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On completeness in $QTAG$-modules

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Abstract The purpose of this paper is essentially to study some elementary concepts of completeness in $QTAG$-modules. We introduce the notion of completeness, which we term semi-complete modules and obtain some interesting results. Certain basic properties of h-pure-complete modules are investigated with the help of h-pure submodules and socles. Also, we define totally quasi-complete modules and study the inter relations between various type of completeness.

1 Introduction and background

The theory of abelian groups is one of the principal motives of new research in module theory. Some results on abelian groups hold good for modules if there are certain restrictions on rings and some hold if the modules satisfy certain conditions. In 1976 Singh [13] started the study of TAG-modules satisfying the following two conditions while the rings were associative with unity.

(I) Every finitely generated submodule of any homomorphic image of M is a direct sum of uniserial modules.

(II) Given any two uniserial submodules U and V of a homomorphic image of M, for any submodule W of U, any non-zero homomorphism $f : W \to V$ can be extended to a homomorphism $g : U \to V$, provided the composition length $d(U/W) \leq d(V/f(W))$.

In 1987 Singh [14] made an improvement and studied the modules satisfying only the condition (I) and called them $QTAG$-modules. Through a number of papers it has been seen that the structure theory of these modules is similar to that of torsion abelian groups and that these modules occur over any ring. Here the rings are almost restriction free and the $QTAG$-modules satisfy a simple condition. Several authors investigated the various notions and structures of $QTAG$-modules. They derived some interesting properties and they characterizes these modules as well. Yet there is much to explore.

All rings examined in the current paper contain unity $(1 \neq 0)$ and modules are unital $QTAG$-modules. A uniserial module M is a module over a ring R, whose submodules are totally ordered by inclusion. This means simply that for any two submodules N_1 and N_2 of M, either $N_1 \subseteq N_2$ or $N_2 \subseteq N_1$. A module M is called a serial module if it is a direct sum of uniserial modules. An element $x \in M$ is uniform, if xR is a non-zero uniform (hence uniserial) module and for any R-module M with a unique decomposition series, $d(M)$ denotes its decomposition length. For a uniform element $x \in M$, $e(x) = d(xR)$ and $H_M(x) = \sup\left\{ d\left(\frac{yR}{xR}\right) \mid y \in M,\ x \in yR \text{ and } y \text{ uniform}\right\}$ are the exponent and height of x in M, respectively. $H_k(M)$ denotes the submodule of M generated by the elements of height at least k and $H^k(M)$ is the submodule of M generated by the elements of exponents at most k. Let us denote by M^1, the submodule of M, containing elements of infinite height. The module M is h-divisible if $M = M^1 = \bigcap_{k=0}^{\infty} H_k(M)$. The module M is called separable if $M^1 = 0$. The module M is said to be bounded, if there exists an integer k such that $H_M(x) \leq k$ for every uniform element $x \in M$. A submodule N of M is h-pure in M if $N \cap H_k(M) = H_k(N)$, for every integer $k \geq 0$. A submodule $B \subseteq M$ is a basic submodule of M, if B is h-pure in M, $B = \oplus B_i$, where each B_i is the direct sum of uniserial modules of length i and M/B is h-divisible.

The sum of all simple submodules of M is called the socle of M, denoted by $Soc(M)$ and a submodule S of $Soc(M)$ is called a subsocle of M. A subsocle S of M is called open if $Soc(H_k(M)) \subseteq S$ for some non-negative integer k. The cardinality of the minimal generating set of M is denoted by $g(M)$. For all ordinals α, $f_M(\alpha)$ is the α^{th}-Ulm invariant of M (see [9]) and it is equal to $g\big(Soc(H_\alpha(M))/Soc(H_{\alpha+1}(M))\big)$.

Imitating [10], the submodules $H_k(M), k \geq 0$ form a neighborhood system of zero, thus a topology known as h-topology arises. Closed modules are also closed with respect to this topology. Thus, the closure of $N \subseteq M$ is defined as $\overline{N} = \bigcap_{k=0}^{\infty} (N + H_k(M))$. Therefore the submodule $N \subseteq M$ is closed with respect to h-topology if $\overline{N} = N$ and h-dense in M if $\overline{N} = M$.

In this paper, we focus on some elementary concepts of completeness in $QTAG$-modules. The object of our study includes, the closure of h-pure submodules of $QTAG$-modules. It is well known that a class of $QTAG$-modules in which the closure of every h-pure submodule is again an h-pure submodule. In [1], Ahmad et al. called such modules as quasi-complete modules and investigated some characterizations of these modules. The concept of h-pure-complete modules, the modules in which every subsocle supports an h-pure submodule, was introduced by Khan [7]. This concept is closely related to the concept of quasi-complete modules. A number of results on h-pure-complete modules and its relation with quasi-complete modules have been given in [1]. Further, Khan et al. [8] introduced the concept of ℓ-quasi-complete modules, a generalization of quasi-complete modules, and developed the study of some other concepts for these modules. Here we continue the similar study of completeness in $QTAG$-modules by generalizing some of the results of [2, 5, 6, 12]. It is interesting to note that almost all the results which hold for TAG-modules are also valid for $QTAG$-modules [11]. In what follows, all notations and terminology are standard and will be in agreement with those used in [3, 4].

2 Semi-completeness

We begin by defining the following.

Definition 2.1. A $QTAG$-module M is called semi-complete if it is the direct sum of a closed module and a direct sum of uniserial modules.

Now we need to prove some elementary but helpful lemmas.

Lemma 2.2. *Let N be an h-pure submodule of the $QTAG$-module M such that M/N is h-divisible. Then M and N have the same Ulm invariants.*

Proof. If K is a submodule of M, the injection $N \to M$ induces a map

$$\psi : (K \cap H_\alpha(N))/(K \cap H_{\alpha+1}(N)) \to (K \cap H_\alpha(M))/(K \cap H_{\alpha+1}(M))$$

for every finite ordinal α. Since N is h-pure, $(K \cap H_{\alpha+1}(M)) \cap (K \cap H_\alpha(N)) = (K \cap H_{\alpha+1}(N))$ and so ψ is injective. This shows that $f_N(\alpha) \leq f_M(\alpha)$.

On the other hand, if x is any uniform element of $K \cap H_\alpha(M)$, by the h-divisibility of M/N there exists an element $y \in H_\alpha(N)$ such that $z = x - y \in H_{\alpha+1}(M)$. Then $z' = -y' \in H_{\alpha+2}(M) \cap N = H_{\alpha+2}(N)$ where $d\left(\frac{zR}{z'R}\right) = d\left(\frac{yR}{y'R}\right) = 1$. Let $t \in N$ such that $t' = -y'$ where $d\left(\frac{yR}{y'R}\right) = 1$ and $d\left(\frac{tR}{t'R}\right) = \alpha+2$. Then $y+t' \in K \cap H_\alpha(N)$ and $z-t' \in K \cap H_{\alpha+1}(M)$ where $d\left(\frac{tR}{t'R}\right) = \alpha + 1$. This shows that the coset $x + K \cap H_\alpha(M)$ is the image of the coset $y + t' + K \cap H_\alpha(N)$ where $d\left(\frac{tR}{t'R}\right) = \alpha + 1$ and so ψ is surjective. Thus $f_N(\alpha) = f_M(\alpha)$. □

Lemma 2.3. *Let $\{x_i\}$ be a sequence of elements in the QTAG-module M such that $e(\{x_i\}) = 1$. If $H_M(x_i) = k_i$ such that $x_i = y_i$ where $d\left(\frac{y_iR}{x_iR}\right) = k_i$, and $k_i \neq k_j$, for $i \neq j$. Then the submodule N generated by y_i's is h-pure and is the direct sum of uniserial modules generated by y_i's.*

Proof. Assuming the contrary N is not the direct sum of uniserial modules generated by y_i's, therefore a relation

$$p_1x_1 + \ldots + p_ix_i = 0,$$

with not every p_i a multiple of prime. But this requires at least two of x_i's to have the same height, which is a contradiction. This substantiates our claim.

If $x \in N$ such that $e(x) = 1$, therefore x is a linear combination of x_i's by the above relation. Then its height in M, and in N, is the smallest of the heights of x_i's which appear nontrivially in the linear combination. Hence N is h-pure. □

Lemma 2.4. *Let M and M' be QTAG-modules without elements of infinite height and ϕ be a homomorphism of M into M' such that for any submodule N of M and for all k, $\phi(N \cap H_k(M)) \neq 0$. Then there exists an h-pure submodule L of M such that the restriction of ϕ to the closure $\overline{L}$ of L is an isomorphism.*

Proof. For all k we have, $\phi(N \cap H_k(M)) \neq 0$, we extract a sequence $\{x_i\}$ of elements of N such that $y_i = \phi(x_i) \neq 0$ and $H_M(x_{i+1}) > H_{M'}(y_i)$ for every i. Let $H_M(x_i) = k_i$ and $H_M(y_i) = m_i$, we have $k_1 \leq m_1 < k_2 \leq m_2 < \ldots$. Suppose $x_i \in M$ such that $x_i = a_i$ where $d\left(\frac{a_iR}{x_iR}\right) = k_i$ and $y_i \in M'$ such that $y_i = b_i$ where $d\left(\frac{b_iR}{y_iR}\right) = m_i$. Let L and T be the submodules of M and M' generated by a_i and b_i, respectively. By Lemma 2.3, L and T are h-pure in M and M' and are the direct sums of uniserial modules generated by a_i and b_i. Then ϕ induces an isomorphism between $N \cap L$ and $K \cap T$, and thus the restriction of ϕ to L is an isomorphism.

Notice that $H_{m_i}(T) \cap K = H_{k_i}(\phi(L)) \cap K \in M'$, and both are generated by the elements y_j for $j \geq i$. Now let $z \in \overline{L}$ such that $e(z) = 1$ and $\phi(z) = 0$. Suppose $z_i \to z$ where $z_i \in L$ and $z - z_i \in H_{m_i}(M)$ for every i. Since L is h-pure, $e(z_i) = 1$ for every i. Thus $-\phi(z_i) = \phi(z - z_i) \in H_{m_i}(M') \cap T \cap K = H_{k_i}(\phi(L)) \cap K \in M'$. Since ϕ is an isomorphism on L, $z_i \in H_{k_i}(L)$. This proves that $z = 0$. □

We are now ready to prove the following theorem.

Theorem 2.5. *Let M be a semi-complete QTAG-module with a decomposition $M = M_1 \oplus M_2$, and let K be an h-pure closed submodule of M. Then there exists an integer t such that $N \cap H_t(K) \subset M_1$, for every submodule N of M.*

Proof. If $N \cap H_t(K) \not\subseteq M_1$ for all t, by Lemma 2.4 there exists an h-pure submodule L in K such that the projection $\phi : M \to M_2$ is an isomorphism when restricted to the closure $\overline{L}$ of L. Now $\overline{L}$ is h-pure, and so the h-topology on $\overline{L}$ is the same as that induced on K. This means that $\overline{L}$ is closed. But $\overline{L}$ is isomorphic to a submodule of M_2 and every submodule of M_2 is a direct sum of uniserial modules, as desired. The proof is finished. □

And so, we proceed to establish a relationship between two decompositions of a semi-complete module.

Theorem 2.6. *Let M be a semi-complete QTAG-module with the decompositions $M = M_1 \oplus M_2 = M_3 \oplus M_4$. Then there exists an integer t such that $H_t(M_1) \cong H_t(M_3)$ and $H_t(M_2) \cong H_t(M_4)$*

Proof. Applying Theorem 2.5, we deduce that for some non-negative t, we have $N \cap H_t(M_1) = N \cap H_t(M_3)$, for any submodule N of M. Hence $N \cap H_{t+i}(M_1) = N \cap H_{t+i}(M_3)$ for every non-negative i. This implies that the Ulm invariants of the closed modules $H_t(M_1)$ and $H_t(M_3)$ are equal, so that $H_t(M_1) \cong H_t(M_3)$.

Next, to show $H_t(M_2) \cong H_t(M_4))$, we need only to show that

$$N \cap H_{t+i}(M_2)/N \cap H_{t+i+1}(M_2) \cong N \cap H_{t+i}(M_4)/N \cap H_{t+i+1}(M_4).$$

It follows that $H_t(M_2)$ and $H_t(M_4)$, as direct sums of uniserial modules with the same Ulm invariants, are isomorphic. Moreover, from $M = M_1 \oplus M_2$ we have $N \cap H_{t+i}(M) = N \cap H_{t+i}(M_1) \oplus N \cap H_{t+i}(M_2)$. Therefore, we define the map $\eta : N \cap H_{t+i}(M) \to N \cap H_{t+i}(M_2)/N \cap H_{t+i+1}(M_2)$ such that $\ker(\eta) = N \cap H_{t+i}(M_1) + N \cap H_{t+i+1}(M)$. Thus

$$(N \cap H_{t+i}(M))/(N \cap H_{t+i}(M_1) + N \cap H_{t+i+1}(M)) \cong (N \cap H_{t+i}(M_2))/(N \cap H_{t+i+1}(M_2)).$$

Repeating the argument for $M = M_3 \oplus M_4$, we obtain a similar isomorphism with M_1 and M_2 replaced by M_3 and M_4. Since $N \cap H_{t+i}(M_1) = N \cap H_{t+i}(M_3)$, the desired isomorphism is evident, ensures our claimed. □

Analysis. Let $M_2 = \Sigma P_i$ and $M_4 = \Sigma Q_i$ where every P_i and Q_i is a direct sum of uniserial modules of exponent i. Also, let M_1 and M_3 be the closed submodules of ΠR_i and ΠS_i where every R_i and S_i is also a direct sum of uniserial modules of exponent i. For every i, $R_i \oplus P_i$ and $S_i \oplus Q_i$ are isomorphic such that each is the direct sum of $f_M(i-1)$ copies of the uniserial modules of exponent i. Furthermore, if t is as in Theorem 2.6, $R_i \cong S_i$ and $P_i \cong Q_i$ for $i \geq t$. This leads to the following result.

Corollary 2.7. *Let M be a semi-complete $QTAG$-module with the decompositions $M = M_1 \oplus M_2 = M_3 \oplus M_4$, then these two decompositions of M possess isomorphic refinements.*

Towards the end of this section, we have the following result which is of particular interest.

Theorem 2.8. *Let M and M' be semi-complete $QTAG$-modules with the same Ulm invariants having decompositions $M = M_1 \oplus M_2$ and $M' = M_3 \oplus M_4$. Then M and M' are isomorphic if and only if there is an integer t such that for every $i \geq t$, $f_{M_1}(i) = f_{M_3}(i)$ and $f_{M_2}(i) = f_{M_4}(i)$.*

3 h-pure-completeness

For facilitating the exposition and for the convenience of the readers, we recall the following definition.

Definition 3.1. A $QTAG$-module M is called h-pure-complete if for every subsocle S of M there exists an h-pure submodule N of M such that $S = Soc(N)$.

To develop the study of h-pure-complete module, we prove the following working lemma.

Lemma 3.2. *Let M and M' be $QTAG$-modules such that M' is bounded and S be a subsocle of $M + M'$. If N is an h-pure submodule of M supported by $S \cap M$, then S supports an h-pure submodule of $M + M'$ which contains N.*

Proof. Clearly, $(S + N)/N$ is an open subsocle of $(M + M')/N$ and hence supports an h-pure submodule K/N of $(M + M')/N$, for any submodule K of M. Then $Soc(K) = S$ and K is an h-pure in $M + M'$, since N and K/N are h-pure submodules of $M + M'$ and $(M + M')/N$, respectively. □

As a consequence, we have the following.

Theorem 3.3. *Let M be an h-pure-complete $QTAG$-module. If M' is a direct sum of uniserial modules, then $M + M'$ is h-pure-complete.*

Proof. Since M' is a direct sum of uniserial modules, $M + M'$ is the union of a monotone sequence of h-pure submodules $M + M'_k$ such that M'_k is bounded. Let S be a subsocle of $M + M'$ and set $S_k = S \cap (M + M'_k)$. Using Lemma 3.2 and the fact that M is h-pure-complete, we construct a monotone sequence of h-pure submodules $N_k \subseteq M + M'_k$ such that $Soc(N_k) = S_k$. Then $N = \bigcup_{k=1}^{\infty} N_k$ is an h-pure submodule of $M + M'$ supported by S. □

Theorem 3.4. *Let M be an h-pure-complete $QTAG$-module. If M' is a closed module with finite Ulm invariants, then $M + M'$ is h-pure-complete.*

Proof. Suppose M is without elements of infinite height, i.e. $H_\omega(M) = 0$. Now let S be a subsocle of $M + M'$ and let P be the projection of S into M. Since M is h-pure-complete, there is an h-pure submodule N of M supported by P. Then S is contained in the h-pure submodule $N + M'$. Let $\overline{C}$ be the closure of $N + M'$ such that $\overline{C} = \overline{N} + M'$, where $\overline{N}$ is the closure of N in $\overline{C}$. Since M' is a closed module with finite Ulm invariants, its socle is a direct summand of P. But $(\overline{S} + Soc(M')) \supset \overline{P} = Soc(\overline{N})$. Therefore, $Soc(\overline{C}) = \overline{P} + Soc(M')$, as desired. □

As immediate consequence, we yield the following corollary.

Corollary 3.5. *Let M be an h-pure-complete $QTAG$-module. If M' is a semi-complete module with finite Ulm invariants, then $M + M'$ is h-pure-complete.*

Recall that a $QTAG$-module M is said to be quasi-complete if the closure $\overline{N}$ of every h-pure submodule N of M, is h-pure in M.

Theorem 3.6. *Let M_k be a quasi-complete $QTAG$-module such that M_k is an h-pure in M_{k+1}, for each positive integer k, then $M = \bigcup_{k=1}^{\infty} M_k$ is h-pure-complete.*

Proof. Let S be a subsocle of M and set $S_k = S \cap M_k$. Now if each S_k is open, then $S = \bigcup_{k=1}^{\infty} S_k$ supports a direct sum of uniserial modules which is h-pure in M. Thus, by what we have just shown above, in view of the Theorem 3.3, we are done. □

The following example demonstrates that the direct sum of two h-pure-complete modules need not be h-pure-complete.

Example. Suppose that C is an unbounded closed module with a countably generated basic submodule B. Then C contains a proper, h-dense h-pure submodule M such that M is a quasi-complete module and $g(M) = c$. Therefore, C contains an h-pure submodule M' such that $Soc(M') = Soc(M)$ and $M' \not\cong M$. Since M' is also quasi-complete and therefore both M and M' are h-pure complete. However, the subsocle S of $M + M'$ consisting all elements of the form (x, x) such that $x \in Soc(M) = Soc(M')$ does not support an h-pure submodule of $M + M'$. Indeed, if N is an h-pure submodule of $M + M'$ such that $Soc(N) = S$, then it is easily seen that N is a sub-direct sum of M and M' with zero kernels, contrary to the fact that M and M' are not isomorphic.

4 Totally quasi-completeness

Following [8], a submodule N of a $QTAG$-module M is called imbedded if there exists a function $\ell : Z^+ \to Z^+$ such that $N \cap H_{\ell(k)}(M) \subseteq H_k(N)$ for each $k \in Z^+$. Here, ℓ is called an imbedding function for N in M. Let ℓ be an imbedding function for N in M then N is called ℓ-imbedded submodule of M. A $QTAG$-module M is called ℓ-quasi-complete if the closure $\overline{N}$ of every ℓ-imbedded submodule N of M, is an imbedded submodule of M.

Let $\mathcal{F}(\ell)$ be the family of ℓ-quasi-complete modules for arbitrary ℓ. We note that $\mathcal{F}(\ell_1) \subset \mathcal{F}(\ell_2)$ if and only if $\ell_2 \leq \ell_1$. In particular $\mathcal{F}(I)$ contains every ℓ-quasi-complete module for every ℓ. Let $\mathcal{C}$ denote the family of closed modules. It is well known that $\mathcal{C} \subset \mathcal{F}(I)$. The following theorem is the rather interesting result that $\mathcal{C} \subset \mathcal{F}(\ell)$ for every ℓ.

Theorem 4.1. *Let M be a closed $QTAG$-module. Then M is ℓ-quasi-complete every ℓ.*

Proof. Let N be ℓ-imbedded in M. We show that $\overline{N} \cap H_{\ell(k)}(M) \subset H_k(\overline{N})$ for each k. Let x be any uniform element in $\overline{N} \cap H_{\ell(k)}(M)$. Then $x \in Soc^m(\overline{N})$ for some m. By [8, Lemma 1.11], $Soc^m(\overline{N}) \subset \overline{Soc^m(N)}$. Thus x is the limit of a Cauchy sequence in $Soc^m(N)$. Let $\{x_n\} \subset Soc^m(N)$ be subsequence of that sequence satisfying

(i) $x_1 \in H_{\ell(k)}(M)$

(ii) $x_{n+1} - x_n \in H_{\ell(k+n)}(M)$ for every n.

Since N is ℓ-imbedded, $x_1 \in H_k(N)$ and $x_{n+1} - x_n \in H_{(k+n)}(N)$. Let $y_1 \in N$ such that $d\left(\frac{y_1R}{x_1R}\right) = k$ and $\{z_n\} \subset N$ such that $d\left(\frac{z_nR}{(x_{n+1}-x_n)R}\right) = k+n$. Then $\{y_n\}$ is a bounded Cauchy sequence in N such that $\{y_n\} \subset Soc^{k+m}(N)$, so there exists $y \in \overline{N}$ such that $\lim y_n = y$, since M is closed. Now $\lim x_n = x$ and, since $d\left(\frac{y_nR}{x_nR}\right) = k$, therefore $\lim x_n = y'$ where $d\left(\frac{yR}{y'R}\right) = k$. But limits are unique since $M^1 = 0$. Thus $x = y' \in H_k(\overline{N})$ where $d\left(\frac{yR}{y'R}\right) = k$, as required. Hence $\overline{N}$ is ℓ-imbedded and M is ℓ-quasi-complete. □

Let $\mathcal{F} = \cap_\ell \mathcal{F}(\ell)$. Then every module in $\mathcal{F}$ has the property that the closure of an ℓ-imbedded submodule is again ℓ-imbedded for every ℓ. This motivates us to make the following definition:

Definition 4.2. A $QTAG$-module M is called totally quasi-complete if it is ℓ-quasi-complete for every ℓ.

Remark 4.3. It is easy to see that every closed module is totally quasi-complete.

We establish the relation among quasi-complete, ℓ-quasi-complete and totally quasi-complete modules.

Theorem 4.4. *Let M be a separable $QTAG$-module. Then the following are equivalent:*

(i) *M is quasi-complete;*

(ii) *M is ℓ-quasi-complete;*

(ii) *M is totally quasi-complete.*

Proof. The implications $(iii) \Rightarrow (ii)$ and $(ii) \Rightarrow (i)$ are obvious, because the identity function is an imbedding function for h-pure submodules.

$(i) \Rightarrow (iii)$. Let M be quasi-complete. If $N \subset M$ such that N is bounded, then $\overline{N} \subset M$. Now, assume that N is unbounded and ℓ-imbedded in M, where $\ell(1) = n$. Let $x \in M$ and $x' \in \overline{N}$ such that $d\left(\frac{xR}{x'R}\right) = n$. Let B be a basic submodule of M and $\overline{B}$ is closure of B in M, $\overline{N}/N = (\overline{B}/N)^1$. Since $\overline{B}$ is totally quasi-complete, $\overline{N} \subset \overline{B}$, so $\overline{N}/N = (\overline{B}/N)^1$ is h-divisible by [8, Lemma 1.10]. Thus there exist $y \in N, z \in \overline{N}$ such that $z' = x' + y$ where $d\left(\frac{xR}{x'R}\right) = d\left(\frac{zR}{z'R}\right) = n$. Now, consider $a \in N$ such that $a' = b'$ where $d\left(\frac{aR}{a'R}\right) = 1$, $d\left(\frac{bR}{b'R}\right) = n$ and $b = z - x$. Therefore, $z' - (x' + a) \in Soc(\overline{B}) = Soc(M) + Soc(\overline{N})$, where $d\left(\frac{xR}{x'R}\right) = d\left(\frac{zR}{z'R}\right) = n-1$. Hence, there exist $u \in Soc(M), v \in Soc(\overline{N})$ such that $z' - v = x' + u + a \in \overline{N} \cap M$ where $d\left(\frac{xR}{x'R}\right) = d\left(\frac{zR}{z'R}\right) = n-1$. Now $H_1((x'+u+a)R) = x' + a'$ where $d\left(\frac{xR}{x'R}\right) = n$ and $d\left(\frac{aR}{a'R}\right) = 1$, therefore $x' + u + a$ where $d\left(\frac{xR}{x'R}\right) = n-1$, is a solution of $w \equiv x' (\text{mod } N)$ where $d\left(\frac{xR}{x'R}\right) = n$. Hence, any element of $(M/N)^1 = (\overline{N} \cap M)/N$ is h-divisible in $(\overline{N} \cap M)/N$, so $(\overline{N} \cap M)/N = H_1((\overline{N} \cap M)/N)$. That is, $(\overline{N} \cap M)/N$ is h-divisible and hence h-pure in M/N. Thus by [8, Lemma 1.5], ℓ is an imbedding function for $\overline{N} \cap M$ in N. The proof is over. □

5 Open problems

In closing, we pose the following questions of interest:

Problem 1. If M and M' are semi-complete, is $M \oplus M'$ as well?

Problem 2. Does every h-dense subsocle of a closed module support an h-pure submodule which is h-pure complete?

Problem 3. Suppose M is a $QTAG$-module with an imbedded submodule N which belongs to the family $\mathcal{F}$ of quasi-complete modules. If $(M/N)^1$ is h-divisible, then whether or not M belongs to $\mathcal{F}$.

Investigate with a priority when $\mathcal{F}$ coincide with the family of semi-complete modules and h-pure-complete modules, respectively.

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