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Geodetic Number Of Circulant Graphs $C_n(\{1, 3\})$

Manal Al-Labadi

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Abstract In this paper, we compute the geodetic set and geodetic number of circulant graphs $C_n(\{S\})$ where $S = \{1,\ 3\}$.

1 Introduction

A graph G is a finite simple connected graph without loops and multiple edges.

The minimum number of a geodetic set of G is called the geodetic number and this number is denoted by $g(G)$.

A graph is called circulant graph $C_n(\{a_1, a_2, \cdots, a_m\})$ where $1 < a_1 < a_2 < \cdots < a_m \leq \lfloor \frac{n}{2} \rfloor$ and two distinct vertices adjacent if $|i - j| \equiv a_l (mod\, n)$.

Also, recently Al-Labadi [1] studied the geodetic number of circulant graphs of $C_m(\{2, 4, \cdots, \lfloor \frac{m}{2} \rfloor - 1, \lfloor \frac{m}{2} \rfloor\})$ and study the other properties of the circulant graph. Fore more application in geodetic number of graph, see [5-12].

2 Preliminary Lemmas

Let $C_n(\{1, 3\})$ be the circulant graphs.
In this section, we will present some crucial significant lemmas.

To light the idea of the following lemma. A vertex u in a graph G is called a extreme point if the subgraph induced by its neighbors is complete. If S is a geodetic, then S contains the set of extreme points.
Now, we give some lemmas of properties $C_n(\{1, 3\})$.

Lemma 2.1. *The circulant graphs $C_n(\{1, 3\})$ has no extreme point.*

Proof. Let v_a be the arbitrary vertex in $C_n(\{1, 3\})$. Observe that v_a is adjacent to v_i and v_j, where $i = \{a + 1, a + 3\}$ and $j = \{a - 1, a - 3\}$. The two vertices v_i and v_j are not adjacent in $C_n(\{1, 3\})$, since the distance between them in $C_n(\{1, 3\})$ is not one or three. So, v_a is not an simplicial vertex.
So, v_a is not an extreme point in the circulant graphs $C_n(\{1, 3\})$ for any vertex v_a. □

The following lemma, determine when the circulant graphs $C_n(\{S\})$ is connected, see [12].

Lemma 2.2. *The circulant graphs $C_n(\{S\})$, where $S = \{a_1,\ ...,\ a_k\}$, is connected if and only if $gcd(a_1,\ ...,\ a_k) = 1$.*

This outcome to the following lemma, we determine the diameter of the circulant graphs $C_n(\{1, 3\})$.

Lemma 2.3. *If $n = 6q + r$ for some positive integer q and $0 \leq r \leq 5$, then the diameter of the circulant graphs $C_n(\{1,3\})$ is*

$$diam(C_n(\{1,3\})) = \begin{cases} q+1 & , \quad if\ n = 6q + r, r \neq 4 \\ q+2 & , \quad if\ n = 6q + 4. \end{cases}$$

Proof. Suppose that $n = 6q + r$, $r \neq 4$. We have the following cases:

- **case 1**: If $n = 6q$, then observe that $d(v_0, v_{\frac{n}{2}}) = q$. For each i with $0 \leq i \leq 2$ the path, $v_0, v_{n\pm(1\times3-2i)}, v_{n\pm(2\times3-2i)}, ..., v_{n\pm((q-1)\times3-2i)}, v_{n\pm(q\times3-2i)}$ is of length q. Therefore the distance between v_0 and v_a is $d(v_0, v_a) \leq q + 1$, where $d(v_0, v_{3q-1}) = q + 1$ the path $v_0, v_3, v_{3\times2}, ..., v_{3q}, v_{3q-1}$ is of length $q + 1$.
- **case 2**:If $n = 6q+1$, then observe that $d(v_0, v_{\lfloor\frac{n}{2}\rfloor-1}) = q+1$. For each i with $0 \leq i \leq 2$ the path, $v_0, v_{1\times3-2i}, v_{2\times3-2i}, ..., v_{(q-1)\times3-2i}, v_{q\times3-2i}, v_{q\times3-2i-1}$ is of length $q + 1$. Therefore the distance between v_0 and v_a is $d(v_0, v_a) \leq q + 1$.
- **case 3**: If $n = 6q + 2$, then observe that $d(v_0, v_{\frac{n}{2}}) = q + 1$. For each i with $0 \leq i \leq 1$ the path, $v_0, v_{n\pm(1\times3-2i)}, v_{n\pm(2\times3-2i)}, ..., v_{n\pm((q-1)\times3-2i)}, v_{n\pm(q\times3-2i)}$ is of length $q+1$. So, the distance between v_0 and v_a is $d(v_0, v_a) \leq q + 1$.
- **case 4**: If $n = 6q + 3$, then observe that $d(v_0, v_{\lfloor\frac{n}{2}\rfloor}) = q + 1$. For each i with $0 \leq i \leq 1$ the path, $v_0, v_{(1\times3-2i)}, v_{(2\times3-2i)}, ..., v_{((q-1)\times3-2i)}, v_{(q\times3-2i)}$ is of length $q + 1$. So, the distance between v_0 and v_a is $d(v_0, v_a) \leq q + 1$.
- **case 5**: If $n = 6q+5$, then observe that $d(v_0, v_{\lfloor\frac{n}{2}\rfloor}) = q+1$. The path, $v_0, v_{n\pm(1\times3)}, v_{n\pm(2\times3)}, ..., v_{n\pm((q-1)\times3)}, v_{n\pm(q\times3)}$ is of length $q+1$. So, the distance between v_0 and v_a is $d(v_0, v_a) \leq q + 1$.

Now if $n = 6q + 4$, then we have $d(v_0, v_{\frac{n}{2}}) = q$. For each i with $0 \leq i \leq 2$ the path $v_0, v_{n\pm(1\times3-2i)}, v_{n\pm(2\times3-2i)}, ..., v_{n\pm((q-1)\times3-2i)}, v_{n\pm(q\times3-2i)}$ is of length $q + 2$. Therefore the distance between v_0 and v_a is $d(v_0, v_a) \leq q + 2$. □

The following lemma is a necessary result to determine the geodetic set in the circulant graph $C_n(\{1,3\})$.

We subtract the vertices of the circulant graph for both sides of the cycle C_n (the side of $\{v_0, v_1, ..., v_{\lfloor\frac{n}{2}\rfloor}\}$ and the side of $\{v_{n-1}, v_{n-2}, ..., v_{\lfloor\frac{n}{2}\rfloor+1}\}$).

Lemma 2.4. *For positive integers n and q if $I_{C_n(\{1,3\})}(v_0, v_a) \bigcap\{v_0, v_1, ..., v_{\lfloor\frac{n}{2}\rfloor}\} \neq \phi$ and $I_{C_n(\{1,3\})}(v_0, v_a) \bigcap\{v_{n-1}, v_{n-2}, ..., v_{\lfloor\frac{n}{2}\rfloor+1}\} \neq \phi$. Then*

$$a = \begin{cases} \frac{n}{2},\ \frac{n}{2} - 1,\ \frac{n}{2} + 1,\ n-2,\ 2 & , \quad if\ n = 6q,\ n = 6q + 4 \\ \frac{n}{2} - 2,\ \frac{n}{2},\ n-2,\ 2 & , \quad if\ n = 6q + 2, \\ \lceil\frac{n}{2}\rceil + \lceil\frac{r}{2}\rceil,\ \lfloor\frac{n}{2}\rfloor - \lceil\frac{r}{2}\rceil,\ n-2,\ 2 & , \quad if\ n\ is\ odd. \end{cases}$$

Proof. First, we begins proof with the trivial two points in all cases are $n - 2$ and 2, since $I_{C_n(\{1,3\})}(v_0, v_2) \bigcap\{v_0, v_1, ..., v_{\lfloor\frac{n}{2}\rfloor}\} = \{v_1\}$ and $I_{C_n(\{1,3\})}(v_0, v_2) \bigcap\{v_{n-1}, v_{n-2}, ..., v_{\lfloor\frac{n}{2}\rfloor+1}\} = \{n-1\}$. Also, $I_{C_n(\{1,3\})}(v_0, v_{\{n-2\}}) \bigcap\{v_0, v_1, ..., v_{\lfloor\frac{n}{2}\rfloor}\} = \{v_1\}$ and $I_{C_n(\{1,3\})}(v_0, v_{\{n-2\}}) \bigcap\{v_{n-1}, v_{n-2}, ..., v_{\lfloor\frac{n}{2}\rfloor+1}\} = \{n-1\}$.
We have the following cases:

- **Case 1**: If $n = 6q + r$ is even, then we have the following subcases
  - **Subcase 1.1**: If $r = 0$, then the vertex $\frac{n}{2} = 3q$ is $d(v_0, v_{3q}) = q$.
    Since $diam(C_{6q}(\{1,3\})) = q + 1$, $I_{C_n(\{1,3\})}(v_0, v_a) \bigcap\{v_0, v_1, ..., v_{\lfloor\frac{n}{2}\rfloor}\} \neq \phi$ and $I_{C_n(\{1,3\})}(v_0, v_a) \bigcap\{v_{n-1}, v_{n-2}, ..., v_{\lfloor\frac{n}{2}\rfloor+1}\} \neq \phi$, we have $d(v_0, v_a) \geq q + 1$. Therefore $a \in \{\frac{n}{2},\ 3q-1,\ 3q+1\}$.

- **Subcase 1.2**: If $r = 2$, then the vertex $\frac{n}{2} = 3q+1$ is $d(v_0, v_{3q+1}) = q+1$. Since $diam(C_{6q+2}(\{1,3\})) = q+1$, $I_{C_n(\{1,3\})}(v_0, v_a) \bigcap \{v_0, v_1, ..., v_{\lfloor \frac{n}{2} \rfloor}\} \neq \phi$ and $I_{C_n(\{1,3\})}(v_0, v_a) \bigcap \{v_{n-1}, v_{n-2}, ..., v_{\lfloor \frac{n}{2} \rfloor+1}\} \neq \phi$, we have $d(v_0, v_a) \geq q+1$. Therefore $a \in \{\frac{n}{2} = 3q+1,\ 3q-1\}$.
- **Subcase 1.3**: If $r = 4$. then the vertex $\frac{n}{2} = 3q+2$ is $d(v_0, v_{3q+2}) = q+2$. Since $diam(C_{6q+4}(\{1,3\})) = q+2$, $I_{C_n(\{1,3\})}(v_0, v_a) \bigcap \{v_0, v_1, ..., v_{\lfloor \frac{n}{2} \rfloor}\} \neq \phi$ and $I_{C_n(\{1,3\})}(v_0, v_a) \bigcap \{v_{n-1}, v_{n-2}, ..., v_{\lfloor \frac{n}{2} \rfloor+1}\} \neq \phi$, we have $d(v_0, v_a) \geq q+1$. Therefore $a \in \{\frac{n}{2} = 3q+1,\ 3q-1\}$.

• **Case 2**: If $n = 6q+r$ is odd, then $\lfloor \frac{n}{2} \rfloor = 3q + r_1$ where $0 \leq r_1 \leq 2$ we have $d(v_0, v_{3q}) = q$. Since the $diam(C_n(\{1,3\})) = q+1$, $I_{C_n(\{1,3\})}(v_0, v_a) \bigcap \{v_0, v_1, ..., v_{\lfloor \frac{n}{2} \rfloor}\} \neq \phi$ and $I_{C_n(\{1,3\})}(v_0, v_a) \bigcap \{v_{n-1}, v_{n-2}, ..., v_{\lfloor \frac{n}{2} \rfloor+1}\} \neq \phi$, we have $d(v_0, v_a) = q+1$. Therefore $a = 3q-1$ i.e $a = 3q \pm \lceil \frac{r}{2} \rceil$.

□

3 The geodetic number of the circulant graphs $C_n(\{1,3\})$

In this section we determine the geodetic number of the circulant graphs $C_n(\{1,3\})$. We also, assume the vertex set of $C_n(\{1,3\})$ is $\{v_0, v_1, ..., v_{n-1}\}$.

Lemma 3.1. *If $n = 6q+r$ for some positive integer q and $0 \leq r \leq 6$, then $g(C_n(\{1,3\})) = 2$ if and only if $r = 4$.*

Proof. Suppose that $n = 6q+4$, then $d(v_0, v_{3q+2}) = 3q+2$. For each i with $0 \leq i \leq 2$ the path $v_0, v_{n\pm(1\times3-2i)}, v_{n\pm(2\times3-2i)}, ..., v_{n\pm((q-1)\times3-2i)}, v_{n\pm(q\times3-2i)}$ is of length $3q+2$ and so it is $v_0 - v_{3q+2}$ geodesic cover all values of i. These paths cover the vertices $v_0, v_1, ..., v_{3q+2}$. Since v_0, and v_{3q+2} are antipodal points in $C_n(\{1,3\})$, we have $S = \{v_0, v_{3q+2}\}$ is geodetic set of $C_n(\{1,3\})$. Now, suppose that $n = 6q+r$ and $g(C_n(\{1,3\})) = 2$. Let $S = \{v_0, v_a\}$ be a minimal geodetic set of $C_n(\{1,3\})$. Then $v_0 - v_a$ geodesic covers all vertices $v_0, v_1, v_2, \cdots, v_a$ and $v_a, v_{a+1}, v_{a+2}, \cdots, v_0$. By using Lemma4, $a = 3q + r_1$ for some positive integer $0 \leq r_1 \leq 2$ and $n - a = 3q + r - r_1$.

On the other hand, since $v_0 - v_a$ geodesic covers all vertices $v_0, v_1, v_2, \cdots, v_a$ and $v_a, v_{a+1}, ...,$ v_0, thus $r - r_1 = r_1$, so $r = 2r_1$. Suppose that $n = 6q+r$ and $r \neq 4$, i.e $r_1 \neq 2$. In this case $n = 6q+2$ or $n = 6q$ by using Lemma 4, for any cases of a not all vertices lie on any $v_0 - v_a$ geodesic. Hence $g(C_n(\{1,3\})) > 2$. □

In the following Lemma, we found the geodetic number of $C_n(\{1,3\})$ when $n = 6q+r$ where $r \neq 4$.

Lemma 3.2. *If $n = 6q+r$ for some positive integer $q > 1$ and $0 \leq r \leq 5$, then $g(C_n(\{1,3\})) = 3$ if and only if $r = 0,\ 1,\ 3$.*

Proof. Suppose that $n = 6q+r$ for some positive integer q and $n \neq 6, 9$ and 11, then:

Case 1: Let $r = 0$. Consider $S = \{v_0, v_{\frac{n}{2}-1}, v_{\frac{n}{2}+1}\}$, the $v_0 - v_{\frac{n}{2}-1}$ geodesics cover all the vertices $\{v_0, v_1, v_2, ..., v_{\frac{n}{2}-1}\}$. And $v_0 - v_{\frac{n}{2}+1}$ geodesics cover all vertices $\{v_0, v_{n-1}, v_{n-2}, ..., v_{\frac{n}{2}+1}\}$. And using Lemma 5, $g(C_n(\{1,3\})) > 2$. Hence S is a geodetic set and $g(C_n(\{1,3\})) = 3$.
Case 2: Let $r = 1$. Consider $S = \{v_0, v_{\lfloor \frac{n}{2} \rfloor-1}, v_{\lceil \frac{n}{2} \rceil+1}\}$, the $v_0 - v_{\lfloor \frac{n}{2} \rfloor-1}$ geodesics cover all vertices $\{v_0, v_1, ..., v_{\frac{n}{2}}, v_{\lfloor \frac{n}{2} \rfloor-1}\}$. And $v_0 - v_{\lceil \frac{n}{2} \rceil+1}$ geodesic cover all vertices $\{v_0, v_{n-1}, v_{n-2}, ..., v_{\lceil \frac{n}{2} \rceil}, v_{\lceil \frac{n}{2} \rceil+1}\}$. And using Lemma 5, $g(C_n(\{1,3\})) > 2$. Hence S is a geodetic set and $g(C_n(\{1,3\})) = 3$.
Case 3: Let $r = 3$. Consider $S = \{v_0, v_{\lfloor \frac{n}{2} \rfloor-2}, v_{\lceil \frac{n}{2} \rceil+2}\}$, the $v_0 - v_{\lfloor \frac{n}{2} \rfloor-2}$ geodesics cover all vertices $\{v_0, v_1, ..., v_{\lfloor \frac{n}{2} \rfloor} - 2, v_{\lfloor \frac{n}{2} \rfloor-1}\}$. And $v_0 - v_{\lceil \frac{n}{2} \rceil+2}$ geodesic cover all vertices $\{v_0, v_{n-1}, v_{n-2}, ..., v_{\lceil \frac{n}{2} \rceil+2}, v_{\lceil \frac{n}{2} \rceil+1}\}$. And $v_{\lfloor \frac{n}{2} \rfloor-2} - v_{\lceil \frac{n}{2} \rceil+1}$ geodesics cover all vertices $\{v_{\lfloor \frac{n}{2} \rfloor-2}, v_{\lfloor \frac{n}{2} \rfloor-1}, v_{\lfloor \frac{n}{2} \rfloor}, ..., v_{\lceil \frac{n}{2} \rceil}, v_{\lceil \frac{n}{2} \rceil+1}, v_{\lceil \frac{n}{2} \rceil+2}\}$. By using Lemma 5, $g(C_n(\{1,3\})) > 2$. Hence S is a geodetic set and $g(C_n(\{1,3\})) = 3$.
Now, suppose that $n = 6q+r$ and $r \neq 0, 1, 3, 4$. In this case, the vertex $v_{\frac{n+1}{2}+1}$ can not lie on any $v_0 - v_a$ geodesic. Hence $g(C_n(\{1,3\})) > 3$.

□

Now, we discuss the cases for the geodetic number when $n = 6q + r$, where $r = 2, 5$.

Lemma 3.3. *For the circulant graph $C_n(\{1,3\})$, suppose that $n = 6q + r$ for some positive integer q and $r = 2, 5$, then $g(C_n(\{1,3\})) = 4$.*

Proof. Suppose that $n = 6q + r$ for some positive integer q and $r = 2, 5$, then:

- **Case 1**: Let $n = 6q + 2$. Then consider $S = \{v_0, v_{\frac{n}{2}-2}, v_{\frac{n}{2}}, v_{n-2}\}$, the $v_0 - v_{\frac{n}{2}-2}$ geodesics cover all vertices $\{v_0, v_1, ..., v_{\frac{n}{2}-2}, v_{\frac{n}{2}-1}\}$. The $v_0 - v_{\frac{n}{2}}$ geodesic cover all vertices $\{v_0, v_{n-1}, v_{n-3}, v_{n-4}, v_{n-6}, ..., v_{\frac{n}{2}+4}, v_{\frac{n}{2}+3}, v_{\frac{n}{2}+1}, v_{\frac{n}{2}}\}$ and $v_{\frac{n}{2}} - v_{n-2}$ geodesics cover all vertices $\{v_{n-2}, v_{n-5}, ..., v_{\frac{n}{2}-2}, v_{\frac{n}{2}}\}$. And using Lemma 6 , $g(C_n(\{1,3\})) > 3$. Hence S is a geodetic set and $g(C_n(\{1,3\})) = 4$.

- **Case 2**: Let $n = 6q + 5$, $q > 1$. Then consider $S = \{v_0, v_{\lfloor\frac{n}{2}\rfloor-3}, v_{\lceil\frac{n}{2}\rceil+3}, v_{\lceil\frac{n}{2}\rceil+1}\}$, the $v_0 - v_{\lfloor\frac{n}{2}\rfloor-3}$ geodesics cover all vertices $\{v_0, v_1, v_2, ..., v_{\lfloor\frac{n}{2}\rfloor-2}, v_{\lfloor\frac{n}{2}\rfloor-3}\}$, the $v_0 - v_{\lceil\frac{n}{2}\rceil+3}$ geodesics all vertices $\{v_0, v_{n-1}, v_{n-2}, ..., v_{\lceil\frac{n}{2}\rceil+2}, v_{\lceil\frac{n}{2}\rceil+3}\}$ and the $v_{\lfloor\frac{n}{2}\rfloor-3} - v_{\lceil\frac{n}{2}\rceil+1}$ geodesics all vertices $\{v_{\lfloor\frac{n}{2}\rfloor-3}, v_{\lfloor\frac{n}{2}\rfloor-2}, v_{\lfloor\frac{n}{2}\rfloor-1}, v_{\lfloor\frac{n}{2}\rfloor}, v_{\lceil\frac{n}{2}\rceil}, v_{\lceil\frac{n}{2}\rceil+1}\}$. And using Lemma 6, $g(C_n(\{1,3\})) > 3$. Hence S is a geodetic set and $g(C_n(\{1,3\})) = 4$.

□

Finally, we agitate the case for when the geodetic number is 5.

Lemma 3.4. *If $n = 9, 11$, then $g(C_n(\{1,3\})) = 5$.*

Proof. If $n = 9$, then consider $S = \{v_0, v_2, v_4, v_6, v_7\}$ is a geodetic set of $C_9(\{1,3\})$. Hence $g(C_9(\{1,3\})) = 5$.
If $n = 11$, then consider $S = \{v_0, v_1, v_2, v_3, v_{10}\}$ is a geodetic set of $C_{11}(\{1,3\})$. Hence $g(C_{11}(\{1,3\})) = 5$. □

4 The girth of the circulant graphs $C_n(\{1,3\})$

In this section we find the girth of the circulant graphs $C_n(\{1,3\})$ and we find the relation between the geodetic number of the circulant graph $C_n(\{1,3\})$ and the girth of the circulant graph $C_n(\{1,3\})$.

Definition 4.1. The smallest cycle in the graph **G** is called the girth of **G** and to simplify we notation by $girth(\mathbf{G})$.

Lemma 4.2. *If $n = 6q+4$ for some positive integer q, then girth of $C_n(\{1,3\})$ is $girth(C_n(\{1,3\})) = 2q + 2$.*

Proof. Suppose that $n = 6q + 4$, then $d(v_0, v_{3q}) = q$. For each i with $0 \leq i \leq 2$ the path $v_0, v_{n\pm(1\times3-2i)}, v_{n\pm(2\times3-2i)}, ..., v_{n\pm((q-1)\times3-2i)}, v_{n\pm(q\times3-2i)}$ is of length $3q + 2$ and so it is $v_0 - v_{3q+2}$ geodesic cover all values of i . This cycle with the vertices $v_0, v_3, v_{2\times3}, ..., v_{\frac{n}{2}-2}$, $v_{\frac{n}{2}+3}, v_{\frac{n}{2}+6}$ ..., v_{n-1}, v_0 is smallest length. Since $v_{\frac{n}{2}-2}$, and $v_{\frac{n}{2}+3}$ are antipodal points in $C_n(\{1,3\})$, we have girth of $C_n(\{1,3\})$ is $girth(C_n(\{1,3\})) = 2q + 2$. □

In the following Lemma, we found the girth of the circulant graph $C_n(\{1,3\})$ when $n = 6q + r$ where $r = 0, 1$ and 3.

Lemma 4.3. *If $n = 6q + r$ for some positive integer $q > 1$ and $r = 0, 1$ and 3, then the girth of the circulant graph is*

$$girth(C_n(\{1,3\})) = \begin{cases} 2q & , \ if\ r = 0, \\ 2q+1 & , \ if\ r = 1 \text{ or } r = 3. \end{cases}$$

.

Proof. Suppose that $n = 6q + r$ for some positive integer q and $n \neq 6, 9$ and 11, then:

> **Case 1**: If $r = 0$, then $d(v_0, v_{3q}) = q$ then this cycle with the vertices $v_0, v_3, ..., v_{3q}$, $v_{3q+3}, v_{3q+6}, ..., v_{n-3}, v_0$ is the smallest length. Since v_0, and v_{3q} are antipodal points in $C_n(\{1, 3\})$, we have girth of $C_n(\{1, 3\})$ is $girth(C_n(\{1, 3\})) = 2q$.
> **Case 2**: Let $r = 1$, then $d(v_0, v_{3q}) = q$ then this cycle with the vertices $v_0, v_3, ..., v_{3q}$, $v_{3(q+1)}, v_{3(q+2)}, ..., v_{n-1}, v_0$ is the smallest length. We have girth of $C_n(\{1, 3\})$ is $girth(C_n(\{1, 3\})) = 2q + 1$.
> **Case 3**: If $r = 3$, then $d(v_0, v_{3q}) = q$ then this cycle with the vertices $v_0, v_3, ..., v_{3q}$, $v_{3q+3}, v_{3q+6}, ..., v_{n-3}, v_0$ is the smallest length. Since v_{3q}, and v_{3q+3} are antipodal points in $C_n(\{1, 3\})$, we have girth of $C_n(\{1, 3\})$ is $girth(C_n(\{1, 3\})) = 2q + 1$.

□

Now, we will discuss the cases for the girth of the circulant graph $C_n(\{1, 3\})$ when $n = 6q+r$, where $r = 2, 5$.

Lemma 4.4. *For the circulant graph $C_n(\{1, 3\})$, suppose that $n = 6q + r$ for some positive integer q and $r = 2, 5$, then the girth of the circulant graph of $C_n(\{1, 3\})$ is*

$$girth(C_n(\{1,3\})) = \begin{cases} 2q+2 & , \quad if\ r = 2, \\ 2q+3 & , \quad if\ r = 5. \end{cases}$$

..

Proof. Suppose that $n = 6q + r$ for some positive integer q and $r = 2, 5$, then:

- **Case 1**: If $n = 6q + 2$, then $d(v_0, v_{3q}) = q$ then this cycle with the vertices $v_0, v_3, ..., v_{3q}$, $v_{3q+3}, v_{3q+6}, ..., v_{n-2}, v_{n-1}, v_0$ is the smallest length. Then we have girth of $C_n(\{1, 3\})$ is $girth(C_n(\{1, 3\})) = 2q + 2$.

- **Case 2**: If $n = 6q + 5$, then $d(v_0, v_{3q}) = q$, $d(v_0, v_{3q+1}) = q + 1$ and $d(v_{3q+1}, v_{n-1}) = 3q$. So,this cycle with the vertices $v_0, v_3, ..., v_{3q}$, $v_{3q+3}, v_{3q+6}, ..., v_{n-2}, v_{n-1}, v_0$ is the smallest length. Then we have girth of $C_n(\{1, 3\})$ is $girth(C_n(\{1, 3\})) = 2q + 3$.

□

Finally, we agitate the case for $n = 9$ or $n = 11$.

Lemma 4.5. *If $n = 9, 11$, then $girth(C_9(\{1, 3\})) = 3$ and $girth(C_9(\{1, 3\})) = 4$.*

Proof. If $n = 9$, then consider the smallest cycle v_0, v_3, v_6, v_0. Hence the girth of the circulant graph is $girth(C_9(\{1, 3\})) = 3$.
If $n = 11$, then consider the smallest cycle v_0, v_3, v_6, v_9, v_0. Hence the girth of the circulant graph is $girth(C_{11}(\{1, 3\})) = 4$.

□

5 Conclusion

In this paper, we determined the geodetic number of circulant graphs $C_n(\{1, 3\})$ and we sum up our calculations in the following theorem.

Theorem 5.1. *If $n = 6q + r$ for some integer q and $n \neq 6$, then*

$$g(C_n(\{1,3\})) = \begin{cases} 2 & , \quad if\ n = 6q+4, \\ 3 & , \quad if\ n = 6q\ or\ n = 6q+1\ or\ n = 6q+3, \\ 4 & , \quad if\ n = 6q+2\ or\ n = 6q+5, \\ 5 & , \quad if\ n = 9\ or\ n = 11. \end{cases}$$

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Author information

Manal Al-Labadi, Department of Mathematics, Faculty of Art and Sciences, University Of Petra, Jordan.
E-mail: manal.allabadi@uop.edu.jo

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