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A NEW FIXED POINT RESULTS ON (Q, P)-CONTRACTIVE MAPPINGS WITH APPLICATIONS

Jayashree Patil, Basel Hardan*, Mohammed S. Abdo and A. Bachhav

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Abstract Depending on previous works about fixed point results in generalized metric spaces, we present a new fixed point theorem by using the generalization (Q, P)-contractive mappings fulfilling μ-admissibility under the condition of Hausdruff b-rectangular space with the invocation of C-functions. Some examples for clarification were introduced. In the end, we apply Corollary 4.3 to establish the existence of a solution for the boundary value problem of a ψ-Caputo type fractional differential equation. The obtained results improved and generalized some recent results.

1 Introduction

Fixed point theorems (FPTs) are significant tools in nonlinear functional analysis. It is notable that the Banach contraction principle (BCP) [12] is an essential result in the FPTs, which has been utilized and reached out in a wide range of bearings. BCP is the most commonly quoted FPT in the literature. It states that if

$$\delta(Tn_1, Tn_2) \leq \omega\delta(n_1, n_2), \forall n_1, n_2 \in M. \tag{1.1}$$

Then T has a unique fixed point, where M is a complete metric space(CMS), and $\omega \in [0, 1)$. The inequality (1.1) included a very primary condition which is uniformly continuous of T. It is normal to search in equation (1.1), and can it be achieved in another way so that the condition of continuity is not of the same strength, which has been studied by Kanan[22], i.e, the author proved that T has a unique fixed point such that

$$\delta(Tn_1, Tn_2) \leq \omega\delta(n_1, n_2) + \eta\delta(n_1, n_2), \tag{1.2}$$

where M is a CMS for all $n_1, n_2 \in M$ and $\omega, \eta \in \mathbb{R}$ and $(\omega + \eta < 1)$. Next, the inequalities (1.1) and (1.2) have been extended and generalized in several ways. Among those ways, we faced some new sorts of metric spaces in literature, like the one established by Branciari [14]. The author showed the idea of a rectangular metric space (RMS), by replacing the triangle inequality of a metric space with another, which is so-called the rectangular inequality. Moreover, the author added the concept of b-metric space (b-MS) as the development of metric space (MS). Also, many researchers started and examined FPT by extending the famous BCP on RMS. For more information on FPT in RMS, we refer to [10, 11, 36]. In principle, RMS can lack the Hausdorffness separation (see examples given in [32]), although it is not beneficial to our theory because the Hausdorffness separation plays a significant role in theorem 3.2 and its corollaries.

A contraction principle in b-RMS appeared by George [19]. Many definitions of various mathematical concepts and terms in (b-RMS) can be found in [4, 35]. Lately, Bari et al., in [13] and Samet et al., in [33] introduced separately some famous FPTs for (Q, P)-weakly functions contractive condition in RMS. They showed that the concept of (Q, P)-contractive mapping is interesting since it does not need the contractive conditions to hold for every pair of points in the domain different from BCP. There is a large development in the literature of transaction for fixed point problems though μ-admissible functions because it includes a type of discontinuous functions, see[8, 9, 23, 27, 28, 29, 31]. Newly, two various generalizations of μ-admissible function were presented by Ansari in [9] which applied the notion of C-type functions, and Budhia et al. in [15] which employed an RMS. In this paper, we will give some results depending on [9] and [15] on generalized RMS. Moreover, we apply our results to establish the existence of a unique solution for the boundary value problem of a ψ-Caputo type of fractional differential equation. The obtained results improve and generalize some recent results.

2 preliminaries

In this part, we establish the foundation of our prime results.

Definition 2.1. [4] Suppose M be a non-empty set and the mapping $\delta : M \times M \to [0, \infty)$ satisfies

(i) $\delta(n_1, n_2) = 0$, if and only if $n_1 = n_2$;

(ii) $\delta(n_1, n_2) = \delta(n_2, n_1)$;

(iii) there exist a real number $s \geq 1$ for all $n_1, n_2, m \in M$ such that

$$\delta(n_1, n_2) \leq s[\delta(n_1, m) + \delta(m, n_2)], (b - triangular\ inequality).$$

Then δ is called a b-metric on M and (M, δ) is called a b-MS with coefficient s.

Remark 2.2. [16] The type of b-MS is greater than the type of MS, where a b-MS is an MS when $s = 1$.

Example 2.3. [16] Let $M = \mathbb{R}$ and $\delta : M \times M \to \mathbb{R}^+$, such that $\delta(n_1, n_2) = |n_1 - n_2|^3$, then (M, δ) is a b-MS, with $s = 3$.

Definition 2.4. [14] Suppose M be a non-empty set. A function $\delta : M \times M \to [0, \infty)$ is a b-RMS on M if, for all $n_1, n_2, m_1, m_2 \in M$ with $m_1 \neq m_2$ and $m_1, m_2 \notin \{n_1, n_2\}$

(i) $\delta(n_1, n_2) = 0$, if and only if $n_1 = n_2$;

(ii) $\delta(n_1, n_2) = \delta(n_2, n_1)$;

(iii) $\delta(n_1, m_1) \leq \delta(n_1, m_2) + \delta(m_2, n_2) + \delta(n_2, m_1)$, (rectangular inequality),

Then (M, δ) is called an RMS.

For more definitions of concept related to this new category of RMS, see [14, 18].

Definition 2.5. [19, 21] Let M be a non-empty set and the mapping $\delta : M \times M \to [0, \infty)$ satisfies

(i) $\delta(n_1, n_2) = 0$, if and only if $n_1 = n_2$ for all $n_1, n_2 \in M$;

(ii) $\delta(n_1, n_2) = \delta(n_2, n_1)$ for all $n_1, n_2 \in M$;

(iii) there exists a real number $s \geq 1$ such that

$$\delta(n_1, m_1) \leq s[\delta(n_1, m_2) + \delta(m_2, n_2) + \delta(n_2, m_1)], (b - rectangularinequality).$$

For all $n_1, n_2 \in M$ and all $(m_1 \neq m_2) \notin \{n_1, n_2\}$.

Then δ is called a b-rectangular metric on M and (M, δ) is called a b-RMS with coefficient s.

Remark 2.6. [19] Every MS is a RMS and every RMS is b-RMS (with $s = 1$). However, the opposite of implying above is not valid.

Example 2.7. [19] Suppose, $M = \mathbb{N}$ and $\delta : M \times M \to M$ as

$$\delta(n_1, n_2) = \begin{cases} 0, & if \quad n_1 = n_2; \\ 4\omega, & if \quad n_1, n_2 \in \{1, 2\}, n_1 \neq n_2; \\ \omega, & if \quad n_1\ or\ n_2 \notin \{1, 2\}, n_1 \neq n_2. \end{cases}$$

Consider, $\omega \in (0, \infty)$. Hence, (M, δ) is a b-RMS ($s = 4/3$), but (M, δ) is not RMS, take

$$\delta(1, 2) = 4\omega > 3\omega = \delta(1, 3) + \delta(3, 4) + \delta(4, 2).$$

We introduce the convergent sequences, Cauchy sequence, and competence of bRMS as

Definition 2.8. [19] Suppose, (M, δ) be a b-RMS, $\{n_i\}$ be a sequence in M and $n \in M$. Then

(i) A sequence $\{n_i\}$ is said to be convergent in (M, δ) and converges to n, if for all $\epsilon > 0$ there exists $i_0 \in \mathbb{N}$ such that $\delta(n_i, n) < \epsilon$, for all $i > i_0$ and this truth is act via $\lim_{i\to\infty} \{n_i\} = n$ *or* $\{n_i\} \to n$ *as* $i \to \infty$.

(ii) A sequence $\{n_i\}$is said to be Cauchy sequence in (M, δ) if for all $\epsilon > 0$ there exists $i_0 \in \mathbb{N}$ such that $\delta(n_i, n_{i+\eta}) < \epsilon$, for all $i > i_0, \eta > 0$ or equivalently, if $\lim_{i\to\infty} \delta(n_i, n_{i+\eta}) = 0$, for all $\eta > 0$.

(iii) (M, δ) is said to be a complete b-RMS if every Cauchy sequence in M converges to some $n \in M$.

The next major lemmas are helpful in providing principle outcomes

Lemma 2.9. *[30] Suppose, (M, δ) be a b-RMS with $s \geq 1$ and let $\{n_i\}$ be a Cauchy sequence in M such that $n_i \neq n_j$ when it was $i \neq j$. Then $\{n_i\}$ be able convergence at most one point.*

Lemma 2.10. *[30] Suppose, (M, δ) be a b-RMS with $s \geq 1$,*

(i) *Suppose that the sequences $\{n_i\}, \{m_i\} \in M$ where $n_i \to n,\ m_i \to m$ as $i \to \infty$, such that $n_i \neq n,\ m_i \neq m$, for all $i \in \mathbb{N}$. Thus we have*

$$\frac{1}{s}\delta(n, m) \leq \lim_{i\to\infty} \inf \delta(n_i, m_i) \leq \lim_{i\to\infty} \sup \delta(n_i, m_i) \leq s\delta(n, m). \tag{2.1}$$

(ii) *Suppose $m \in M$ and $\{n_i\}$ is a Cauchy sequence in M where $n_i \neq n_j$, for all $i, j \in \mathbb{N}, i \neq j$. where $n_i \to n,\ n_j \to m$ as $i \to \infty,\ n \neq m$. Thus we have*

$$\frac{1}{s}\delta(n, m) \leq \lim_{i\to\infty} \inf \delta(n_i, m) \leq \lim_{i\to\infty} \sup \delta(n_i, m) \leq s\delta(n, m). \tag{2.2}$$

Definition 2.11. [33] Suppose, T be a self mapping on a metric space (M, δ) and suppose $\mu : M \times M \to [0, \infty)$ be a function. T is called a μ-admissible function if $\mu(Tn_1, Tn_2) \geq 1$ whenever $\mu(n_1, n_2) \geq 1, \forall n_1, n_2 \in M$.

Definition 2.12. [33] Suppose, T be a self mapping on a metric space (M, δ). A map T is called a (μ, Q)-contractive mapping if there exist two functions $\mu : M \times M \to [0, \infty)$ and $Q : [0, +\infty) \to [0, +\infty)$ such that

$$\mu(n_1, n_2)\delta(Tn_1, Tn_2) \leq Q(n_1, n_2) \quad \forall n_1, n_2 \in M, \tag{2.3}$$

where Q is a non-decreasing functions such that $\sum_{i=1}^{\infty} Q^i(t) < +\infty$, for all $t > 0$, and Q^i is the i^{th} iteration of Q.

For examples to μ-admissible and (μ, Q)-contractive mappings, see [24, 33].

Definition 2.13. [31] Suppose, T be a self-mapping on a metric space (M, δ) and suppose $\mu, \upsilon : M \times M \to [0, \infty)$ are two mappings. A map T is called μ-admissible with respect to υ if $\mu(Tn_1, Tn_2) \geq \upsilon(Tn_1, Tn_2)$ where $\mu(n_1, n_2) \geq \upsilon(n_1, n_2), \forall n_1, n_2 \in M$. Observe that, if $\upsilon(n_1, n_2) = 1$ for all $n_1, n_2 \in M$, thus this definition led to Definition 2.11. Likewise, if we pick $\mu(n_1, n_2) = 1$, then we say that T is a υ-sub admissible functions.

Ansari in [9] introduced the definition for the type of C-function as

Definition 2.14. A type of C-function $g : [0, \infty) \times [0, \infty) \to \mathbb{R}$ is a continuous functions such that,

(i) $g(\alpha_1, \alpha_2) \leq \alpha_1$;

(ii) $g(\alpha_1, \alpha_2) = \alpha_1 \Rightarrow \alpha_1 = 0$ *or* $\alpha_2 = 0$.

$\forall \alpha_1, \alpha_2 \in [0, \infty)$.

For examples of the type of C-function, see [9].

Definition 2.15. [25] A non-decreasing continuous map $Q : [0, \infty) \to [0, \infty)$ is called an altering distance mapping whenever $Q(\alpha) = 0 \Leftrightarrow \alpha = 0$.

Remark 2.16. Type of altering distance mapping, we denoted it by sympole $\mho$.

In the next section, we present a new fixed point theorem using the generalization (μ, Q)-contractive mappings fulfilling μ-admissibility under the condition of Hausdruff b-rectangular space with the invocation of type C- functions.

3 Main Results

In this section, we shall introduce new results of fixed point in b-RMS. Then it is applied to obtain the uniqueness of a solution for generalized fractional boundary value problem. Let us start with the following

Definition 3.1. Let (M, δ) be a b-RM,S, with $s \geq 1$ and let μ, υ as in Definition 2.13. M is said to be μ-orderly with respect to υ if for a sequence $\{n_i\}$ in M with $\mu(n_i, n_{i+1}) \geq \upsilon(n_i, n_{i+1})$ for all $i \geq N$ and $n_i \to n$ as $i \to \infty$, therefore $\mu(n_i, n) \geq \upsilon(n_i, n)$, for all $i \geq N$.

Our primary result is

Theorem 3.2. *Let (M, δ) be a complete Hausdorff b-RMS, $s \geq 1$ and $T : M \to M$ be an μ-admissible function with respect to υ. Let $g \in$ C- type functions and $Q, P \in \mho$ such that*

$$\mu(n_1, n_2) \geq \upsilon(n_1, n_2) \Rightarrow Q\big(\frac{1}{s}\delta(Tn_1, Tn_2)\big) \leq g\big[Q\big(\rho(n_1, n_2)\big), P\big(\rho(n_1, n_2)\big)\big], \tag{3.1}$$

for all $n_1, n_2 \in M$. Where,

$$\rho(n_1, n_2) = \sup \left\{ \begin{array}{c} \frac{1}{s}\delta(n_1, n_2), \frac{1}{s}\delta(n_1, Tn_1), \frac{1}{s}\delta(n_2, Tn_2), \\ \\ \frac{\delta(n_1, Tn_1)\delta(n_2, Tn_2)}{s + s\delta(n_1, n_2)}, \frac{\delta(n_1, Tn_1)\delta(n_2, Tn_2)}{s + s\delta(Tn_1, Tn_2)} \end{array} \right\}. \tag{3.2}$$

Suppose that

a $\mu(n_0, Tn_0) \geq \upsilon(n_0, Tn_0)$ *for some $n_0 \in M$;*

b $\mu(m_{i(n)-1}, m_{j(n)-1}) \geq \upsilon(m_{i(n)-1}, m_{j(n)-1})$*, for all $m_i \neq m_j$;*

c *either T is continuous or $\mu(m_{i(n)-1}, m_{j(n)-1}) \geq \upsilon(m_{i(n)-1}, m_{j(n)-1})$, for all $m_i \subset M$.*

Then there exists $n \in M$ such that $T^k n = n$, for some $k \in \mathbb{N}$ that is, n is a periodic point, but if for each periodic point n satisfying $\mu(n, Tn) \geq \upsilon(n, Tn)$, then T has a fixed point. Moreover, the fixed point is unique if
$\forall a, b \in g(T) = \{n \in M : Tn = n\}$, such that $\mu(a, b) \geq \upsilon(a, b)$.

Proof. Given $m_0 \in M$, such that

$$\mu(m_0, Tm_0) \geq \upsilon(m_0, Tm_0). \tag{3.3}$$

Consider the iteration

$$T^i m_0 = Tm_{i-1} = m_i, \tag{3.4}$$

such that $m_i \neq m_{i+1}$, for all $i \in \mathbb{N}$. So, by (3.4) and since T satisfied Definition 2.13 and by using (3.3) we have

$$\mu(n_1, n_2) = \mu(Tn_0, T^2 n_0) \geq \upsilon(Tn_0, T^2 n_0) = \upsilon(n_1, n_2)$$

By induction, we obtain

$$\mu(m_i, m_{i+1}) \geq \upsilon(m_i, m_{i+1}), \forall i \in \mathbb{N}$$

In the beginning, we shall show that $\frac{1}{s}\delta(m_i, m_{i+1}) \to 0,\ as\ i \to \infty$, i.e. $\delta(m_i, m_{i+1})$ is non-increasing.
By (3.1),we get

$$Q(\frac{1}{s}\delta(m_i, m_{i+1})) = Q(\frac{1}{s}\delta(Tm_{i-1}, Tm_i)) \leq g[Q(\rho(m_{i-1}, m_i)), P(\rho(m_{i-1}, m_i))], \tag{3.5}$$

such that,

$$\rho(m_{i-1}, m_i) = \sup \left\{ \begin{array}{c} \frac{1}{s}\delta(m_{i-1}, m_i), \frac{1}{s}\delta(m_{i-1}, Tm_{i-1}), \frac{1}{s}\delta(m_i, Tm_i), \\ \\ \frac{\delta(m_{i-1}, Tm_{i-1})\delta(m_i, Tm_i)}{s + s\delta(m_{i-1}, m_i)}, \frac{\delta(m_{i-1}, Tm_{i-1})\delta(m_i, Tm_i)}{s + s\delta(Tm_{i-1}, Tm_i)} \end{array} \right\}$$

$$= \sup \left\{ \begin{array}{c} \frac{1}{s}\delta(m_{i-1}, m_i), \frac{1}{s}\delta(m_{i-1}, m_i), \frac{1}{s}\delta(m_i, m_{i+1}), \\ \\ \frac{\delta(m_{i-1}, m_i)\delta(m_i, m_{i+1})}{s + s\delta(m_{i-1}, m_i)}, \frac{\delta(m_{i-1}, m_i)\delta(m_i, m_{i+1})}{s + s\delta(m_i, m_{i+1})} \end{array} \right\}$$

$$\leq \sup \left\{ \ \frac{1}{s}\delta(m_{i-1}, m_i), \frac{1}{s}\delta(m_{i-1}, m_i), \frac{1}{s}\delta(m_i, m_{i+1}) \ \right\},$$
$$= \sup \left\{ \ \frac{1}{s}\delta(m_{i-1}, m_i), \frac{1}{s}\delta(m_i, m_{i+1}) \ \right\}.$$

We have two cases:

Case (i) If $\rho(m_{i-1}, m_i) = \frac{1}{s}\delta(m_{i-1}, m_i)$ for some $i \in \mathbb{N}$ therefore, (3.5) it will become

$$Q\big(\frac{1}{s}\delta(m_i, m_{i+1})\big) \leq g\big[Q\big(\frac{1}{s}\delta(m_{i-1}, m_i)\big), P\big(\frac{1}{s}\delta(m_{i-1}, m_i)\big)\big],$$
$$\leq Q\big(\frac{1}{s}\delta(m_{i-1}, m_i)\big).$$

Since, Q is non-decreasing function, then

$$\delta(m_i, m_{i+1}) \leq \delta(m_{i-1}, m_i).$$

SO, $\{\delta(m_i, m_{i+1})\}$ is a non-increasing sequence which $w \geq 0$ and fulling

$$\lim_{i\to\infty} \delta(m_i, m_{i+1}) = sw$$
$$\lim_{i\to\infty} \rho(m_{i-1}, m_i) = sw.$$

Also, due to Q, P are continuous functions, we have

$$\lim_{i\to\infty} Q\big(\delta(m_i, m_{i+1})\big) \leq \lim_{i\to\infty} g\big[Q\big(\rho(m_{i-1}, m_i)\big), P\big(\rho(m_{i-1}, m_i)\big)\big],$$
$$= g\big[\lim_{i\to\infty} Q\big(\rho(m_{i-1}, m_i)\big), \lim_{i\to\infty} P\big(\rho(m_{i-1}, m_i)\big)\big],$$
$$\leq \lim_{i\to\infty} Q\big(\delta(m_{i-1}, m_i)\big).$$

Consequently,

$$Q(sw) \leq g[Q(ws), P(ws)] \leq Q(sw).$$

By Definition 2.15, we get $sw = 0$. Hence $\lim_{i\to\infty} \delta(m_i, m_{i+1}) = 0$.

Case (ii) If $\rho(m_{i-1},m_i)=\frac{1}{s}\delta(m_i,m_{i+1})$ for some $i\in\mathbb{N}$, therefore, (3.5) it will become

$$Q\big(\frac{1}{s}\delta(m_i,m_{i+1})\big)\le g\big[Q\big(\frac{1}{s}\delta(m_{i+1},m_i)\big),P\big(\frac{1}{s}\delta(m_{i+1},m_i)\big)\big]\le Q\big(\frac{1}{s}\delta(m_{i+1},m_i)\big).$$

By Definition 2.15, we get either $Q\big(\frac{1}{s}\delta(m_{i+1},m_i)\big)=0$ or $P\big(\frac{1}{s}\delta(m_{i+1},m_i)\big)=0$, implies $\delta(m_{i+1},m_i)=0$, but this is contradiction with $m_{i+1}\neq m_i$.

Based on the previous steps, we shall show that $\delta(m_i,m_{i+2})\to 0,\ as\ i\to\infty$.
By (3.1), we have

$$\begin{aligned}Q\big(\frac{1}{s}\delta(m_i,m_{i+2})\big)&=Q\big(\frac{1}{s}\delta(Tm_{i-1},Tm_{i+1})\big)\\&\le g\big[Q\big(\rho(m_{i-1},m_{i+1})\big),P\big(\rho(m_{i-1},m_{i+1})\big)\big]\\&\le Q\big(\rho(m_{i-1},m_{i+1})\big).\end{aligned}\tag{3.6}$$

Well it could be

$$\frac{1}{s}\delta(m_i,m_{i+2})\le\rho(m_{i-1},m_{i+1}).$$

Since, Q is altering distance function, which gives

$$\begin{aligned}&\frac{1}{s}\delta(m_i,m_{i+2})\le\rho(m_{i-1},m_{i+1})\\&=\sup\left\{\begin{array}{c}\frac{1}{s}\delta(m_{i-1},m_{i+1}),\frac{1}{s}\delta(m_{i-1},Tm_{i-1}),\frac{1}{s}\delta(m_{i+1},Tm_{i+1}),\\ \frac{\delta(m_{i-1},Tm_{i-1})\delta(m_{i+1},Tm_{i+1})}{s+s\delta(m_{i-1},m_{i+1})},\frac{\delta(m_{i-1},Tm_{i-1})\delta(m_{i+1},fm_{i+1})}{s+s\delta(Tm_{i-1},Tm_{i+1})}\end{array}\right\}\\&=\sup\left\{\begin{array}{c}\frac{1}{s}\delta(m_{i-1},m_{i+1}),\frac{1}{s}\delta(m_{i-1},m_i),\frac{1}{s}\delta(m_{i+1},m_{i+2}),\\ \frac{\delta(m_{i-1},m_i)\delta(m_{i+1},m_{i+2})}{s+s\delta(m_{i-1},m_{i+1})},\frac{\delta(m_{i-1},m_i)\delta(m_{i+1},m_{i+2})}{s+s\delta(m_i,m_{i+2})}\end{array}\right\}\\&\le\sup\left\{\begin{array}{c}\frac{1}{s}\delta(m_{i-1},m_{i+1}),\frac{1}{s}\delta(m_{i-1},m_i),\frac{1}{s}\delta(m_{i+1},m_{i+2}),\\ \frac{\delta(m_{i-1},m_i)\delta(m_{i+1},m_{i+2})}{s},\frac{\delta(m_{i-1},m_i)\delta(m_{i+1},m_{i+2})}{s}\end{array}\right\}\\&\le\sup\left\{\begin{array}{c}\frac{1}{s}\delta(m_{i-1},m_{i+1})+\frac{1}{s}\delta(m_{i-1},m_i)+\frac{1}{s}\delta(m_{i+1},m_{i+2}),\\ \frac{1}{s}\delta(m_{i-1},m_i)\delta(m_{i+1},m_{i+2})\end{array}\right\}\\&=\sup\left\{\ s^2\delta(m_i,m_{i+2}),\frac{1}{s}\delta(m_{i-1},m_i)\delta(m_{i+1},m_{i+2})\ \right\}.\end{aligned}$$

From case (i), we obtain $\rho(m_{i-1},m_{i+1})=s^2\delta(m_i,m_{i+2})$, when $i\to\infty$. Then

$$\lim_{i\to\infty}Q\big(s^2\delta(m_i,m_{i+2})\big)=\lim_{i\to\infty}Q\big(\rho(m_{i-1},m_{i+1})\big).$$

Therefore, by (3.6)

$$\begin{aligned}Q\big(s^2\lim_{i\to\infty}\delta(m_i,m_{i+2})\big)&\le g\big[Q\big(\lim_{i\to\infty}\rho(m_{i-1},m_{i+1})\big),P\big(\lim_{i\to\infty}\rho(m_{i-1},m_{i+1})\big)\big],\\&\le Q\big(\lim_{i\to\infty}\rho(m_{i-1},m_{i+1})\big)=\lim_{i\to\infty}Q\big(s^2\delta(m_i,m_{i+2})\big).\end{aligned}$$

Consequently, with Definition 2.15, we obtain

$$\lim_{i\to\infty} \delta(m_i, m_{i+2}) \to 0, \quad as\ i \to \infty. \tag{3.7}$$

Clearly, $m_i \in M$ does not need to be sequential in order for the convergence of b-RMS. The next lemma is useful for the rest and its proof is classical. We omit it.

Lemma 3.3. *Suppose, (M, δ) be a b-RMS with $s \geq 1$ and let $\{m_i\}$ be a sequence in M such that*

$$\lim_{i\to\infty} (m_i, m_{i+1}) = \lim_{i\to\infty} (m_i, m_{i+2}) = 0,$$

where, $m_i \neq m_j$, for all $i \neq j$. If $\{m_i\}$ is not a b-Cauchy sequence, then there exist $\epsilon > 0$ and two sub-sequences $m_{i(k)}, m_{j(k)} \subset \{m_i\}$, where $i(k) > j(k) > k,\ k \in \mathbb{N}$. Also,

$$\delta(m_{i(k)}, m_{j(k)}) \geq \epsilon, \quad \delta(m_{i(k)}, m_{j(k)-1}) \leq \epsilon.$$

Such that for the next sequences

$$\delta(m_{i(k)}, m_{j(k)}), \delta(m_{i(k)-1}, m_{j(k)}), \delta(m_{i(k)}, m_{j(k)-1}), \delta(m_{i(k)+1}, m_{j(k)+1}),$$

it satisfies

$$\begin{aligned}
\epsilon &\leq \lim_{i\to\infty} \inf \delta(m_{i(k)}, m_{j(k)}) \leq \lim_{i\to\infty} \sup \delta(m_{i(k)}, m_{j(k)}) \leq s\epsilon \\
\epsilon &\leq \lim_{i\to\infty} \inf \delta(m_{i(k)-1}, m_{j(k)}) \leq \lim_{i\to\infty} \sup \delta(m_{i(k)-1}, m_{j(k)}) \leq s^3\epsilon \\
\epsilon &\leq \lim_{i\to\infty} \inf \delta(m_{i(k)}, m_{j(k)}) \leq \lim_{i\to\infty} \sup \delta(m_{i(k)}, m_{j(k)}) \leq s^3\epsilon \\
\epsilon &\leq \lim_{i\to\infty} \inf \delta(m_{i(k)-1}, m_{j(k)-1}) \leq \lim_{i\to\infty} \sup \delta(m_{i(k)-1}, m_{j(k)-1}) \leq s^5\epsilon.
\end{aligned}$$

Now, we shall prove that T has periodic points,i.e., $T^k n = n$ for some $n \in M$ and some $k \in \mathbb{N}$.
To this end, we assume a sequence $\{m_i\} \in M$ is not a Cauchy sequence and let $\{m_{i(n)}\}, \{m_{j(n)}\} \subset \{m_i\}$ such that $i(n) > j(n) > n, \quad n \in \mathbb{N}$. We have

$$\lim_{n\to\infty} \delta(m_{i(n)}, m_{j(n)}) = \lim_{n\to\infty} \delta(Tm_{i(n)-1}, Tm_{j(n)-1}).$$

Thus

$$\rho(m_{i(n)-1}, m_{j(n)-1}) = \sup \left\{ \begin{array}{l} \frac{1}{s}\delta(m_{i(n)-1}, m_{j(n)-1}), \frac{1}{s}\delta(m_{i(n)-1}, Tm_{i(n)-1}), \\ \\ \frac{1}{s}\delta(m_{j(n)-1}, Tm_{j(n)-1}), \frac{\delta(m_{i(n)-1}, Tm_{i(n)-1})\delta(m_{j(n)-1}, Tm_{j(n)-1})}{s+s\delta(m_{i(n)-1}, m_{j(n)-1})}, \\ \\ \frac{\delta(m_{i(n)-1}, Tm_{i(n)-1})\delta(m_{j(n)-1}, Tm_{j(n)-1})}{s+s\delta(Tm_{i(n)-1}T, m_{j(n)-1})} \end{array} \right\}$$

$$\rho(m_{i(n)-1}, m_{j(n)-1}) = \sup \left\{ \begin{array}{l} \frac{1}{s}\delta(m_{i(n)-1}, m_{j(n)-1}), \frac{1}{s}\delta(m_{i(n)-1}, m_{i(n)}), \\ \\ \frac{1}{s}\delta(m_{j(n)-1}, m_{j(n)}), \frac{\delta(m_{i(n)-1}, m_{i(n)})\delta(m_{j(n)-1}, m_{j(n)})}{s+s\delta(m_{j(n)-1}, m_{i(n)-1})}, \\ \\ \frac{\delta(m_{i(n)-1}, m_{i(n)})\delta(m_{j(n)-1}, m_{j(n)})}{s+s\delta(m_{i(n)}, m_{j(n)})} \end{array} \right\}$$

Then, from Lemma 3.3, there is $\epsilon > 0$ such that

$$\lim_{n\to\infty}(\rho(m_{i(n)-1}, m_{j(n)-1}) = s^4\epsilon.$$

From condition ***b*** we have

$$\mu(m_{i(n)-1}, m_{j(n)-1}) \geq \upsilon(m_{i(n)-1}, m_{j(n)-1}).$$

By (3.1), we get

$$Q\big(\frac{1}{s}\delta(m_{i(n)-1}, m_{j(n)-1})\big) \leq g\big[Q\big(\rho(m_{i(n)-1}, m_{j(n)-1})\big), P\big(\rho(m_{i(n)-1}, m_{j(n)-1})\big)\big].$$

Since g, Q and P are continuous functions, accordingly as $n, i \to \infty$

$$Q(s^4\epsilon) \leq g[Q(s^4\epsilon), P(s^4\epsilon)] \leq Q(s^4\epsilon).$$

So, $Q(s^4\epsilon) = 0$ *or* $P(s^4\epsilon) = 0$, thus $\epsilon = 0$ which is a contradiction. We conclude that $\{m_i\}$ is a b-Cauchy sequence. Since M is complete then a b-Cauchy sequence $\{m_i\}$ is converged to $n \in M$.
In case that T is continuous and by (3.4), we get

$$\lim_{i\to\infty} Tm_i = \lim_{i\to\infty} m_{i+1} \to Tn.$$

Hence, $Tn = n$ (M is Hausdorff) so, T has a periodic points.
In the other hand, we assume that M is μ-orderly with respect to υ. From condition ***c***, we get

$$\mu(m_i, n) \geq \upsilon(m_i, n), \quad \forall i \in \mathbb{N},$$

which implies,

$$Q\big(\frac{1}{s}\delta(Tm_i, Tn)\big) \leq g\big[Q\big(\rho(m_i, n)\big), P\big(\rho(m_i, n)\big)\big], \tag{3.8}$$

then,

$$\rho(m_i, n) = \sup\left\{\begin{array}{c} \frac{1}{s}\delta(m_i, n), \frac{1}{s}\delta(m_i, Tm_i), \frac{1}{s}\delta(n, Tn), \\ \\ \frac{\delta(m_i, Tm_i)\delta(n, Tn)}{s+s\delta(m_i, n)}, \frac{\delta(m_i, Tm_i)\delta(n, Tn)}{s+s\delta(Tm_i, Tn)} \end{array}\right\}$$

$$= \sup\left\{\begin{array}{c} \frac{1}{s}\delta(m_i, n), \frac{1}{s}\delta(m_i, m_{i+1}), \frac{1}{s}\delta(n, Tn), \\ \\ \frac{\delta(m_i, m_{i+1})\delta(n, Tn)}{s+s\delta(m_i, n)}, \frac{\delta(m_i, m_{i+1})\delta(n, Tn)}{s+s\delta(m_{i+1}, Tn)} \end{array}\right\}.$$

Since, $\{m_i\} \to n \; as \; i \to \infty$ then

$$\rho(m_i, n) = \frac{1}{s}\delta(n, Tn). \tag{3.9}$$

Replacing (3.9) into (3.8), we obtain

$$Q\frac{1}{s}\delta(n, Tn) \leq g[Q(\frac{1}{s}\delta(n, Tn)), P(\frac{1}{s}\delta(n, Tn))].$$

Subsequently, we get $Q(\frac{1}{s}\delta(n, Tn) = 0$ *or* $P(\frac{1}{s}\delta(n, Tn)) = 0$, thus $\delta(n, Tn)) = 0$. Hence, T has periodic points. Now, we must prove that T has a fixed point from a periodic points. So, let T has a fixed point, say w. Thus $T^k w = w, \quad k \in \mathbb{N}, w$ is a fixed point of T where $k = 1$. Now we will show that $T^{k-1}w = z$ is a fixed point of T such that $k > 1$. Assume $T^{k-1}n \neq T^k n$, for all $k > 1$. Furthermore, $\mu(w, Tw) \geq \upsilon(w, Tw)$ for a periodic point n. Therefore, from (3.1) we get

$$Q\big(\frac{1}{s}\delta(T^{k-1}w, T^k w)\big) \leq g[Q\big(\rho(T^{k-2}w, T^{k-1}w)\big), P\big(\rho(T^{k-2}w, T^{k-1}w)\big)], \tag{3.10}$$

where,

$$\rho(T^{k-2}w, T^{k-1}w) = \sup \left\{ \begin{array}{c} \frac{1}{s}\delta(T^{k-2}w, T^{k-1}w), \frac{1}{s}\delta(T^{k-2}w, T^{k-1}w), \\ \frac{1}{s}\delta(T^{k-1}w, T^k w), \frac{\delta(T^{k-2}w, T^{k-1}w)\delta(T^{k-1}w, T^k w)}{s + s\delta(T^{k-2}w, T^{k-1}w)}, \\ \frac{\delta(T^{k-2}w, T^{k-1}w)\delta(T^{k-1}w, T^k w)}{s + s\delta(T^{k-1}w, T^k w)} \end{array} \right\}$$

$$= \sup \left\{ \begin{array}{c} \frac{1}{s}\delta(T^{k-2}w, T^{k-1}w), \frac{1}{s}\delta(T^{k-2}w, T^{k-1}w), \\ \frac{1}{s}\delta(T^{k-1}w, T^k w), \frac{\delta(T^{k-2}w, T^{k-1}w)\delta(T^{k-1}w, T^k w)}{s + s\delta(T^{k-2}w, T^{k-1}w)}, \\ \frac{\delta(T^{k-2}w, T^{k-1}w)\delta(T^{k-1}w, T^k w)}{s + s\delta(T^{k-1}w, T^k w)} \end{array} \right\}$$

$$= \left\{ \frac{1}{s}\delta(T^{k-2}w, T^{k-1}w), \frac{1}{s}\delta(T^{k-1}w, T^k w) \right\}.$$

Again we have two cases:

Case (a) $\rho(T^{k-2}w, T^{k-1}w) = \frac{1}{s}\delta(T^{k-2}w, T^{k-1}w), \forall k > 1$, then with return to $\mu(w, Tw) \geq \upsilon(w, Tw)$ we attain

$$\begin{aligned} Q\big(\frac{1}{s}\delta(T^{k-1}w, T^k w)\big) &\leq g\big[Q\big(\frac{1}{s}(\delta(T^{k-2}w, T^{k-1}w))\big), P\big(\frac{1}{s}(\delta(T^{k-2}w, T^{k-1}w))\big)\big] \\ &\leq Q\big(\frac{1}{s}\delta(T^{k-2}w, T^{k-1}w)\big). \end{aligned}$$

Thus, $\{(T^{k-1}w, T^k w)\}$ a non-increasing sequence on $[0, \infty)$. Follows that

$$\begin{aligned} Q\big(\frac{1}{s}\delta(w, fw)\big) =& Q\big(\frac{1}{s}\delta(T^k w, T^{k+1}w)\big) \\ \leq& Q\big(\frac{1}{s}\delta(T^{k-1}w, T^k w)\big) \\ \leq& g\big[Q\big(\frac{1}{s}(\delta(T^{k-2}w, T^{k-1}w))\big), P\big(\frac{1}{s}(\delta(T^{k-2}w, T^{k-1}w))\big)\big] \\ \leq& Q\frac{1}{s}(\delta(T^{k-2}w, T^{k-1}w)) \\ \leq& g\big[Q\big(\frac{1}{s}(\delta(T^{k-3}w, T^{k-2}w))\big), P\big(\frac{1}{s}(\delta(T^{k-3}w, T^{k-2}w))\big)\big] \\ \leq& Q\big(\frac{1}{s}(\delta(T^{k-3}w, T^{k-2}w))\big) \\ &\vdots \\ \leq& g\big[Q\big(\frac{1}{s}(\delta(w, Tw))\big), P\big(\frac{1}{s}(\delta(w, Tw))\big)\big] \\ \leq& Q\big(\frac{1}{s}\delta(w, Tw)\big). \end{aligned}$$

As a result, $Q\big(\frac{1}{s}\delta(T^{k-2}w, T^{k-1}w)\big) = 0$ or $P\big(\frac{1}{s}\delta(T^{k-2}w, T^{k-1}w)\big) = 0$, thus $\delta(T^{k-2}w, T^{k-1}w)) = 0$ also, $T^{k-2}w = T^{k-1}w$, which is contradiction, due to $T^{k-1}w \neq T^k w$. This conforms that there is no other fixed point $z = T^{k-1}w$.

Case(b) $\rho(T^{k-2}w, T^{k-1}w) = \frac{1}{s}\delta(T^{k-1}w, T^k w)$, for some $k > 1$, then from (3.10), we get

$$Q\big(\frac{1}{s}\delta(T^{k-1}w, T^k w)\big) \leq g\big[Q\big(\frac{1}{s}(\delta(T^{k-1}w, T^k w))\big), P\big(\frac{1}{s}(\delta(T^{k-1}w, T^k w))\big)\big]$$
$$\leq Q\big(\frac{1}{s}\delta(T^{k-1}w, T^k w)\big).$$

Therefore, $Q(\frac{1}{s}(\delta(T^{k-1}w, T^k w) = 0$ or $P(\frac{1}{s}\delta(T^{k-1}w, T^k w)) = 0$, which is contradiction.

Finally, we need to prove that the fixed point is a unique. Suppose that, $w_1, w_2 \in M$ are two separate fixed points of T. By using $\mu(w_1, w_2) \geq \upsilon(w_1, w_2)$ and inequality (3.1), we get

$$Q\big(\frac{1}{s}\delta(w_1, w_2)\big) = Q\big(\frac{1}{s}\delta(Tw_1, Tw_2)\big) \leq g\big[Q\big(\rho(w_1, w_2)\big), P\big(\rho(w_1, w_2)\big)\big], \tag{3.11}$$

where,

$$\rho(w_1, w_2) = \sup\left\{\begin{array}{c} \frac{1}{s}\delta(w_1, w_2), \frac{1}{s}\delta(w_1, Tw_1), \frac{1}{s}\delta(w_2, Tw_2), \\ \frac{\delta(w_1, Tw_1)\delta(w_2, Tw_2)}{s + s\delta(w_1, w_2)}, \frac{\delta(w_1, Tw_1)\delta(w_2, Tw_2)}{s + s\delta(Tw_1, Tw_2)} \end{array}\right\}$$
$$= \frac{1}{s}\delta(w_1, w_2).$$

By (3.11),

$$Q\big(\frac{1}{s}\delta(w_1, w_2)\big) = Q\big(\frac{1}{s}\delta(w_1, w_2)\big) \leq g\big[Q\big(\frac{1}{s}\delta(w_1, w_2)\big), P\big(\frac{1}{s}\delta(w_1, w_2)\big)\big],$$
$$\leq Q\big(\frac{1}{s}\delta(w_1, w_2)\big).$$

Hence, $Q(\frac{1}{s}\delta(w_1, w_2)) = 0$ or $P(\frac{1}{s}\delta(w_1, w_2)) = 0$, which implies that $w_1 = w_2$. This proves the uniqueness of a fixed point of T on M. □

4 Examples

We give the following examples

Example 4.1. Suppose $M = [0, 1], T : M \to M$ such that

$$Tn = \begin{cases} n + \frac{1}{3}, & n \in [0, \frac{1}{3}), \\ \frac{1}{3}, & n \in [\frac{1}{3}, 1]. \end{cases}$$

Consider, $\mu, \upsilon : M \times M \to [0, \infty)$ since,$\mu(n, m) = 3, \upsilon(n, m) = 2, \forall n, m \in M$. And assume $\delta : M \times M \to [0, 1)$ be a b-rectangular metric space, as

$$\delta(n, m) = \begin{cases} 6/5, & n, m \in [0, \frac{1}{3}], \\ 6/20, & n, m \in [\frac{1}{3}, 0], \\ 2/3, & n \in [0, \frac{1}{3}], m \in [\frac{1}{3}, 1]. \end{cases}$$

It very well may be the following

(i) $\delta(Tn, Tm) = 6/20$ and $\rho(n, m) = 6/5$, if $n, m \in [0, \frac{1}{3}]$;

(ii) $\delta(Tn, Tm) = 0$ and $\rho(n, m) = 6/20$, if $n, m \in [\frac{1}{3}, 1]$;

(iii) $\delta(Tn, Tm) = 6/20$ and $\rho(n, m) = 1/3$, if $n \in [0, \frac{1}{3}], m \in [\frac{1}{3}, 1]$.

Examine, $g : [0,\infty) \times [0,\infty) \to \mathbb{R}$ and $Q, P : [0,\infty) \to [0,\infty)$ defined as $g(r,\alpha) = r - \alpha, Q(\alpha) = 4\alpha/5$ and $P(\alpha) = \alpha/3$. Then Theorem 3.2 has been fulfilled, (with $s = 2$). Hence, $w = \frac{1}{3}$ be a unique fixed point of T.

In the following example, we will explain that the b-rectangular metric space condition must be Hausdorff as an indispensable prerequisite, without which the Theorem 3.2 results cannot be achieved

Example 4.2. Assume that $M_1 = \{0, 2\}, M_2 = \{\frac{1}{i}, i \in \mathbb{N}\},\ M = M_1 \cup M_2$. Consider $\delta : M \times M \to [0,\infty)$ as

$$\delta(n,m) = \begin{cases} n, & n \in M_2, m \in M_1, \\ m, & n \in M_1, m \in M_2, \\ 1, & n \neq m, \\ 0, & n = m. \end{cases}$$

M is complete b-rectangular metric. And define $\mu, \upsilon : M \times M \to [0,\infty)$ such

$$\mu(n,m) = \begin{cases} 2, & n = 0, m = \frac{1}{i}, \\ 4, & n \neq m, \text{ or } m \neq \frac{1}{i}, \end{cases} \qquad \upsilon(n,m) = 3 \quad n, m \in M \times M.$$

Consider, $T : M \to M$ as

$$T(0) = \frac{1}{2}, \quad T(2) = 0 \quad and \quad T(\frac{1}{i}) = 0, \quad \forall \frac{1}{i} \in M_2.$$

For our enjoyment, use the following token

$$C_1 = \frac{1}{s}\delta(n,m), \quad C_2 = \frac{1}{s}\delta(n,Tn), \quad C_3 = \frac{1}{s}\delta(m,Tm)$$
$$C_4 = \frac{\delta(n,Tn)\delta(m,Tm)}{s + s\delta(n,m)}, \quad and \quad C_5 = \frac{\delta(n,Tn)\delta(m,Tm)}{s + s\delta(Tn,Tm)}.$$

We have,

	$n = 0, m = 2,$ $Tn = \frac{1}{2}, Tm = 0$	$n = 0, m = \frac{1}{k},$ $Tn = \frac{1}{2}, Tm = 0$	$n = 2, m = \frac{1}{k}$ $Tn = 0, Tm = 0$	$n = \frac{1}{k}, m = \frac{1}{p},$ $Tn = 0, Tm = 0$
C_1	3	3m	3m	3
C_2	$\frac{3}{2}$	$\frac{3}{2}$	3	3n
C_3	3	3m	3m	m
C_4	$\frac{3}{4}$	$\frac{3}{2(k+1)}$	$\frac{3}{(k+1)}$	$\frac{3}{2}$ nm
C_5	1	m	3m	3nm
$\delta(Tn,Tm)$	$\frac{3}{2}$	$\frac{3}{2}$	0	0
$\rho(n,m)$	1	1, k=1. $\frac{1}{2}$, $p \geq 2$	1	1

For all $k, p \in \mathbb{N}$ and $k \neq p$, (with $s = 3$). Now consider $g : [0,\infty) \times [0,\infty) \to \mathbb{R},$ $also$ $Q, P : [0,\infty) \to [0,\infty)$ as $g(\alpha, r) = 5\alpha/6, Q(r) = 2r/3$ and $P(r) = r/4$. Through the previous table, we can show the validity of following

$$Q(\frac{1}{s}\delta(Tn,Tm)) \leq g[Q(\rho(n,m), P(\rho(n,m)))],$$

when, $\mu(n,m) \geq \upsilon(n,m)$, although $g(T) = \{w \in M : Tw = w\} = \phi$. Hence, there is no a fixed point of T. This is because b-RMS is not a Husdorrf space. Since, M_1, M_2 have the same closure, that is there does not exist any reduces $a_1, a_2 > 0$ such that $B_{a_1}(0) \cap B_{a_2}(2) = \phi$, where B is open ball.

Corollary 4.3. *Suppose, $(M, , \delta)$ be a Hausdorff and complete (b-RMS). Let $T : M \to M$ be a μ-admissible mapping with respect to υ. Assume there exist $P \in \mho$ such that for $n_1, n_2 \in M$.*

$$\mu(n_1, n_2) \geq \upsilon(n_1, n_2) \Rightarrow \delta(Tn_1, Tn_2) \leq \rho(n_1, n_2) - P((\rho(n_1, n_2)))$$

where,

$$\rho(n_1, n_2) = \sup \left\{ \begin{array}{c} \frac{1}{s}\delta(n_1, n_2), \frac{1}{s}\delta(n_1, Tn_1), \frac{1}{s}\delta(n_2, Tn_2), \\ \\ \frac{\delta(n_1,Tn_1)\delta(n_2,Tn_2)}{s+s\delta(n_1,n_2)}, \frac{\delta(n_1,Tn_1)\delta(n_2,Tn_2)}{s+s\delta(Tn_1,Tn_2)} \end{array} \right\}.$$

Assume also that the following conditions hold:

a *there exists $n_0 \in M$ such that $\mu(n_0, Tn_0) \geq \upsilon(n_0, Tn_0)$;*

b $\forall n_1, n_2, n_3 \in M, \mu(n_1, n_2) \geq \upsilon(n_1, n_2)$ *and* $\mu(n_2, n_3) \geq \upsilon(n_2, n_3) \Rightarrow \mu(n_1, n_3) \geq \upsilon(n_1, n_3)$*;*

c *either M is μ-orderly with respect υ or T is continuous.*

Then there exist $n \in M$ such that $T^k n = n$,that is, n is periodic point, but if for periodic point n satisfying $\mu(n, Tn) \geq \upsilon(n, Tn)$ we can decide that T has a fixed point. The fixed point is unique if $\forall n_1, n_2 \in g(T) = \{n \in M : Tn = n\}$, such that $\mu(n_1, n_2) \geq \upsilon(n_1, n_2)$.

Corollary 4.4. *Suppose, $(M, , \delta)$ be a Hausdorff and complete (b-RMS). Let $T : M \to M$ be a μ-admissible mapping with respect to υ. Such that for $n_1, n_2 \in M$.*

$$\mu(n_1, n_2) \geq \upsilon(n_1, n_2) \Rightarrow \delta(Tn_1, Tn_2) \leq \omega(\rho(n_1, n_2)), \quad 0 < \omega < 1.$$

Where,$\rho(n_1, n_2)$ is the same as in Corollary 4.3. Assume also that the following conditions hold:

a *there exists $n_0 \in M$ such that $\mu(n_0, Tn_0) \geq \upsilon(n_0, Tn_0)$;*

b $\forall n_1, n_2, n_3 \in M, \mu(n_1, n_2) \geq \upsilon(n_1, n_2)$ *and* $\mu(n_2, n_3) \geq \upsilon(n_2, n_3) \Rightarrow \mu(n_1, n_3) \geq \upsilon(n_1, n_3)$*;*

c *either M is μ-orderly with respect υ or T is continuous.*

Then there exist $n \in M$ such that $T^k n = n$, that is, n is periodic point, but if for periodic point n satisfying $\mu(n, Tn) \geq \upsilon(n, Tn)$ we can decide that T has a fixed point. The fixed point is unique if $\forall n_1, n_2 \in g(T) = \{n \in M : Tn = n\}$, such that $\mu(n_1, n_2) \geq \upsilon(n_1, n_2)$.

Remark 4.5. In Theorem 3.2, if we replace $s = 1$, then our results reduces to Theorem 1 in [15].

5 Application

Fractional calculus (FC) has as of late been of extraordinary intrigue as a result of both the concentrated improvement of the hypothesis of FC itself and the uses of such developments in different applied sciences, engineering, and so on. For subtleties, see the monographs of Kilbas [26], Diethelm [17], Samko et al. [34], and the series of papers [1, 2, 3, 5, 6, 7, 20, 37, 38, 39]. On a basic level, one may reduce a fractional differential equation (FDE) to a fractional integral equation (FIE) and apply to it the fundamental strategy of fixed point theory. In this part, we apply the fixed point result inferred in Corollary 4.3 to ensure the existence of a unique solution of a boundary value problem for a ψ-Caputo-type FDE.

In the remainder of this paper, we will use the following notations: $\Delta = [0, 1]$; $\mathbb{R}^+ = [0, \infty)$; $C(\Delta)$ be the space of all continuous functions on Δ with the supremum (uniform) norm; ψ be an increasing function, having a continuous derivative ψ' on Δ with $\psi'(\varrho) \neq 0$ for all $\varrho \in \Delta$; and , $\Gamma(\cdot)$ is the Gamma function.

Consider the ψ-Caputo type of fractional boundary value problem ψ-Caputo type of FBVP for short.

$$\begin{cases} {}^C D_{0^+}^{\vartheta,\psi}\sigma(\varkappa) + g(\varkappa, \sigma(\varkappa)) = 0, \quad \varkappa \in (0, 1), \\ \sigma(0) = \sigma(1) = 0, \end{cases} \tag{5.1}$$

where $1 < \vartheta \leq 2$, $g : \Delta \times \mathbb{R} \to \mathbb{R}^+$ is a continuous function and ${}^C D_{0+}^{\vartheta,\psi}$ is ψ-Caputo fractional derivative of order ϑ introduced by Almeida [6], that is

$$ {}^C D_{a^+}^{\vartheta;\psi} \sigma(\varrho) = I_{a^+}^{n-\vartheta,\psi} \ D_{\varrho,\psi}^n \sigma(\varrho), $$

where $n = [\vartheta] + 1$, $D_{\varrho,\psi}^n = \left(\frac{1}{\psi'(\varrho)} \frac{d}{d\varrho} \right)^n$, $\sigma : \Delta \to \mathbb{R}$ is an integrable function, and

$$ I_{a^+}^{n-\vartheta,\psi} \sigma(\varrho) = \frac{1}{\Gamma(n-\vartheta)} \int_a^{\varrho} \psi'(\varsigma) \left[\psi(\varrho) - \psi(\varsigma) \right]^{n-\vartheta-1} \sigma(\varsigma) d\varsigma, \ \varrho > a, $$

is called ψ -Riemann-Liouville fractional integral of order $n - \vartheta$ introduced by Kilbas [26]. Note that when $\psi(\varrho) = \varrho$, we obtain the known classical Riemann-Liouville fractional integral

$$ I_{a^+}^{n-\vartheta} \sigma(\varrho) = \frac{1}{\Gamma(n-\vartheta)} \int_a^{\varrho} (\varrho - \varsigma)^{n-\vartheta-1} \sigma(\varsigma) d\varsigma, \ \varrho > a. $$

Lemma 5.1. *[37] Let $g \in C(\Delta)$ and $1 < \vartheta \leq 2$. Then the ψ-Caputo type FBVP*

$$ \begin{cases} {}^C D_{a+}^{\vartheta,\psi} \sigma(\varkappa) + g(\varkappa, \sigma(\varkappa)) = 0, \ \varkappa \in (0,1), \\ \sigma(0) = 0, \ \sigma(1) = 0, \end{cases} $$

is equivalent to

$$ \sigma(\varkappa) = \int_0^1 \mathcal{G}(\varkappa, \varsigma) \psi'(\varsigma) g(\varsigma, \sigma(\varsigma)) d\varsigma, $$

where

$$ \mathcal{G}(\varkappa, \varsigma) = \frac{(\psi(\varkappa) - \psi(0))^{\vartheta-1}}{(\psi(1) - \psi(0))^{\vartheta-1} \Gamma(\vartheta)} \begin{cases} (\psi(1) - \psi(\varsigma))^{\vartheta-1} \\ -\frac{(\psi(1)-\psi(0))^{\vartheta-1}}{(\psi(\varkappa)-\psi(0))^{\vartheta-1}} (\psi(\varkappa) - \psi(\varsigma))^{\vartheta-1}, \\ 0 \leq \varsigma \leq \varkappa \leq 1, \\ \\ (\psi(1) - \psi(\varsigma))^{\vartheta-1}, \ 0 \leq \varkappa \leq \varsigma \leq 1. \end{cases} $$

Let $M = C(\Delta)$ and $\delta : M \times M \to \mathbb{R}$ be given by

$$ \delta(\sigma, w) = \parallel \sigma - w) \parallel_\infty = \sup_{\varkappa \in \Delta} (\sigma(\varkappa) - w(\varkappa)). $$

Then, (M, δ) is a RMS.

Theorem 5.2. *Let $g : \Delta \times \mathbb{R} \to \mathbb{R}$ and $F : \Delta \times \Delta \to \mathbb{R}$ be given functions. Suppose that*

(i) *there exists $\kappa > 0$ such that*

$$ |g(\varkappa, \sigma(\varkappa)) - g(\varkappa, w(\varkappa))| \leq \kappa \left| \sigma - w \right|, \ \ \varkappa \in \Delta, \ \sigma, w \in \mathbb{R}; $$

(ii) *the following inequality holds: $\frac{(\psi(1)-\psi(0))^{\vartheta}}{\Gamma(\vartheta+1)} \kappa := \lambda < 1$.*

(iii) *For $\varkappa \in \Delta$ and $\sigma, w \in C(\Delta)$, $F(\sigma(\varkappa), w(\varkappa)) \geq 0$ implies*

$$ F\left(T\sigma(\varkappa), Tw(\varkappa) \right) \geq 0; $$

(iv) *there exists $\sigma_0 \in C(\Delta)$ and $F(\sigma_0(\varkappa), T\sigma_0(\varkappa)) \geq 0$ for all $\varkappa \in \Delta$ where the operator $T : C(\Delta) \to C(\Delta)$ is defined by*

$$ T\sigma(\varkappa) = \int_0^1 \mathcal{G}(\varkappa, \varsigma) \psi'(\varsigma) g(\varsigma, \sigma(\varsigma)) d\varsigma. \tag{5.2} $$

(v) *If $\{\sigma_n\}$ is a sequence in $C(\Delta)$ with $\sigma_n \to \sigma$ and $F(\sigma_n, \sigma_{n+1}) \geq 0 \ \forall n \in$, then $F(\sigma_n, \sigma) \geq 0$, $\forall n \in$.*
Then the problem (5.1) has unique solution.

Proof. By Lemma 5.1, $\sigma \in C(\Delta)$ is a solution of (5.1) if and only if σ is a solution of the integral equation $\sigma(\varkappa) = \int_0^1 \mathcal{G}(\varkappa,\varsigma)\psi'(\varsigma)g(\varsigma,\sigma(\varsigma))d\varsigma$, $\varkappa \in \Delta$. Define the operator T defined by (5.2). We find a fixed point of T. Now, let $\sigma, w \in C(\Delta)$ be such that $F(\sigma(\varkappa), w(\varkappa)) \geq 0$. On one hand we have

$$|T\sigma(\varkappa) - Tw(\varkappa)| = \left| \int_0^1 \mathcal{G}(\varkappa,\varsigma)\psi'(\varsigma)g(\varsigma,\sigma(\varsigma))d\varsigma - \int_0^1 \mathcal{G}(\varkappa,\varsigma)\psi'(\varsigma)g(\varsigma,w(\varsigma))d\varsigma \right|$$

$$\leq \int_0^1 \mathcal{G}(\varkappa,\varsigma)\psi'(\varsigma)|g(\varsigma,\sigma(\varsigma)) - g(\varsigma,w(\varsigma))|d\varsigma.$$

By Lemma 5.1, for $0 < \varkappa < \varsigma < 1$ we have

$$\begin{aligned}
\int_0^1 \mathcal{G}(\varkappa,\varsigma)\psi'(\varsigma)d\varsigma &= \frac{(\psi(\varkappa)-\psi(0))^{\vartheta-1}}{(\psi(1)-\psi(0))^{\vartheta-1}\Gamma(\vartheta)} \int_0^1 (\psi(1)-\psi(\varsigma))^{\vartheta-1}\psi'(\varsigma)d\varsigma \\
&= \frac{(\psi(\varkappa)-\psi(0))^{\vartheta-1}}{(\psi(1)-\psi(0))^{\vartheta-1}\Gamma(\vartheta)} \left[\frac{-(\psi(1)-\psi(\varsigma))^{\vartheta}}{\vartheta}\right]_0^1 \\
&\leq \frac{(\psi(1)-\psi(0))^{\vartheta-1}}{(\psi(1)-\psi(0))^{\vartheta-1}\Gamma(\vartheta+1)} (\psi(1)-\psi(0))^{\vartheta} \\
&= \frac{(\psi(1)-\psi(0))^{\vartheta}}{\Gamma(\vartheta+1)}.
\end{aligned}$$

For $0 < \varsigma < \varkappa < 1$, same estimates can be proved in analogous way to the previous one. So we will omit it.

Using (i) and (ii), we get

$$\begin{aligned}
|T\sigma(\varkappa) - Tw(\varkappa)| &\leq \int_0^1 \mathcal{G}(\varkappa,\varsigma)\psi'(\varsigma)|g(\varsigma,\sigma(\varsigma)) - g(\varsigma,w(\varsigma))|d\varsigma \\
&\leq \int_0^1 \mathcal{G}(\varkappa,\varsigma)\psi'(\varsigma)\kappa|\sigma(\varsigma) - w(\varsigma)|d\varsigma \\
&\leq \frac{(\psi(1)-\psi(0))^{\vartheta}}{\Gamma(\vartheta+1)}\kappa \parallel \sigma - w \parallel_\infty .
\end{aligned}$$

That is,

$$\parallel T\sigma - Tw \parallel_\infty \leq \lambda \parallel \sigma - w \parallel_\infty,$$

i.e.

$$\delta(T\sigma, Tw) \leq \lambda\delta(\sigma, w) \leq \lambda\rho(\sigma, w)$$

where

$$\rho(\sigma, w) = \sup\left\{ \tfrac{1}{s}\delta(\sigma, w), \tfrac{1}{s}\delta(\sigma, T\sigma), \tfrac{1}{s}\delta(w, Tw), \tfrac{\delta(\sigma,T\sigma)\delta(w,Tw)}{s+s\delta(\sigma,w)}, \tfrac{\delta(\sigma,T\sigma)\delta(w,Tw)}{s+s\delta(T\sigma,Tw)} \right\}.$$

Put, $\alpha : C(\Delta) \times C(\Delta) \to [0, +\infty)$ and $\upsilon : C(\Delta) \times C(\Delta) \to [0, +\infty)$ by

$$\alpha(\sigma, w) = \begin{cases} 1 & F(\sigma(\vartheta), w(\vartheta)) \geq 0, \quad \vartheta \in \Delta, \\ 0 & else. \end{cases}$$

and $\upsilon(\sigma, w) = 1$ for all $\sigma, w \in C(\Delta)$. Now by condition (iii), we get

$$\begin{aligned}
\alpha(\sigma, w) &\geq \upsilon(\sigma, w) \\
&\Rightarrow F(\sigma(\varsigma), w(\varsigma)) \geq 0 \\
&\Rightarrow F(T\sigma(\varsigma), Tw(\varsigma)) \geq 0,
\end{aligned}$$

which implies

$$\alpha(T\sigma, Tw) \geq \upsilon(T\sigma, Tw).$$

Thus, T is α-admissible mapping. From (iv) there is $\sigma_0 \in C(\Delta)$ such that $\alpha(\sigma_0, T\sigma_0) \geq \upsilon(\sigma_0, T\sigma_0)$. Moreover, from (v), the condition (iii) of Corollary 4.3 holds. Thus, all conditions of Corollary 4.3 are satisfied. Hence, T has a unique fixed point. □

6 Conclusion

Depending on previous works about fixed point results in generalized metric spaces, we have presented a new fixed point theorem by using the generalization (Q, P)- contractive mappings fulfilling μ-admissibility under the condition of Hausdruff b-rectangular space with the invocation of C-functions. Some illustrative examples were introduced. In the end, we have applied Corollary 4.3 to establish the existence of solution for the boundary value problem of a generalized fractional differential equation involving ψ-Caputo operator. The acquired results have extended and generalized some recent results in the literature.

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Author information

Jayashree Patil, Department of Mathematics, Vasantrao Naik Mahavidyalaya, Cidco, Aurangabad 431003, India.
E-mail: `jv.patil29@gmail.com`

Basel Hardan*, Department of Mathematics, Dr. Babasaheb Ambedkar Marathwada University, Aurangabad 431004, India.
E-mail: `bassil2003@gmail.com`

Mohammed S. Abdo, Department of Mathematics, Hodeidah University, Al-Hodeidah, Yemen.
E-mail: `msabdo1977@gmail.com`

A. Bachhav, Navin Jindal School of Management, University of Texas at Dallas, Dallas, 75080, U.S.A.
E-mail: amol.bachhav@utdallas.edu

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