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ON COMBINATORIAL IDENTITIES OF PELL-CHEBYSHEV TWIN NUMBER PAIRS

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Abstract The pair of Lucas and Fibonacci numbers and Pell-Lucas and Pell numbers and their numerous elegant combinatorial identities occupy a significant area in the study of combinatorial identities. With the application of Chebyshev polynomials of the first and second kind, the two pairs may be further split into two pairs of twin number pairs. In the present paper, one such twin number pair is defined, and we call them as Pell-Chebyshev twin number pairs. Many of their combinatorial identities are derived which show remarkable interconnections among them.

1 Introduction

Pascals triangle, continued fractions, partition of numbers, matrices and determinants, linear and nonlinear difference equations, convolution sums and so on, produce many beautiful combinatorial identities of several combinatorial entities like pair of sequences of numbers, pair of sequences of polynomials and so on ([10], [13], [14], [17], [18], [19]). One of the major portions of these identities is spread on the two well known pairs of numbers, namely, the pair of Lucas and Fibonacci numbers[17], [19] as well as Pell-Lucas and Pell numbers[18]. They have practical applications in computer science such as quantum key distribution[7], cryptography[15], Coding theory [4] and so on. There are identities which highlight the striking similarities among the two pairs. For the sake of clarity the standard notations are listed in Table(1):

No.	Entity	Notation and definition
1	Golden Ratio and its reciprocal	$\Phi = \frac{\sqrt{5}+1}{2}, \phi = \frac{\sqrt{5}-1}{2}$
2	Pythagorean algebraic integer and its reciprocal	$\alpha = \sqrt{2}+1, \beta = \sqrt{2}-1$
3	Fibonacci numbers	$F_n = \frac{1}{\sqrt{5}}(\Phi^n - (-\phi)^n)$ $= 0, 1, 1, 2, 3, 5, 8, \ldots$
4	Lucas numbers	$L_n = \Phi^n + (-\phi)^n$ $=2, 1, 3, 4, 7, 11, 18, \ldots$
5	Pell numbers	$q_n = \frac{1}{2\sqrt{2}}(\alpha^n - (-\beta)^n)$ $= 0, 1, 2, 5, 12, 29, \ldots$
6	Pell-Lucas numbers	$p_n = \frac{1}{2}(\alpha^n + (-\beta)^n)$ $= 1, 1, 3, 7, 17, 41, \ldots$

Table 1. Basic notations and definitions

The two pairs of numbers (L_n, F_n) and (p_n, q_n) appear to be very simple but they have profound mathematical ideas behind the variety of their combinatorial identities. The following Table 2 shows their striking similarities:

No.	(L_n, F_n)	(p_n, q_n)
1	$F_n = F_{n+1} - F_{n-1}$ $L_n = L_{n+1} - L_{n-1}$	$2q_n = q_{n+1} - q_{n-1}$ $2p_n = p_{n+1} - p_{n-1}$
2	$L_n = F_{n+1} + F_{n-1}$ $5F_n = L_{n+1} + L_{n-1}$	$2p_n = q_{n+1} + q_{n-1}$ $4q_n = p_{n+1} + p_{n-1}$
3	$\frac{F_{n-1}}{F_n} = [0; 1, 1, \ldots, 1]_n$	$\frac{q_{n-1}}{q_n} = [0; 2, 2, \ldots, 2]_n$
4	$F_{2n+1} = F_n^2 + F_{n+1}^2$ $F_{2n} = L_n\ F_n$	$q_{2n+1} = q_n^2 + q_{n+1}^2$ $q_{2n} = 2\ p_n\ q_n$
5	$(2\ F_n\ F_{n+1})^2\ +\ (F_{n-1}\ F_{n+1})^2 = F_{2n+1}^2$	$(2\ q_n\ q_{n+1})^2\ +\ (p_n\ p_{n+1})^2 = q_{2n+1}^2$ where the terms on the left hand side have two consecutive numbers.
6	$F_{n+1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-k}{k}$ -rising diagonal sum of Pascal triangle.	$q_{n+1} = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n+1}{2k+1} 2^k$ -weighted row sum of second alternate terms of the Pascal triangle.

Table 2. Similarities between (L_n, F_n) and (p_n, q_n)

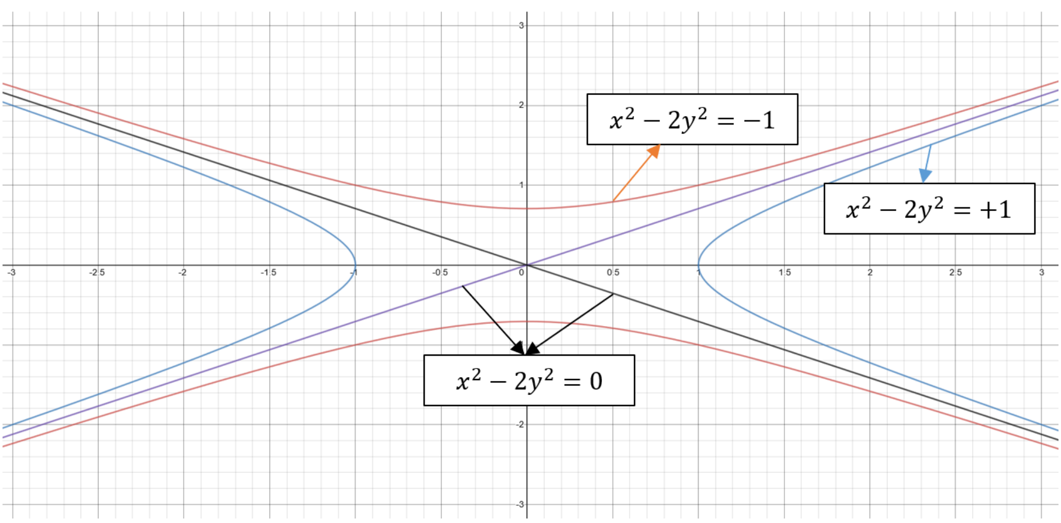

Figure 1. Graph of pair of hyperbolas

Diophantine equations[2][3] such as $ax + by = \pm 1$, $x^2 - 2y^2 = \pm 1$, $x^2 - 5y^2 = \pm 4$ and so on have great history in number theory with interesting connection to continued fractions, acceleration of convergence technique and many more. Let us consider the Pell's equation[2][3][9] $x^2 - Ny^2 = \pm 1$ when $N \geq 2$ is a square free positive number. The solutions are exactly integer coordinates on the four parts of pair of hyperbolic curves namely, $y = \pm\sqrt{\frac{x^2 \mp 1}{N}}$. Geometrically $x^2 - 2y^2 = \pm 1$ form an envelope or kernel for all those four parts of pair of hyperbolas refer fig(1). Any $M-$ data points placed on such hyperbolas naturally are inside the kernel

$x^2 - 2y^2 = \pm 1$. The kernel is separated by the pair of lines $x^2 - 2y^2 = 0$. This model has motivated many engineers who wish to design data analysis tools to work on multi variable functions with values ± 1. Given any $M-$ data in the form of $n-$ dimensional vectors, it is a challenging problem to design kernel and the separating hyper planes which is a serious mathematical discipline in machine learning[5][8]. In this context the Pell-Lucas and Pell numbers being solution of simplest Pell's equation have a practical significance.

Chebyshev polynomials of first and second kind[10][13] play a very significant role in deriving many combinatorial identities of certain sequences of numbers such as Binet forms, recurrence relations, generating functions, convolutional identities and so on. In the standard notations:

$$T_n(x) = \frac{1}{2}[x + \sqrt{x^2-1}]^n + [x - \sqrt{x^2-1}]^n, n = 0, 1, 2, 3, \ldots$$
$$\text{and} U_{n-1}(x) = \frac{1}{2\sqrt{x^2-1}}[x + \sqrt{x^2-1}]^n - [x - \sqrt{x^2-1}]^n, n = 1, 2, 3, \ldots$$

define the Chebyshev polynomials of first and second kind. They are also regarded as trigonometric polynomials because if $-1 < x < 1$,

$$T_n(cos\theta) = cosn\theta;\ U_n(cos\theta) = \frac{sin(n+1)\theta}{sin\theta}$$

As a result, they exhibit many combinatorial identities as well as orthogonality properties. In the recent literature one can find many applications of Chebyshev polynomials in the field of computer science such as cryptography[1], google page ranking[12], wavelets[11] and so on. In the present paper Chebyshev polynomials are applied to define Pell-Chebyshev twin pairs and derive their combinatorial identities.

2 Twin pairs of Pell-Chebyshev numbers

The whole idea begins with writing the golden ratio in the form

$$\Phi = \frac{\sqrt{5}}{2} + \frac{1}{2} = \left(\frac{\Phi+\phi}{2}\right) + \sqrt{\left(\frac{\Phi+\phi}{2}\right)^2 - 1}$$

and its reciprocal in the form

$$\phi = \frac{\sqrt{5}}{2} - \frac{1}{2} = \left(\frac{\Phi+\phi}{2}\right) - \sqrt{\left(\frac{\Phi+\phi}{2}\right)^2 - 1}.$$

An immediate result is

$$\begin{aligned}\Phi^n + \phi^n &= 2T_n\left(\frac{\Phi+\phi}{2}\right)\\ \Phi^{n+1} - \phi^{n+1} &= U_n\left(\frac{\Phi+\phi}{2}\right)\end{aligned}$$

(Because $2\sqrt{\left(\frac{\Phi+\phi}{2}\right)^2 - 1} = 1 = \Phi - \phi$).
As a direct consequence we can split (L_n, F_n) into the following twin pairs:

$$\begin{aligned}(\sqrt{5}F_{2n-1}, \sqrt{5}F_{2n}) &= \left(2T_{2n-1}\left(\frac{\sqrt{5}}{2}\right), U_{2n-1}\left(\frac{\sqrt{5}}{2}\right)\right)\\ (L_{2n-1}, L_{2n}) &= \left(U_{2n-2}\left(\frac{\sqrt{5}}{2}\right), 2T_{2n}\left(\frac{\sqrt{5}}{2}\right)\right)\end{aligned}$$

It is an amazing result that the constant $\alpha = \sqrt{2}+1$ and the reciprocal $\beta = \sqrt{2}-1$ also have similar expressions in terms of Chebyshev polynomials.

$$\begin{aligned}\alpha^n + \beta^n &= 2T_n\left(\frac{\alpha+\beta}{2}\right)\\ \alpha^{n+1} - \beta^{n+1} &= 2U_n\left(\frac{\alpha+\beta}{2}\right)\end{aligned}$$

These beautiful expressions have motivated us to define two pairs of sequences $\{(A_n, B_n), (C_n, D_n)\}$ given by Binet form:

Definition 2.1.

$$A_n = \alpha^{2n-1} + \beta^{2n-1} = 2T_{2n-1}(\sqrt{2}) \tag{2.1}$$

$$B_n = \frac{\alpha^{2n-1} - \beta^{2n-1}}{\alpha-\beta} = U_{2n-2}(\sqrt{2}) \tag{2.2}$$

$$C_n = \alpha^{2n} + \beta^{2n} = 2T_{2n}(\sqrt{2}) \tag{2.3}$$

$$D_n = \frac{\alpha^{2n} - \beta^{2n}}{\alpha-\beta} = U_{2n-1}(\sqrt{2}) \tag{2.4}$$

are defined as twin Pell-Chebyshev pairs.

We may write each one of the above equations (2.1) to (2.4) in the form

$$x_n = k_1(\alpha^2)^n + k_2(\beta^2)^n$$

As a result, each one of the sequences $\{A_n\}$ to $\{D_n\}$ must satisfy the following relation:

$$x_{n+1} = (\alpha^2+\beta^2)x_n - (\alpha^2\beta^2)x_{n-1}$$

$n = 1, 2, 3\ldots$, $x_0 = k_1 + k_2$ and $x_1 = k_1\alpha^2 + k_2\beta^2$.

Theorem 2.2. *For* $n = 1, 2, 3\ldots$

$$\begin{aligned}(i) \quad A_{n+1} &= 6A_n - A_{n-1}, A_0 = A_1 = 2\sqrt{2}\\ (ii) \quad B_{n+1} &= 6B_n - B_{n-1}, B_0 = -1, B_1 = 1\\ (iii) \quad C_{n+1} &= 6C_n - C_{n-1}, C_0 = 2, C_1 = 6\\ (iv) \quad D_{n+1} &= 6D_n - D_{n-1}, D_0 = 0, D_1 = 2\sqrt{2}\end{aligned}$$

The Pell-Chebyshev twin pairs can also be written as

$$\begin{aligned}(A_n, B_n) &= (2\sqrt{2}q_{2n-1}, p_{2n-1}) = \{(2\sqrt{2}, -1), (2\sqrt{2}, 1), (10\sqrt{2}, 7), \ldots\}\\ (C_n, D_n) &= (2p_{2n}, \sqrt{2}q_{2n}) = \{(2, 0), (6, 2\sqrt{2}), (34, 12\sqrt{2}), \ldots\}\end{aligned}$$

Using the basic property of (p_n, q_n), $p_n^2 - 2q_n^2 = (-1)^n$, we may rewrite similar expressions for $\{A_n\}$ to $\{D_n\}$ in the following theorem:

Theorem 2.3. *For* $n = 1, 2, 3, \ldots$

$$\begin{aligned}(i)A_n^2 &= 4[1 + B_n^2].\\ (ii)C_n^2 &= 4[1 + D_n^2].\end{aligned}$$

Let us note that for $n = 1, 2, 3, \ldots$

$$\begin{aligned}\frac{A_n}{2} &= T_{2n-1}(\sqrt{2}), \frac{C_n}{2} = T_{2n}(\sqrt{2})\\ B_{n+1} &= U_{2n}(\sqrt{2}), D_n = U_{2n-1}(\sqrt{2})\end{aligned}$$

and by applying standard identities satisfied by $T_n(x)$ and $U_n(x)$

$$\begin{aligned}T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x), T_0(x) = 1, T_1(x) = x, n = 1, 2, 3, \ldots\\ U_{n+1}(x) &= 2xU_n(x) - U_{n-1}(x), U_0(x) = 1, U_1(x) = 2x, n = 1, 2, 3, \ldots\end{aligned}$$

we prove the following theorem :

Theorem 2.4. *For* $n = 1, 2, 3, \ldots$

$$\begin{aligned}(i)\quad 2\sqrt{2}A_n &= C_n + C_{n-1}\\ (ii)\quad 2\sqrt{2}B_n &= D_n + D_{n-1}\\ (iii)\quad 2\sqrt{2}C_n &= A_n + A_{n+1}\\ (iv)\quad 2\sqrt{2}D_n &= B_n + B_{n+1}\end{aligned}$$

Let us write $p_{2n-1} = B_n, p_{2n} = \frac{C_n}{2}, q_{2n-1} = \frac{A_n}{2\sqrt{2}}, q_{2n} = \frac{D_n}{\sqrt{2}}, n = 1, 2, 3, \ldots$ and let us apply the following standard 3-term recurrence relations satisfied by (p_n, q_n): For $n = 1, 2, 3, \ldots$,

$$p_{n+1} = 2p_n + p_{n-1},\ q_{n+1} = 2q_n + q_{n-1},\ p_n = q_n + q_{n-1},\ 2q_n = p_n + p_{n-1}$$

one can prove the following theorem:

Theorem 2.5. *For* $n = 1, 2, 3, \ldots$

$$\begin{aligned}(i)\quad A_n &= A_{n-1} + 4D_{n-1}\\ (ii)\quad B_n &= B_{n-1} + C_{n-1}\\ (iii)\quad C_n &= C_{n-1} + 4B_n\\ (iv)\quad D_n &= D_{n-1} + A_n\\ (v)\quad \sqrt{2}A_n &= 2B_n + C_{n-1}\\ (vi)\quad 2\sqrt{2}B_n &= A_n + 2D_{n-1}\\ (vii)\quad \sqrt{2}C_n &= A_n + 2D_n\\ (viii)\quad 2\sqrt{2}D_n &= 2B_n + C_n\end{aligned}$$

By combining theorem 2.2, 2.4 and 2.5 one obtains the following corollary.

Corollary 2.6. For $n = 1, 2, 3, \ldots$

$$\begin{aligned}(i)\quad A_n &= \frac{A_{n+1} + A_{n-1}}{6} = \frac{B_{n+1} - B_{n-1}}{2\sqrt{2}} = \frac{C_n + C_{n-1}}{2\sqrt{2}} = D_n - D_{n-1}\\ (ii)\quad B_n &= \frac{A_{n+1} - A_{n-1}}{8\sqrt{2}} = \frac{B_{n+1} + B_{n-1}}{6} = \frac{C_n - C_{n-1}}{4} = \frac{D_n + D_{n-1}}{2\sqrt{2}}\\ (iii)\quad C_n &= \frac{A_{n+1} + A_n}{2\sqrt{2}} = B_{n+1} - B_n = \frac{C_{n+1} + C_{n-1}}{6} = \frac{D_{n+1} - D_{n-1}}{2\sqrt{2}}\\ (iv)\quad D_n &= \frac{A_{n+1} - A_n}{4} = \frac{B_{n+1} + B_n}{2\sqrt{2}} = \frac{C_{n+1} - C_{n-1}}{8\sqrt{2}} = \frac{D_{n+1} + D_{n-1}}{6}\end{aligned}$$

There is a determinant connection between $\{X_n\}(X_n = A_n$ or B_n or C_n or $D_n)$ and $\{Y_n\}(Y_n = F_{2n-1}$ or F_{2n} or L_{2n-1} or $L_{2n})$ given by

Theorem 2.7.

$$\begin{vmatrix} X_{n+1} & X_n \\ Y_{n+1} & Y_n \end{vmatrix} + \begin{vmatrix} X_{n-1} & X_n \\ Y_{n-1} & Y_n \end{vmatrix} = 3 \begin{vmatrix} X_n & 0 \\ 0 & Y_n \end{vmatrix}$$

The proof is a simple consequence of the 3-term recurrence relations

$$\begin{aligned} X_{n+1} + X_{n-1} &= 6X_n \\ Y_{n+1} + Y_{n-1} &= 3Y_n. \end{aligned}$$

If $X(t) = \sum_{n=0}^{\infty} X_n t^n$ and if $X_{n+1} = 6X_n - X_{n-1}$, then

$$X(t) = \frac{X_0 + (X_1 - 6X_0)t}{1 - 6t + t^2}.$$

As a result we obtain the generating functions of $\{A_n\}, \{B_n\}, \{C_n\}, \{D_n\}$ given in the following theorem:

Theorem 2.8.

$$\begin{aligned} (i) \quad \textstyle\sum_{n=0}^{\infty} A_n t^n &= \frac{2\sqrt{2}(1-5t)}{1-6t+t^2} \\ (ii) \quad \textstyle\sum_{n=0}^{\infty} B_n t^n &= \frac{-1+7t}{1-6t+t^2} \\ (iii) \quad \textstyle\sum_{n=0}^{\infty} C_n t^n &= \frac{2-6t}{1-6t+t^2} \\ (iv) \quad \textstyle\sum_{n=0}^{\infty} D_n t^n &= \frac{2\sqrt{2}t}{1-6t+t^2} \end{aligned}$$

3 Convolutional Identities

In the literature we can find many convolution identities of Fibonacci Numbers and Lucas Numbers [17][19]. The first three such identities are [6][16]:

$$\begin{aligned} (i) \sum_{k=0}^{n} L_k L_{n-k} &= (n+1)L_n + 2F_{n+1} \\ (ii) \sum_{k=0}^{n} L_k F_{n-k} &= \sum_{k=0}^{n} F_k L_{n-k} = (n+1)F_n \\ (iii) \sum_{k=0}^{n} F_k F_{n-k} &= \frac{1}{5}(n+1)L_n - \frac{2}{5}F_{n+1} \end{aligned}$$

In the present section many similar identities of the twin pair are worked out.

Theorem 3.1.

$$(i) \quad \sum_{k=0}^{n} A_k = D_n + D_1$$

$$(ii) \quad 4\sum_{k=0}^{n} B_k = C_n - C_1$$

$$(iii) \quad \sum_{k=0}^{n} C_k = B_{n+1} + B_1$$

$$(iv) \quad 4\sum_{k=0}^{n} D_k = A_{n+1} - A_1$$

Proof. The theorem follows by splitting the left hand side sum into two geometric series in the power of α^2 and β^2 respectively and applying the definition of D_n, C_n, B_n and A_n respectively. □

Theorem 3.2.

$$(i) \quad \sum_{k=0}^{n} \binom{n}{k} A_k = \begin{cases} (2\sqrt{2})^{2m+1} C_m & \text{if } n = 2m+1 \\ (2\sqrt{2})^{2m} A_m & \text{if } n = 2m \end{cases}$$

$$(ii) \quad \sum_{k=0}^{n} \binom{n}{k} B_k = \begin{cases} (2\sqrt{2})^{2m+1} D_m & \text{if } n = 2m+1 \\ (2\sqrt{2})^{2m} B_m & \text{if } n = 2m \end{cases}$$

$$(iii) \quad \sum_{k=0}^{n} \binom{n}{k} C_k = \begin{cases} (2\sqrt{2})^{2m-1} A_m & \text{if } n = 2m-1 \\ (2\sqrt{2})^{2m} C_m & \text{if } n = 2m \end{cases}$$

$$(iv) \quad \sum_{k=0}^{n} \binom{n}{k} D_k = \begin{cases} (2\sqrt{2})^{2m-1} B_m & \text{if } n = 2m-1 \\ (2\sqrt{2})^{2m} D_m & \text{if } n = 2m \end{cases}$$

Proof. The idea of proof for this theorem is quite similar to that of previous theorem. Here one has to make sum using binomial theorem with powers in α^2 or β^2 and one has to split each sum into even and odd number of terms which show directly what is suitable among $\{A_n, B_n, C_n, D_n\}$ to fit into the situation. □

Theorem 3.3.

$$(i) \quad \sum_{k=0}^{n} A_k\, A_{n-k} = (n+1)\, C_{n-1} + \frac{1}{\sqrt{2}} D_{n+1}$$

$$(ii) \quad 4\sum_{k=0}^{n} B_k\, B_{n-k} = (n+1)\, C_{n-1} - \frac{1}{\sqrt{2}} D_{n+1}$$

$$(iii) \quad \sum_{k=0}^{n} C_k\, C_{n-k} = (n+1)\, C_n + \frac{1}{\sqrt{2}} D_{n+1}$$

$$(iv) \quad 4\sum_{k=0}^{n} D_k\, D_{n-k} = (n+1)\, C_n - \frac{1}{\sqrt{2}} D_{n+1}$$

$$(v) \quad \sum_{k=0}^{n} A_k\, B_{n-k} = \sum_{k=0}^{n} B_k\, A_{n-k} = (n+1) D_{n-1}$$

$$(vi) \quad \sum_{k=0}^{n} C_k\, D_{n-k} = \sum_{k=0}^{n} D_k\, C_{n-k} = (n+1) D_n$$

$$(vii) \quad \sum_{k=0}^{n} A_k\, C_{n-k} = \sum_{k=0}^{n} C_k\, A_{n-k} = (n+1) A_n + D_{n+1}$$

$$
\begin{aligned}
(viii)\quad & 4\sum_{k=0}^{n} B_k\, D_{n-k} &&= 4\sum_{k=0}^{n} D_k\, B_{n-k} = (n+1)A_n - D_{n+1}\\
(ix)\quad & \sum_{k=0}^{n} A_k\, D_{n-k} &&= \sum_{k=0}^{n} D_k\, A_{n-k} = (n+1)B_n + \frac{1}{2\sqrt{2}}D_{n+1}\\
(x)\quad & \sum_{k=0}^{n} B_k\, C_{n-k} &&= \sum_{k=0}^{n} C_k\, B_{n-k} = (n+1)B_n - \frac{1}{2\sqrt{2}}D_{n+1}
\end{aligned}
$$

Proof. The left hand side of each convolution identity takes one of the following forms:

$$\sum_{k=0}^{n} C(\alpha,\beta)[\alpha^r \pm \beta^r][\alpha^s \pm \beta^s]$$

Where $C(\alpha,\beta) = 1$ or $\frac{1}{\alpha-\beta}$ or $\frac{1}{(\alpha-\beta)^2}$ and $(r,s) = (2k-1, n-2k-1)$ or $(2k-1, n-2k)$ or $(2k, n-2k-1)$ or $(2k, n-2k)$. The sum splits into three terms. Where $X_n = C_{n-1}$ or C_n or D_{n-1} or D_n or A_n or B_n. The otherwise terms involve summing a finite geometric series with powers of either $\frac{\alpha}{\beta} or \frac{\beta}{\alpha}$. A duly simplified form takes to right hand side of each identity. □

Theorem 3.4.

$$
\begin{aligned}
(i)\quad & \sum_{k=0}^{n} \binom{n}{k} A_k\, A_{n-k} = 2^n C_{n-1} + 2\, 6^n\\
(ii)\quad & 4\sum_{k=0}^{n} \binom{n}{k} B_k\, B_{n-k} = 2^n C_{n-1} - 2\, 6^n\\
(iii)\quad & \sum_{k=0}^{n} \binom{n}{k} C_k\, C_{n-k} = 2^n C_n + 2\, 6^n\\
(iv)\quad & 4\sum_{k=0}^{n} \binom{n}{k} D_k\, D_{n-k} = 2^n C_n - 2\, 6^n\\
(v)\quad & \sum_{k=0}^{n} \binom{n}{k} A_k\, B_{n-k} = \sum_{k=0}^{n} \binom{n}{k} B_k\, A_{n-k} = 2^n D_{n-1}\\
(vi)\quad & \sum_{k=0}^{n} \binom{n}{k} C_k\, D_{n-k} = \sum_{k=0}^{n} \binom{n}{k} D_k\, C_{n-k} = 2^n D_n\\
(vii)\quad & \sum_{k=0}^{n} \binom{n}{k} A_k\, C_{n-k} = \sum_{k=0}^{n} \binom{n}{k} C_k\, A_{n-k} = 2^n A_n + 2\sqrt{2}\, 6^n\\
(viii)\quad & 4\sum_{k=0}^{n} \binom{n}{k} B_k\, D_{n-k} = \sum_{k=0}^{n} \binom{n}{k} D_k\, B_{n-k} = 2^n A_n - 2\sqrt{2}\, 6^n\\
(ix)\quad & \sum_{k=0}^{n} \binom{n}{k} A_k\, D_{n-k} = \sum_{k=0}^{n} \binom{n}{k} D_k\, A_{n-k} = 2^n B_n + 6^n\\
(x)\quad & \sum_{k=0}^{n} \binom{n}{k} B_k\, C_{n-k} = \sum_{k=0}^{n} \binom{n}{k} C_k\, B_{n-k} = 2^n B_n - 6^n
\end{aligned}
$$

Proof. The proof is similar and simpler than that of previous theorem and the only difference is instead of the geometric sum we do the sum using the binomial theorem. □

It gives always a rich experience when one works convolution identities with classical Fibonacci and Lucas numbers. Finite power series with coefficients F_k and L_k, $k = 0, 1, 2, \ldots, n$ can be summed with the help of 3−term recurrence relation satisfied by them in the following forms:

$$\sum_{k=0}^{n} X_k\, t^k = \frac{X_0 + (X_1 - X_0)t - (X_n + X_{n-1})t^{n+1} - X_n t^{n+2}}{1 - t - t^2}$$

where $X_k = F_k or L_k$. Similar proof techniques adopted to this nontrivial situation and a careful simplification yields the following beautiful theorem.

Theorem 3.5.

$$(i) \qquad \sum_{k=0}^{n} F_k A_{n-k} = \sum_{k=0}^{n} A_k F_{n-k} = \frac{1}{31}[2A_{n+1} - 5A_n + 6\sqrt{2}F_{n+1} + 46\sqrt{2}F_n]$$

$$(ii) \qquad \sum_{k=0}^{n} F_k B_{n-k} = \sum_{k=0}^{n} B_k F_{n-k} = \frac{1}{31}[2B_{n+1} - 5B_n - 7F_{n+1} - 33F_n]$$

$$(iii) \qquad \sum_{k=0}^{n} F_k C_{n-k} = \sum_{k=0}^{n} C_k F_{n-k} = \frac{1}{31}[2C_{n+1} - 5C_n - 2F_{n+1} + 26F_n]$$

$$(iv) \qquad \sum_{k=0}^{n} F_k D_{n-k} = \sum_{k=0}^{n} D_k F_{n-k} = \frac{1}{31}[2D_{n+1} - 5D_n - 4\sqrt{2}F_{n+1} - 10\sqrt{2}F_n]$$

$$(v) \qquad \sum_{k=0}^{n} L_k A_{n-k} = \sum_{k=0}^{n} A_k L_{n-k} = \frac{1}{31}[12A_{n+1} + A_n + 6\sqrt{2}L_{n+1} + 46\sqrt{2}L_n]$$

$$(vi) \qquad \sum_{k=0}^{n} L_k B_{n-k} = \sum_{k=0}^{n} B_k L_{n-k} = \frac{1}{31}[12B_{n+1} + B_n - 7L_{n+1} - 33L_n]$$

$$(vii) \qquad \sum_{k=0}^{n} L_k C_{n-k} = \sum_{k=0}^{n} C_k L_{n-k} = \frac{1}{31}[12C_{n+1} + C_n - 2L_{n+1} + 26L_n]$$

$$(viii) \qquad \sum_{k=0}^{n} L_k D_{n-k} = \sum_{k=0}^{n} D_k L_{n-k} = \frac{1}{31}[12D_{n+1} + D_n - 4\sqrt{2}L_{n+1} - 10\sqrt{2}L_n]$$

The above theorem shows more scope to workout many more convolution identities involving convolution sum of more than two factors of the same or different entities.

In closing, we indicate two possible directions for future work. In the abstract we have mentioned that one twin number pair is defined, and we called them as Pell-Chebyshev twin number pairs. It is possible to consider some more such twin pairs motivated by Fibonacci and Lucas numbers. Also in theorem 2.2 we have selected particular initial conditions. One can consider other interesting initial conditions and work out the results of this paper.

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