) = n(\Delta + 1) - 2(\Delta + 1) - n + 3 = n(\Delta + 1) - 2\Delta + 1 - n = n(\Delta + 1) - 4 - 2\Delta + 5 - n$ if $-2\Delta + 5 - n \geq 0$ then $2\Delta - 5 + n \leq 0 \Rightarrow 2\Delta + n \leq 5$, this is a contradiction and as $-2\Delta - n + 5 < 0$. Therefore $d^D(G) < n(\Delta+1) - 4$ which is contradiction to our assumption and

so p can exclude at most one vertex of G. Now in order to prove the assumption, it is enough to prove that $p(G)$ does not contain all the vertices of G. Suppose $p(G)$ contain all vertices of G. If there is a vertex w such that $deg(w) \geq 3$, then $p(G)$ will not contain all vertices of neighbor of w and so $\Delta(G) \leq 2$. Since $\Delta(G) \geq 2$, we have $\Delta(G) = 2$. Further if $deg(u) = 1$ for some u, then $d^D(G) < n(\Delta+1)-4$ and so G must be cycle. since D-path between any two vertices of a cycle C_n will not include all the vertices of C_n and $p(G = C_n)$ contains all the vertices of $G = C_n$. we get a contradiction and hence the claim follows. Now $d^D(G) \leq (n-1)(\Delta+1)+n-2 = (n-1)(\Delta+1)-1$. By hypothesis $d^D(G) = n(\Delta+1)-4 \leq (n-1)(\Delta+1)-1 \Rightarrow \Delta = 2$. Therefore $d^D(G) = n(\Delta+1)-4$ by theorem (3.2) in [6]. $n = 3$ or 4. Thus G is either C_3 or C_4. Converse is obvious. □

Theorem 2.3. *For any graph G, then $r^D(G) = 2\delta+1$, if and only if $G \approx K_n$.*

Proof. Let $r^D(G) = 2\delta+1$ and $e^D(u) = 2\delta+1, \forall u \in V(G)$, let $d^D(u,v) = 2\delta+1$.
Claim: u and v are adjacent vertices. Suppose u and v are not adjacent is an internal vertex w in the $u-v$ D-path. Now $d^D(u,v) = d(u,v)+deg(u)+deg(v)+deg(w) = 2+3\delta$. $2\delta+1 = 3\delta+1$ which is contradiction. Therefore there is no internal vertex in $u-v$ path and hence u and v are adjacent vertices. Therefore G is a complete graph. converse is obvious. □

Next, we prove bounds for difference of radius and diameters.

Theorem 2.4. *For any graph G, $r^D(G) - r(G) \leq \Delta(r(G)+1)$.*

Proof. Let v be a vertex of G such that $e(v) = r(G)$. Clearly $r^D(G) \leq e^D(v)$. Now u be farthest vertex from v with respect to D-distance and $d(u,v) = m$ then $e^D(u) \leq (m+1)\Delta+m = m(\Delta+1)+\Delta < e(v)(\Delta+1)+\Delta$. $r^D(G)-r(G) \leq e(v)(\Delta+1)+\Delta e(v) = \Delta(e(v)+1) = \Delta(r(G)+1)$. Therefore $r^D(G) - r(G) \leq \Delta(r(G)+1)$. □

Theorem 2.5. *For any graph G, $d^D(G) - d(G) \leq \Delta(d(G)+1)$.*

Proof. Let v be a vertex of G such that $e^D(v) = d^D(G)$ and u be the farthest vertex v with respect to D-distance. let $d(u,v) = m$ then $d^D(v) = e^D(v) \leq (m+1)\Delta+m = m(\Delta+1)+\Delta = m(\Delta+1)+\Delta$
$d^D(G) \leq d(G)(\Delta+1)+\Delta-d(G) = \Delta(d(G)+1)$. And hence $d^D(G)-d(G) \leq \Delta(d(G)+1)$. □

Theorem 2.6. *For any graph G, the following are equivalent.*
(i) $r^D(G) - r(G) = \Delta(r(G)+1)$
(ii) $d^D(G) - d(G) = \Delta(d(G)+1)$
(iii) $G = C_n$.

Proof. $(i) \Rightarrow (ii)$
Let us assume that $r^D(G) - r(G) = \Delta(r(G)+1)$.
$r(G) = \lceil \frac{n-1}{2} \rceil$.
Suppose $r(G) < \lceil \frac{n-1}{2} \rceil$, let u be the vertex of G such that $e(u) = r(G)$ and v be a farthest vertex of u with respect to D-distance then $r^D(G) \leq e^D(u) \leq (d(u,v)+1)\Delta + d(u,v) = d(u,v)(\Delta+1)+\Delta \leq e(u)(\Delta+1)+\Delta = r(G)(\Delta+1)+\Delta$
$r^D(G)-r(G) \leq r(G)(\Delta+1)+\Delta-r(G) = \Delta(r(G)+1) < \Delta(\lceil \frac{n-1}{2} \rceil+1)$, which is contradiction and hence the assumption follows. Therefore G must be either a path or cycle. Suppose G is path on n vertices.
Case(1) n is even $r^D(G) = \frac{3n+2}{2}$ and $r(G) = \frac{n}{2}$
$r^D(G) - r(G) = \frac{3n+2}{2} - \frac{n}{2} = n+1$
$(n+1) < \Delta(r(G)+1)$ which is contradiction.
Case(2) n is odd $r^D(G) = \frac{3n-1}{2}$ and $r(G) = \frac{n-1}{2}$
$r^D(G) - r(G) = \frac{3n-1}{2} - \frac{n-1}{2} = n$
$n < \Delta(r(G)+1)$ which is also contradiction. Therefore G must be a cycle.
$(iii) \Rightarrow (i)$
If n is even $r^D(C_n) = \frac{3n+4}{2}$ and $r(C_n) = \frac{n}{2}$
$r^D(C_n) - r(C_n) = \frac{3n+4}{2} - \frac{n}{2} = n+2$
$\Delta(r(G)+1) = 2(\frac{n}{2}+1) = n+2$.

when n is odd $r^D(C_n) = \frac{3n+1}{2}$ and $r(C_n) = \frac{n-1}{2}$
$r^D(C_n) - r(C_n) = \frac{3n+1}{2} - \frac{n_1}{2} = n+1$
$\Delta(r(G)+1) = 2(\frac{n-1}{2}+1) = n+1.$
$(iii) \Rightarrow (ii)$
Let G be a cycle on n vertices then, $d^D(C_n) = \frac{3n+4}{2}$ and $d(C_n) = \frac{n}{2}$
$d^D(C_n) - d(C_n) = \frac{3n+4}{2} - \frac{n}{2} = n+2$, if n is even
$\Delta(d(G)+1) = 2(\frac{n}{2}+1) = n+2$. And
$d^D(C_n) = \frac{3n+1}{2}$ and $d(C_n) = \frac{n-1}{2}$, if n is odd
$d^D(C_n) - d(C_n) = \frac{3n+1}{2} - \frac{n_1}{2} = n+1$
$\Delta(d(G)+1) = 2(\frac{n-1}{2}+1) = n+1$. Hence (ii) follows.
$(ii) \Rightarrow (iii)$
We have to prove that $G = C_n$.
Claim: $deg(v_i) = \Delta(G) \forall v_i \in p$. p be the D-path suppose $deg(v_i) < \Delta$, for some vertex v_i in D-path p. Then $d^D(G) = \sum deg(v_i) + d(u,v) < d(u,v)\Delta + d(u,v) = d(u,v)(\Delta+1) < d(G)(\Delta+1)$, where $d(G)$ is the diameter of G. Therefore $d^D(G) - d(G) < d(G)(\Delta+1) - d(G) = \Delta d(G)$ and so $\Delta(d(G)+1) < \Delta d(G)$. This is contradiction as $n > d(G)$. Hence $deg(v_i) = \Delta$, for every v_i in p. Therefore $G = C_n$. □

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