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PAIRWISE CONNECTEDNESS IN BI-ISOTONIC SPACES

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Abstract In this paper, we introduce the pairwise separated sets in bi-isotonic spaces and study some of their properties. Based on this we define the pairwise connectedness (disconnectedness) and also total disconnectedness in bi-isotonic spaces. In addition to describing their properties, we also give some characterizations of these notions. We provide some examples and also counterexamples to express their differences with the corresponding notions in the biclosure or bitopological spaces.

1 Introduction

There are some equivalent ways to construct a topological structure on a non-empty set with different starting points and selected axioms. For instance, Kuratowski closure axioms allow us to define closed sets $F \subset X$ by use of $\mathrm{cl}(F) = F$ provided that the closure operator $\mathrm{cl} : P(X) \to P(X)$ is satisfying the properties

K0) $\mathrm{cl}(\emptyset) = \emptyset$ (grounded)

K1) $A \subseteq B \Rightarrow \mathrm{cl}(A) \subseteq \mathrm{cl}(B)$ (Isotony)

K2) $A \subseteq \mathrm{cl}(A)$ (Expansive)

K3) $\mathrm{cl}(A \cup B) = \mathrm{cl}(A) \cup \mathrm{cl}(B)$ (Preservation of binary union)

K4) $\mathrm{cl}(\mathrm{cl}(A)) = \mathrm{cl}(A)$ (Idempotency).

for all $A, B \in P(X)$. Here $P(X)$ denotes the power set of the non-empty set X.
Furthermore, Kuratowski extended the topological spaces by removing the property $\mathrm{cl}(A \cup B) \subseteq \mathrm{cl}(A) \cup \mathrm{cl}(B)$ and defined closure spaces [14]. Closure spaces and in other words extended topological spaces were also studied by Day [3], Gnilka [5, 6, 7], Hammer [10, 11], Hausdorff [12], and Tabor [19] along similar lines. An actual and more general concept is isotonic space which is a closure space satisfying only the grounded and the isotony axioms. In recent years, Habil and Elzenati [8, 9] and Stadler et al. [17, 18] studied the isotonic spaces and determine some basic principles of these spaces.
In the meantime, Ersoy and Erol [4] introduced bi-isotonic spaces and studied the separation axioms in these spaces. Thus, they took a step in the direction extending the theory of the bitopological spaces which is a broad studying area. The systematic study of the bitopological spaces was initiated by Kelly in [13]. Afterwards, Pervin introduced the concept of connectedness for bitopological spaces [15]. The descriptions in [4] and discoveries on the concepts of connectedness in bitopological space given in [15, 16] have inspired us to study connectedness in bi-isotonic spaces.

2 Preliminaries

In this section, we briefly review some concepts and some related results of isotonic spaces and bi-isotonic spaces which are needed to use in the current paper.

2.1 Isotonic Spaces

Definition 2.1. A pair (X, cl) is called generalized closure space such that cl $: P(X) \to P(X)$ is an arbitrary set-valued set-function, called a closure operator, where $P(X)$ is the power set of the non-empty set X. Especially if the operator cl satisfies the axioms (K0) and (K1) then (X, cl) is called isotonic space [17, 18].

Definition 2.2. The dual of a closure operator cl on X is called interior operator int : $P(X) \to P(X)$ defined by $\mathrm{int}(A) = X \backslash \mathrm{cl}(X \backslash A)$ for all $A \in P(X)$.
In a generalized closure space (X, cl), a set $F \in P(X)$ is closed if $\mathrm{cl}(F) = F$ and $A \in P(X)$ is open if $X \backslash A$ is closed or $\mathrm{int}(A) = A$ [17, 18].

Definition 2.3. Let cl and int be closure and interior operators on X, respectively. Then the neighborhood operator $\mathcal{N} : X \to P(P(X))$ is defined by $\mathcal{N}(x) = \{N \in P(X) : x \in \mathrm{int}(N)\}$ for $x \in X$ [17, 18].

Proposition 2.4. *Let (X, cl) be a generalized closure space and $A \subseteq X$. $\mathcal{N} : X \to P(P(X))$ if and only if $A \subseteq \mathrm{int}(V)$ [17, 18].*

Proposition 2.5. *Let $\mathcal{N} : X \to P(P(X))$ be a neighborhood function on a generalized closure space(X, cl). Then $x \in \mathrm{cl}(A)$ if and only if $\mathcal{N} : X \to P(P(X))$ [17, 18].*

Proposition 2.6. *Let (X, cl) and (Y, cl') be isotonic spaces. A function $f : X \to Y$ is continuous if and only if $\mathrm{cl}\left(f^{-1}(B)\right) \subseteq f^{-1}(\mathrm{cl}'(B))$ for all $B \in P(Y)$ [17, 18].*

2.2 Bi-isotonic Spaces

Definition 2.7. A triple $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is called generalized bi-closure space such that $\mathrm{cl}_1 : P(X) \to P(X)$ and $\mathrm{cl}_2 : P(X) \to P(X)$ are arbitrary set-valued operators from the power set of non-empty set X [1, 2].

Bi-closure spaces were introduced and studied with two closure operators satisfying the grounded, isotony and expansive axioms given by (K0), (K1) and (K2) in [1, 2]. Recently, these spaces were generalized by weakening these axioms and a new space called bi-isotonic space were introduced with two isotonic operators satisfying the axioms (K0) and (K1) in [4].

Definition 2.8. Let cl_1 and cl_2 be two isotonic operators satisfying the axioms (K0) and (K1) on X , then the triple $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is called bi-isotonic space [4].

Definition 2.9. A subset A of a bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is called closed if $\mathrm{cl}_1\mathrm{cl}_2(A) = A$. The complement of a closed set is called open [4].

Under the light of this definition, the following proposition is obvious.

Proposition 2.10. *A subset A of a bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is closed if and only if $\mathrm{cl}_1(A) = A$ and $\mathrm{cl}_2(A) = A$ [4].*

It is obvious that a subset A of a bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is open if $X \backslash A = \mathrm{cl}_i(X \backslash A)$or $\mathrm{int}_{\,i}(A) = A$ for all $i \in \{1, 2\}$.

Proposition 2.11. *Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space and. Then for any $A \subseteq X$, the followings are held.*

- ***i)*** *A is an open set if and only if $A = X \backslash \mathrm{cl}_1\mathrm{cl}_2(X \backslash A)$.*
- ***ii)*** *If A is an open set and $A \subseteq G$, then $A \subseteq X \backslash \mathrm{cl}_1\mathrm{cl}_2(X \backslash G)$ [4].*

Proposition 2.12. *Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space and $Y \subseteq X$. The induced operators $\mathrm{cl}_i^Y : P(Y) \to P(Y)$ are isotonic provided that $\mathrm{cl}_i^Y(A) = \mathrm{cl}_i(A) \cap Y$ for all $A \subseteq Y$ and $i \in \{1, 2\}$ [4].*

Proposition 2.13. *Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space and $Y \subseteq X$. $\left(Y, \mathrm{cl}_1^Y, cl_2^Y\right)$ is called a subspace of $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ such that cl_1^Y and cl_2^Y are induced isotonic operators [4].*

Proposition 2.14. *Let* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *and* $(Y, \mathrm{cl}'_1, \mathrm{cl}'_2)$ *be bi-isotonic spaces. A map* $f : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to (Y, \mathrm{cl}'_1, \mathrm{cl}'_2)$ *is bi-continuous if and only if* $\mathrm{cl}_i \left(f^{-1}(B)\right) \subseteq f^{-1}(\mathrm{cl}'_i(B))$ *for all* $B \in P(Y)$ *and* $i \in \{1, 2\}$ *[4].*

Proposition 2.15. *Let* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *and* $(X, \mathrm{cl}'_1, \mathrm{cl}'_2)$ *be bi-isotonic spaces. Then the following conditions (for bi-continuity) are equivalent:*

i) $f : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to (Y, \mathrm{cl}'_1, \mathrm{cl}'_2)$ *is bi-continuous.*

ii) $f^{-1}(\mathrm{int}'_i(B)) \subseteq \mathrm{int}_i\left(f^{-1}(B)\right)$ *for all* $B \in P(Y)$ *and* $i \in \{1, 2\}$.

iii) $\mathcal{N} : X \to P(P(X))$ *provided* $\mathcal{N} : X \to P(P(X))$ *for all* $B \in P(Y)$ *and* $i \in \{1, 2\}$ *[4].*

Let $f : (X, \mathrm{cl}_1, \mathrm{cl}_2) \to (Y, \mathrm{cl}'_1, \mathrm{cl}'_2)$ be a bijective map. If f is bi-continuous and f^{-1} bi-continuous, then it is called bi-homeomorphism [4].

Definition 2.16. A generalized bi-closure space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is called

i) pairwise S_T_1-space if there is $N_x \in \mathcal{N}_1(x)$ such that $y \notin N_x$ and there is $N_y \in \mathcal{N}_2(y)$ such that $x \notin N_y$ for all distinct points $x, y \in X$,

ii) pairwise R_T_1-space if (X, cl_1) and(X, cl_2) are both T_1-spaces [4].

Also, it is known from [18] that an isotonic space (X, cl) is a T_1-space if and only if $\mathrm{cl}(\{x\}) \subseteq \{x\}$ for all $x \in X$.

Proposition 2.17. *If a bi-isotonic space* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *is a pairwise* R_T_1-*space then it is pairwise* S_T_1-*space [4].*

3 Pairwise Separated Sets in Bi-isotonic Spaces

Definition 3.1. Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space and $A, B \subset X$ be non-empty subsets. A and B are said to be pairwise separated sets if and only if $A \cap \mathrm{cl}_1(B) = \varnothing$ and $\mathrm{cl}_2(A) \cap B = \varnothing$.

Example 3.2. Let $X = \{a, b\}$ and define cl_1 and cl_2 as follows:

$$\mathrm{cl}_1(\varnothing) = \varnothing,\ \mathrm{cl}_1(\{b\}) = \{b\},\ \mathrm{cl}_1(\{a\}) = \mathrm{cl}_1\ (X) = X,$$

and

$$\mathrm{ccl}_2(\varnothing) = \varnothing\,,\ \mathrm{cl}_2(\{a\}) = \{a\}\ ,\ \mathrm{cl}_2(\{b\}) = \mathrm{cl}_2(X) = X.$$

Then $\{a\}$ and $\{b\}$ are pairwise separated sets in $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ since $\{a\} \cap \mathrm{cl}_1(\{b\}) = \mathrm{cl}_2\ (\{a\}) \cap \{b\} = \varnothing$ while they are not separated in (X, cl_1) and (X, cl_2).

Two pairwise separated sets in a bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ do not have to be disjoint as can be seen in the following counterexample.

Example 3.3. Let $X = \{a, b, c\}$ and define cl_1 and cl_2 as follows:

$$\begin{aligned}&\mathrm{cl}_1(\emptyset) = \emptyset,\ \mathrm{cl}_1(\{a\}) = \{a\}\,,\ \mathrm{cl}_1(\{b\}) = \{b\}\,,\ \mathrm{cl}_1(\{c\}) = \{a\}\,,\\&\mathrm{cl}_1(\{a, b\}) = \{a, b\}\,,\ \ \mathrm{cl}_1(\{b, c\}) = \{a, b\}\,,\ \mathrm{cl}_1(\{a, c\}) = \mathrm{cl}_1(X) = X,\end{aligned}$$

and

$$\begin{aligned}&\mathrm{cl}_2(\emptyset) = \emptyset,\ \ \mathrm{cl}_2(\{b\}) = \{b\}\,,\ \ \mathrm{cl}_2(\{c\}) = \{a\}\,,\ \ \mathrm{cl}_2(\{b, c\}) = \{a, b\}\\&\mathrm{cl}_2(\{a\}) = \mathrm{cl}_2(\{a, b\}) = \mathrm{cl}_2(\{a, c\}) = \mathrm{cl}_2(X) = X.\end{aligned}$$

Then it is obvious that $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a bi-isotonic space. In this space $\{c\}$ and $\{b, c\}$are pairwise separated since $\{c\} \cap \mathrm{cl}_1(\{b, c\}) = \{c\} \cap \{a, b\} = \varnothing$ and $\mathrm{cl}_2(\{c\}) \cap \{b, c\} = \{a\} \cap \{b, c\} = \varnothing$ but they are not disjoint.

Theorem 3.4. *Let* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *be a bi-closure space,* A *and* B *be two disjoint subsets of* X*. If* $\mathrm{cl}_1(B) \subset B$ *and* $\mathrm{cl}_2(A) \subset A$ *(or* $A \subset \mathrm{int}_1(A)$ *and* $B \subset \mathrm{int}_2(B)$*) then* A *and* B *are pairwise separated sets.*

Proof. First, assume that $A, B \subset X$, $\text{cl}_1(B) \subset B$, $\text{cl}_2(A) \subset A$ and $A \cap B = \varnothing$ in bi-isotonic space X. It is easily seen that $A \cap \text{cl}_1(B) \subset A \cap B$ and $B \cap \text{cl}_2(A) \subset A \cap B$. Then $A \cap \text{cl}_1(B) = \varnothing$ and $B \cap \text{cl}_2(A) = \varnothing$ are satisfied since $A \cap B = \varnothing$. This proves that A and B are pairwise separated sets.
On the other hand, assume that $A \subset \text{int}_1(A)$, $B \subset \text{int}_2(B)$ and $A \cap B = \varnothing$. Thereby, $\text{cl}_1(B) \subset \text{cl}_1(X \backslash A)$ since $B \subset X \backslash A$ and cl_1 is an isotonic operator. In this case, we find $\text{int}_1(A) = X \backslash \text{cl}_1(X \backslash A) \subset X \backslash \text{cl}_1(B)$. Under the assumption, $A \subset \text{int}_1(A)$ we get $A \subset X \backslash \text{cl}_1(B)$ that is $A \cap \text{cl}_1(B) = \varnothing$. Similarly, we know $\text{cl}_2(A) \subset \text{cl}_2(X \backslash B)$ since $A \subset X \backslash B$ and cl_2 is isotonic. Hence $\text{int}_2(B) = X \backslash \text{cl}_2(X \backslash B) \subset X \backslash \text{cl}_2(A)$ and this gives $B \subset X \backslash \text{cl}_2(A)$ because of $B \subset \text{int}_2(B)$. Then we deduce $\text{cl}_2(A) \cap B = \varnothing$. These complete the proof. □

As in bitopological spaces the following corollary becomes true in bi-closure spaces since cl_1 and cl_2 hold the axioms (K0), (K1) and (K2).

Corollary 3.5. *Let $(X, \text{cl}_1, \text{cl}_2)$ be a bi-closure space, A and B be two disjoint subsets of X. If $\text{cl}_1(B) \subset B$ and $\text{cl}_2(A) \subset A$ (or $A \subset \text{int}_1(A)$ and $B \subset \text{int}_2(B)$) then A and B are pairwise separated sets.*

Theorem 3.6. *Let $(X, \text{cl}_1, \text{cl}_2)$ be a bi-isotonic space. If two subsets $A, B \subset X$ satisfy $\text{cl}_1(B) \subset B$ and $\text{cl}_2(A) \subset A$ (or equivalently $A \subset \text{int}_1(A)$ and $B \subset \text{int}_2(B)$) then $A \backslash B$ and $B \backslash A$ are pairwise separated sets.*

Proof. Suppose that $(X, \text{cl}_1, \text{cl}_2)$ is a bi-isotonic space and A, B are any subsets of X with $\text{cl}_1(B) \subset B$ and $\text{cl}_2(A) \subset A$. It is known from [8] the condition $U \subseteq V \Rightarrow \text{cl}(U) \subseteq \text{cl}(V)$ is equivalent to $\text{cl}(U \cap V) \subseteq \text{cl}(U) \cap \text{cl}(V)$ for an arbitrary closure operator $\text{cl} : P(X) \to P(X)$. So, we get

$$\begin{aligned}(A \backslash B) \cap \text{cl}_1(B \backslash A) &= (A \cap (X \backslash B)) \cap \text{cl}_1(B \cap (X \backslash A)) \\ &\subset A \cap (X \backslash B) \cap \text{cl}_1(B) \cap \text{cl}_1(X \backslash A) \\ &= A \cap \text{cl}_1(X \backslash A) \cap (X \backslash B) \cap \text{cl}_1(B)\end{aligned}$$

and by the fact that $\text{cl}_1(X \backslash A) = X \backslash \text{int}_1(A)$ we find $(A \backslash B) \cap \text{cl}_1(B \backslash A) \subset (X \backslash \text{int}_1(A)) \cap A \cap (X \backslash B) \cap \text{cl}_1(B)$. We have $(X \backslash B) \cap \text{cl}_1(B) = \varnothing$ from the hypothesis $\text{cl}_1(B) \subset B$ (or $(X \backslash \text{int}_1(A)) \cap A = \varnothing$ from the hypothesis $A \subset \text{int}_1(A)$). Thus, we deduce $(A \backslash B) \cap \text{cl}_1(B \backslash A) = \varnothing$. Similarly, we can prove $\text{cl}_2(A \backslash B) \cap (B \backslash A) = \varnothing$, too. Finally, the last two equalities mean that $A \backslash B$ and $B \backslash A$ are pairwise separated sets. □

Then the following corollary is immediate in bi-closure spaces.

Corollary 3.7. *Let $(X, \text{cl}_1, \text{cl}_2)$ be a bi-closure space, A and B be two disjoint subsets of X. If B is cl_1-closed and A is cl_2-closed (or A is int_1-open and B is int_2-open) then $A \backslash B$ and $B \backslash A$ are pairwise separated sets.*

Theorem 3.8. *Let $(X, \text{cl}_1, \text{cl}_2)$ be a bi-isotonic space. Then the following statements are equivalent for any subsets $A, B \subset X$:*

- ***i)*** *A and B are pairwise separated sets.*
- ***ii)*** *There are $U \in \mathcal{N}_1(A)$ and $V \in \mathcal{N}_2(B)$ such that $U \cap B = \varnothing$ and $A \cap V = \emptyset$.*

Proof. First, assume that A and B are pairwise separated subsets in the bi-isotonic space. So, $A \cap \text{cl}_1(B) = \varnothing$ and $\text{cl}_2(A) \cap B = \varnothing$ require $A \subset \text{int}_1(X \backslash B)$ and $B \subset \text{int}_2(X \backslash A)$, respectively. If $X \backslash B$ and $X \backslash A$ are designated with the terms U and V, respectively, we see the existence of the neighborhoods $U \in \mathcal{N}_1(A)$ and $V \in \mathcal{N}_2(B)$ satisfying $U \cap B = \varnothing$ and $A \cap V = \emptyset$, respectively.
Secondly, assume that there are $U \in \mathcal{N}_1(A)$ and $V \in \mathcal{N}_2(B)$ such that $U \cap B = \varnothing$ and $A \cap V = \emptyset$. Thus it is known that

$$U \in \mathcal{N}_1(A) \text{ if and only if } A \subseteq \text{int}_1(U)$$

and

$$V \in \mathcal{N}_2(B) \text{ if and only if } B \subseteq \text{int}_2(V).$$

Moreover, int_1 and int_2 are two isotonic operators since cl_1 and cl_2 are two isotonic operators and they satisfy $\text{int}_1(U) \subset \text{int}_1(X \backslash B)$ and $\text{int}_2(V) \subset \text{int}_2(X \backslash A)$ from the assumptions $U \subset X \backslash B$ and $V \subset X \backslash A$. So we find $A \subset \text{int}_1(X \backslash B)$ and $B \subset \text{int}_2(X \backslash A)$ which imply that $A \cap \text{cl}_1(B) = \varnothing$ and $\text{cl}_2(A) \cap B = \varnothing$ by Definition 2.2. Consequently, these prove that A and B are pairwise separated sets. □

Theorem 3.9. *Let* $(X, \text{cl}_1, \text{cl}_2)$ *and* $(Y, \text{cl}'_1, \text{cl}'_2)$ *be two bi-isotonic spaces and* $f : X \to Y$ *be a bi-continuous function. If* A *and* B *are pairwise separated subsets of* Y *then* $f^{-1}(A)$ *and* $f^{-1}(B)$ *are pairwise separated in* X.

Proof. Assume that f is a bi-continuous function from X to Y and two subsets $A, B \subset Y$ are pairwise separated subsets. Thus, $A \cap \text{cl}'_1(B) = \varnothing$ and $\text{cl}'_2(A) \cap B = \varnothing$. The pre-image of first equality gives us $f^{-1}(A \cap \text{cl}'_1(B)) = f^{-1}(A) \cap f^{-1}(\text{cl}'_1(B)) = \varnothing$, that is, $f^{-1}(\text{cl}'_1(B)) \subset X \backslash f^{-1}(A)$. Also from Proposition 2.14, it is known $\text{cl}_1(f^{-1}(B)) \subset f^{-1}(\text{cl}'_1(B))$ since f is bi-continuous. Then we get $\text{cl}_1(f^{-1}(B)) \subset X \backslash f^{-1}(A)$ and this implies that $\text{cl}_1(f^{-1}(B)) \cap f^{-1}(A) = \varnothing$.
In the same manner, we prove $\text{cl}_2(f^{-1}(A)) \cap f^{-1}(B) = \varnothing$, too. These are sufficient to complete the proof that $f^{-1}(A)$ and $f^{-1}(B)$ are pairwise separated subsets of X. □

Theorem 3.10. *Let* $(X, \text{cl}_1, \text{cl}_2)$ *be a bi-isotonic space and* $A, B \subset Y \subset X$. *A and B are pairwise separated subsets in* $(X, \text{cl}_1, \text{cl}_2)$ *if and only if A and B are pairwise separated subsets in the subspace* $\left(Y, \text{cl}_1^Y, \text{cl}_2^Y\right)$.

Proof. (⇒:) Let A and B be pairwise separated subsets in $(X, \text{cl}_1, \text{cl}_2)$. Then$A \cap \text{cl}_1(B) = \varnothing$ and $\text{cl}_2(A) \cap B = \varnothing$. By considering Proposition 2.13, it is easy to see that $A \cap \text{cl}_1^Y(B) = A \cap (\text{cl}_1(B) \cap Y) = (A \cap \text{cl}_1(B)) \cap Y = \varnothing$ and $\text{cl}_2^Y(A) \cap B = Y \cap (\text{cl}_2(A) \cap B) = Y \cap (\text{cl}_2(A) \cap B) = \varnothing$, i.e., A and B are pairwise separated subsets in the subspace $\left(Y, \text{cl}_1^Y, \text{cl}_2^Y\right)$.
(⇐:) Let A and B be pairwise separated subsets of $\left(Y, \text{cl}_1^Y, \text{cl}_2^Y\right)$. Then $A \cap \text{cl}_1^Y(B) = \varnothing$ and $\text{cl}_2^Y(A) \cap B = \varnothing$. By considering $A, B \subset Y \subset X$ and Proposition 2.12, we get

$$A \cap \text{cl}_1^Y(B) = A \cap (\text{cl}_1(B) \cap Y) = (A \cap Y) \cap \text{cl}_1(B) = A \cap \text{cl}_1(B) = \varnothing$$

and

$$\text{cl}_2^Y(A) \cap B = (\text{cl}_2(A) \cap Y) \cap B = \text{cl}_2(A) \cap (Y \cap B) = \text{cl}_2(A) \cap B = \varnothing.$$

Consequently, we deduce A and B are pairwise separated subsets in X. □

4 Pairwise Connectedness in Bi-isotonic Spaces

Definition 4.1. A bi-isotonic space $(X, \text{cl}_1, \text{cl}_2)$ is called pairwise connected if and only if X cannot be expressed as the union of two non-empty disjoint sets A and B such that $[A \cap \text{cl}_1(B)] \cup [\text{cl}_2(A) \cap B] = \varnothing$.

If a bi-isotonic space $(X, \text{cl}_1, \text{cl}_2)$ is not pairwise connected, then it is said to be pairwise disconnected. Equivalently, $(X, \text{cl}_1, \text{cl}_2)$ is said to be pairwise disconnected, if it is possible to express X as a union of pairwise separated disjoint non-empty two sets.

Example 4.2. Let $X = \{a, b, c\}$ and define cl_1 and cl_2 as follows:

$$\text{cl}_1(\emptyset) = \emptyset, \text{cl}_1(\{a\}) = \text{cl}_1(\{b\}) = \text{cl}_1(\{a, b\}) = \{a, b\}, \text{cl}_1(\{c\}) = \{c\},$$
$$\text{cl}_1(\{a, c\}) = \text{cl}_1(\{b, c\}) = \text{cl}_1(X) = X,$$

and

$$\text{cl}_2(\emptyset) = \emptyset, \text{cl}_2(\{a\}) = \text{cl}_2(\{c\}) = \text{cl}_2(\{a, c\}) = \{a, c\}, \text{cl}_2(\{b\}) = \{b\}$$
$$\text{cl}_2(\{a, b\}) = \text{cl}_2(\{b, c\}) = \text{cl}_2(X) = X.$$

Here the only pairwise separated disjoint non-empty sets in X are $\{b\}$ and $\{c\}$ however $X \neq \{b\} \cup \{c\}$. Then $(X, \text{cl}_1, \text{cl}_2)$ is a pairwise connected bi-isotonic space while (X, cl_1) and (X, cl_2) are disconnected isotonic spaces from the facts that $\{a, b\}$ and $\{c\}$ are separated in(X, cl_1) and also $\{a, c\}$ and $\{b\}$ are separated in (X, cl_2).

Example 4.3. Let $X=\{a,b,c\}$ and define cl_1 and cl_2 as follows:

$$\begin{aligned}&\mathrm{cl}_1(\emptyset)=\emptyset,\ \mathrm{cl}_1(\{a\})=\{a\},\ \mathrm{cl}_1(\{b\})=\{b\},\ \mathrm{cl}_1(\{c\})=\{c\},\\&\mathrm{cl}_1(\{a,b\})=\{a,b\},\ \mathrm{cl}_1(\{b,c\})=\{b,c\},\ \mathrm{cl}_1(\{a,c\})=\mathrm{cl}_1(X)=X,\end{aligned}$$

and

$$\begin{aligned}&\mathrm{cl}_2(\emptyset)=\emptyset,\ \mathrm{cl}_2(\{a\})=\{a\},\ \mathrm{cl}_2(\{b\})=\{b\},\ \mathrm{cl}_2(\{c\})=\{c\},\\&\mathrm{cl}_2(\{a,c\})=\{a,c\},\ \mathrm{cl}_2(\{b,c\})=\{b,c\},\ \mathrm{cl}_2(\{a,b\})=\mathrm{cl}_2(X)=X.\end{aligned}$$

It is obvious that X is a pairwise disconnected bi-isotonic space since there exist pairwise separated and disjoints sets $\{a\}$ and $\{b,c\}$ such that $X=\{a\}\cup\{b,c\}$.

Theorem 4.4. *A bi-isotonic space* $(X,\mathrm{cl}_1,\mathrm{cl}_2)$ *is pairwise connected if and only if* X *cannot be expressed as* $X=A\cup B$ *for any non-empty disjoint sets* A *and* B *such that* $\mathrm{cl}_1(B)\subset B$ *and* $\mathrm{cl}_2(A)\subset A$.

Proof. Suppose that there exists non-empty sets A and B satisfying $X=A\cup B$ such that $A\cap B=\varnothing$, $\mathrm{cl}_1(B)\subset B$ and $\mathrm{cl}_2(A)\subset A$. Then it is easy to get $A\cap\mathrm{cl}_1(B)=\varnothing$ and $\mathrm{cl}_2(A)\cap B=\varnothing$ from $\mathrm{cl}_1(B)\subset X\backslash A$ and $\mathrm{cl}_2(A)\subset X\backslash B$. These means that X is a pairwise disconnected bi-isotonic space.
Conversely, suppose that X is a pairwise disconnected bi-isotonic space. Then X can be expressed as a union of for two pairwise separated disjoint non-empty sets two sets A and B. These sets satisfy $\mathrm{cl}_1(B)\subset X\backslash A=B$ $\mathrm{cl}_2(A)\subset X\backslash B=A$ from $X=A\cup B$,$A\cap B=\varnothing$, $A\cap\mathrm{cl}_1(B)=\varnothing$ and $\mathrm{cl}_2(A)\cap B\neq\varnothing$. □

In bi-closure spaces, the following corollary becomes obvious and corresponds to pairwise connectedness in bitopological spaces.

Corollary 4.5. *Let* $(X,\mathrm{cl}_1,\mathrm{cl}_2)$ *be a bi-closure space.* X *is pairwise connected if and only if* X *cannot be expressed as a union of any non-empty disjoint* cl_1-*closed set and* cl_2-*closed set.*

Example 4.6. Let us consider that

$$\mathrm{cl}_1(A)=\begin{cases}\varnothing, & A=\varnothing\\(-\infty,a], & \sup A=a\\\mathbb{R}, & \sup A=\infty\end{cases}\quad\text{and}\quad\mathrm{cl}_2(A)=\begin{cases}\varnothing, & A=\varnothing\\ [b,\infty), & \inf A=b\\\mathbb{R}, & \inf A=-\infty\end{cases}$$

be two operators on $\mathbb{R}$, then $(\mathbb{R},\mathrm{cl}_1,\mathrm{cl}_2)$ is a bi-closure space and it cannot be expressed as a union of any non-empty disjoint cl_1-closed set and cl_2-closed set. Thus, it is pairwise connected.

The assertion of Corollary 4.5 needs not be true in general for all bi-isotonic spaces. Although a bi-isotonic space $(X,\mathrm{cl}_1,\mathrm{cl}_2)$ is pairwise disconnected, it can be expressed as a union of pairwise separated disjoint non-empty two sets which are not have to be a cl_1-closed or cl_2-closed set in X as can be seen in the following example.

Example 4.7. Let $X=\{a,b,c\}$ and define cl_1 and cl_2 as follows:

$$\begin{aligned}&\mathrm{cl}_1(\emptyset)=\emptyset,\ \mathrm{cl}_1(\{b\})=\mathrm{cl}_1(\{c\})=\mathrm{cl}_1(\{b,c\})=\{b\}\\&\mathrm{cl}_1(\{a\})=\mathrm{cl}_1(\{a,b\})=\mathrm{cl}_1(\{a,c\})=\mathrm{cl}_1(X)=X,\end{aligned}$$

and

$$\begin{aligned}&\mathrm{cl}_2(\emptyset)=\emptyset,\ \mathrm{cl}_2(\{a\})=\mathrm{cl}_2(\{c\})=\mathrm{cl}_2(\{a,c\})=\{a\},\\&\mathrm{cl}_2(\{b\})=\mathrm{cl}_2(\{b,c\})=\mathrm{cl}_2(\{a,b\})=\mathrm{cl}_2(X)=X.\end{aligned}$$

Then X is a pairwise disconnected bi-isotonic space since $\{a\}\cap\{b,c\}=\{a\}\cap\mathrm{cl}_1\{b,c\}=\mathrm{cl}_2\{a\}\cap\{b,c\}=\varnothing$ and $X=\{a\}\cup\{b,c\}$, besides $\{b,c\}$ is not cl_1-closed.

The connectedness of any subspace of an isotonic space X hasnt been characterized by the subsets of X, yet. In this regard, we define and characterize the connectedness of sub-isotonic spaces.

Definition 4.8. Let (X, cl) be an isotonic space. A subset $A \subset X$ is connected if it is a connected isotonic space when endowed with the induced isotonic operator cl_A.
Otherwise, (A, cl_A) is called disconnected if it is not connected. In other words, if that there exist non-empty disjoint subsets P' and Q' of A satisfying $A = P' \cup Q'$ such that $\mathrm{cl}_A(P') \cap Q' = \varnothing$ and $P' \cap \mathrm{cl}_A(Q') = \varnothing$ then the subspace (A, cl_A) is disconnected.

Theorem 4.9. *Let (X, cl) be an isotonic space and $A \subset X$. A is a disconnected subspace if and only if there exist non-empty sets P and Q of X such that*

$$P \cap A \neq \emptyset,\ \ Q \cap A \neq \emptyset,\ \ P \cap Q \subset X \backslash A,\ \ A \subset P \cup Q,$$
$$\mathrm{cl}(P \cap A) \cap Q \subset X \backslash A,\ \ P \cap \mathrm{cl}(Q \cap A) \subset X \backslash A.$$

Proof. ($\Rightarrow$:) Let A be a disconnected sub-isotonic space. Then there exist non-empty subsets $P', Q' \subset A$ satisfying $A = P' \cup Q'$, $P' \cap Q' = \varnothing$, $\mathrm{cl}_A(P') \cap Q' = \varnothing$ and $P' \cap \mathrm{cl}_A(Q') = \varnothing$. Thus, there exist non-empty subsets $P, Q \subset X$ such that $P' = P \cap A$ and $Q' = Q \cap A$. Moreover, the order of the following relations give us the desired results as follows
$P' \cap Q' = \varnothing \Rightarrow (P \cap A) \cap (Q \cap A) = \varnothing \Rightarrow (P \cap Q) \cap A = \varnothing \Rightarrow P \cap Q \subset X \backslash A$,
$A = P' \cup Q' \subset P \cup Q \Rightarrow A \subset P \cup, Q$
$\mathrm{cl}_A(P') \cap Q' = \mathrm{cl}(P \cap A) \cap A \cap (A \cap Q) = \emptyset \Rightarrow \mathrm{cl}(P \cap A) \cap Q \subset X \backslash, A$
$P' \cap \mathrm{cl}_A(Q') = (P \cap A) \cap \mathrm{cl}(Q \cap A) \cap A = \emptyset \Rightarrow P \cap \mathrm{cl}(Q \cap A) \subset X \backslash A$.
($\Leftarrow$:) Suppose that there exist non-empty sets P and Q in X such that

$$P \cap A \neq \emptyset,\ Q \cap A \neq \emptyset,\ P \cap Q \subset X \backslash A,\ A \subset P \cup Q,$$
$$\mathrm{cl}(P \cap A) \cap Q \subset X \backslash A,\ \ P \cap \mathrm{cl}(Q \cap A) \subset X \backslash A.$$

If we call $P' = P \cap A$ and $Q' = Q \cap A$. It easy to get that there exist non-empty disjoint subsets P' and Q' of A satisfying $A = P' \cup Q'$ such that $\mathrm{cl}_A(P') \cap Q' = \varnothing$ and $P' \cap \mathrm{cl}_A(Q') = \varnothing$. In this case, the subspace (A, cl_A) is disconnected as desired. □

Definition 4.10. Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space and $A \subset X$. Then the bi-isotonic subspace $\left(A, \mathrm{cl}_1^A, \mathrm{cl}_2^A\right)$ is called pairwise disconnected if there exist non-empty disjoint subsets P' and Q' of A satisfying $A = P' \cup Q'$ such that $\mathrm{cl}_1^A(P') \cap Q' = \varnothing$ and $P' \cap \mathrm{cl}_2^A(Q') = \varnothing$.

Then the following theorem is an immediate consequence of this definition.

Theorem 4.11. *Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space and $A \subset X$. A is a pairwise disconnected subspace if and only if there exist non-empty sets P and Q of X such that*

$$P \cap A \neq \emptyset,\ Q \cap A \neq \emptyset,\ P \cap Q \subset X \backslash A,\ A \subset P \cup Q,$$
$$\mathrm{cl}_1^A(P \cap A) \cap Q \subset X \backslash A,\ \ P \cap \mathrm{cl}_2^A(Q \cap A) \subset X \backslash A.$$

By this theorem, it becomes easy to prove pairwise connectedness is preserved under bi-continuous functions.

Theorem 4.12. *Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ and $\left(Y, \mathrm{cl}_1', \mathrm{cl}_2'\right)$ be bi-isotonic spaces and $f : X \to Y$ be a bi-continuous function. If X is pairwise connected, then $f(X) \subset Y$ is pairwise connected.*

Proof. Let $f(X) \subset Y$ be pairwise disconnected. Then there are the subsets $P, Q \subset Y$ such that

$$P \cap f(X) \neq \emptyset,\ Q \cap f(X) \neq \emptyset,\ P \cap Q \subset Y \backslash f(X),\ f(X) \subset P \cup Q,$$
$$\mathrm{cl}_1'(P \cap f(X)) \cap Q \subset Y \backslash f(X),\ \ P \cap \mathrm{cl}_2'(Q \cap f(X)) \subset Y \backslash f(X).$$

Then it is straightforward to see $f^{-1}(P) \cap X \neq \varnothing$, $f^{-1}(Q) \cap X \neq \varnothing$, $f^{-1}(P) \cap f^{-1}(Q) = \varnothing$, $X = f^{-1}(P) \cup f^{-1}(Q)$. Moreover, $\mathrm{cl}_1'(P \cap f(X)) \cap Q \subset Y \backslash f(X)$ and $\mathrm{cl}_1'(Q \cap f(X)) \cap P \subset Y \backslash f(X)$ imply $f^{-1}\left(\mathrm{cl}_1'(P \cap f(X))\right) \subset X \backslash f^{-1}(Q)$ and $f^{-1}\left(\mathrm{cl}_2'(Q \cap f(X))\right) \subset X \backslash f^{-1}(P)$, respectively.
On the other hand, by Proposition 2.14, we get $\mathrm{cl}_1\left(f^{-1}(P \cap f(X))\right) \subset f^{-1}\left(\mathrm{cl}_1'(P \cap f(X))\right)$ and $\mathrm{cl}_2\left(f^{-1}(Q \cap f(X))\right) \subset f^{-1}\left(\mathrm{cl}_2'(Q \cap f(X))\right)$ since f is bi-continuous. Therefore, we obtain $\mathrm{cl}_1\left(f^{-1}(P \cap f(X))\right) \subset X \backslash f^{-1}(Q)$ and $\mathrm{cl}_2\left(f^{-1}(Q \cap f(X))\right) \subset X \backslash f^{-1}(P)$. Finally, these give us $\mathrm{cl}_1\left(f^{-1}(P)\right) \cap f^{-1}(Q) = \varnothing$ and $f^{-1}(P) \cap \mathrm{cl}_2\left(f^{-1}(Q)\right) = \varnothing$ which complete the proof. □

Corollary 4.13. *Let* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *and* $(Y, \mathrm{cl'}_1, \mathrm{cl'}_2)$ *be bi-isotonic spaces and* $f : X \to Y$ *be a bi-homeomorphism. Then* X *is pairwise connected if and only if* Y *is pairwise connected.*

Theorem 4.14. *Let* A *and* B *be subsets of an isotonic space* (X, cl) *such that* $A \subset B \subset \mathrm{cl}\,(A)$. *If* A *is a connected subset then* B *is connected.*

Proof. Consider the subsets A and B such that $A \subset B \subset \mathrm{cl}\,(A)$ and assume that A is connected and B is not connected. Then there exist non-empty sets P and Q of X such that

$$P \cap B \neq \emptyset,\ Q \cap B \neq \emptyset,\ P \cap Q \subset X \backslash B,\ B \subset P \cup Q,$$
$$\mathrm{cl}\,(P \cap B) \cap Q \subset X \backslash B,\ P \cap \mathrm{cl}\,(Q \cap B) \subset X \backslash B.$$

Thus, it is evident that $P \cap Q \subset X \backslash A$ and $A \subset P \cup Q$ since $A \subset B$.
Now lets prove $P \cap A \neq \varnothing$ and $Q \cap A \neq \varnothing$. Assume that $P \cap A = \varnothing$ or $Q \cap A = \varnothing$. $(P \cap A) \cup (Q \cap A) = (P \cup Q) \cap A = A$ since $A \subset P \cup Q$. At this stage $A \subset Q$ if$P \cap A = \varnothing$ or $A \subset P$ if $Q \cap A = \varnothing$.
In the case of $A \subset Q$ we obtain $A \cap B \subset Q \cap B \Rightarrow A \subset Q \cap B \Rightarrow \mathrm{cl}\,(A) \subset \mathrm{cl}\,(Q \cap B)$. Then $B \subset \mathrm{cl}\,(Q \cap B)$ because of $B \subset \mathrm{cl}\,(A)$. This implies that $P \cap B \subset P \cap \mathrm{cl}\,(Q \cap B) = \varnothing$ which contradicts with $B \cap P \neq \varnothing$.
In the same way, $A \subset P$ requires to $Q \cap B \subset Q \cap \mathrm{cl}\,(P \cap B) = \varnothing$ and this contradicts with $B \cap Q \neq \varnothing$. So $P \cap A \neq \varnothing$ and $Q \cap A \neq \varnothing$.
Moreover, we see

$$\begin{aligned} P \cap A \subset P \cap B &\Rightarrow \mathrm{cl}\,(P \cap A) \subset \mathrm{cl}\,(P \cap B) \\ &\Rightarrow \mathrm{cl}\,(P \cap A) \cap Q \subset \mathrm{cl}\,(P \cap B) \cap Q \subset X \backslash B \subset X \backslash A \\ &\Rightarrow \mathrm{cl}\,(P \cap A) \cap Q \subset X \backslash A. \end{aligned}$$

In the same manner, $P \cap \mathrm{cl}\,(P \cap A) \subset X \backslash A$. All these relations indicate A is disconnected. This contradiction leads that B is a connected set. □

By the last theorem, the following corollary is obvious and it was proved by an alternative method in [8].

Corollary 4.15. *In an isotonic space if a subset* A *is connected and* $A \subset \mathrm{cl}\,(A)$ *then* $\mathrm{cl}\,(A)$ *is connected. Additionally in a neighborhood space closure of connected sets are connected, too.*

The proof of the following theorem is in the same vein as that of Theorem 4.14.

Theorem 4.16. *Let* A *and* B *be given with relations* $A \subset B \subset \mathrm{cl}_i\,(A)$ *for all* $i \in \{1, 2\}$ *in a bi-isotonic space* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$. *If* A *is pairwise connected then* B *is pairwise connected.*

In a bi-closure space, $A \subset \mathrm{cl}_1\,(A)$, $A \subset \mathrm{cl}_2\,(A)$ and hence $\mathrm{cl}_2\,(A) \subset \mathrm{cl}_2\,(\mathrm{cl}_1\,(A))$ and $\mathrm{cl}_1\,(A) \subset \mathrm{cl}_1\,(\mathrm{cl}_2\,(A))$ are satisfied. So, we can give the following corollary.

Corollary 4.17. *Let* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *be a bi-closure space. If* $A \subset X$ *is pairwise connected then* $\mathrm{cl}_1\,(A)$, $\mathrm{cl}_2\,(A)$, $\mathrm{cl}_1\,(\mathrm{cl}_2\,(A))$, *and* $\mathrm{cl}_2\,(\mathrm{cl}_1\,(A))$*are pairwise connected sets, too.*

Theorem 4.18. *Let* (X, cl) *be an isotonic space and* $A \subset X$ *be a connected subset. If* $A \subset P \cup Q$ *for pairwise separated and disjoint subsets* P *and* Q *then* $A \subset P$ *or* $A \subset Q$.

The method of aforementioned proof and the condition of the pairwise connectedness of a subset given in Theorem 4.14 prove the following theorem.

Theorem 4.19. *Let* $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ *be a bi-isotonic space and* $A \subset X$ *be a pairwise connected subset. If* $A \subset P \cup Q$ *for pairwise separated and disjoint subsets* P *and* Q *then* $A \subset P$ *or* $A \subset Q$.

The following theorem was proved in [8]. Now we give an alternative method to prove it.

Theorem 4.20. *Let* (X, cl) *be an isotonic space and* I *be a non-empty indexing set. If* $A_i \subset X$ *are connected for all* $i \in I$ *and* $A_i \cap A_j \neq \varnothing$ *for all* $i, j \in I$ *where* $i \neq j$, *then* $\bigcup_{i \in I} A_i$ *is connected.*

Proof. Assume that $A_i \subset X$ are connected for all $i \in I$ and $A_i \cap A_j \neq \varnothing$ for $i \neq j$ but $A = \bigcup_{i \in I} A_i$ is disconnected. Then there exist non-empty disjoint subsets P' and Q' of A satisfying $A = P' \cup Q'$ such that $\mathrm{cl}_A(P') \cap Q' = \varnothing$ and $P' \cap \mathrm{cl}_A(Q') = \varnothing$. So $A_i \subset P' \cup Q'$ for all $i \in I$. By the hypothesis and Theorem 4.18 $A_i \subset P'$ or $A_i \subset Q'$. Even if $A_i \subset P'$ for all $i \in I$ then $Q' = \varnothing$ since $A_i \cap A_j \neq \varnothing$ for $i \neq j$. Otherwise, if $A_i \subset Q'$ for all $i \in I$ then $P' = \varnothing$. Thus, these contradictions complete the proof. □

So by following the same steps in the last proof and by Theorem 4.19 we can prove the following theorem.

Theorem 4.21. *Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space and I be a nonempty indexing set. If $A_i \subset X$ are pairwise connected for all $i \in I$ and $A_i \cap A_j \neq \varnothing$ for all $i, j \in I$ where $i \neq j$ then $\bigcup_{i \in I} A_i$ is pairwise connected.*

It is clear from the above theorem that even if a bi-isotonic space is pairwise disconnected, it can be written as a disjoint union of maximal pairwise connected pieces.

Definition 4.22. The maximal pairwise connected subsets (ordered by inclusion) of a non-empty bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ are called the pairwise connected components of the space.

Corollary 4.23. *If $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a bi-isotonic space is pairwise connected then it has exactly one component itself.*

Definition 4.24. In a bi-isotonic space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$, the pairwise connected component $C(x)$ of $x \in X$ is the union of all pairwise connected subsets of X containing x.

Theorem 4.25. *Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-closure space. Then all pairwise connected components of X are closed.*

Proof. Consider arbitrary pairwise connected component A of a bi-closure space $(X, \mathrm{cl}_1, \mathrm{cl}_2)$. By the definition of pairwise connected components, the subset A is a maximal pairwise connected component set in $(X, \mathrm{cl}_1, \mathrm{cl}_2)$. $\mathrm{cl}_i(A)$ is pairwise connected for all $i \in \{1, 2\}$ from Corollary 4.17. Also, $A \subset \mathrm{cl}_i(A)$since X is a bi-closure space. So $A = \mathrm{cl}_i(A)$ for all $i \in \{1, 2\}$ and this completes the proof. □

So the last theorem is also valid for bi-topological spaces however, if $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a bi-isotonic space it is not true in general as can be seen in the following example.

Example 4.26. $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ given in Example 3.2 is a pairwise disconnected bi-isotonic space since $X = \{a\} \cup \{b\}$, $\{a\} \cap \{b\} = \varnothing$, $\{a\} \cap \mathrm{cl}_1(\{b\}) = \varnothing$, $\{b\} \cap \mathrm{cl}_2(\{a\}) = \varnothing$. $\{a\}$ and $\{b\}$ are pairwise connected components of this space however they are not closed in $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ since $\mathrm{cl}_1(\mathrm{cl}_2(\{a\})) = X$ and $\mathrm{cl}_1(\mathrm{cl}_2(\{b\})) = X$.

Definition 4.27. Let $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ be a bi-isotonic space with more than one element. X is called totally pairwise disconnected if the pairwise connected components in X are the singleton sets.

Theorem 4.28. *Let (X, cl_1, cl_2)be a bi-closure space. If X is totally pairwise disconnected then it is $R - T_1-$space and hence $S - T_1-$ space.*

Proof. Assume that $(X, \mathrm{cl}_1, \mathrm{cl}_2)$ is a totally pairwise disconnected bi-closure space. Then $C(x) = \{x\}$ for all $x \in X$, that is by the virtue of Theorem 4.25 we say $\mathrm{cl}_i(\{x\}) = \{x\}$ for all $i \in \{1, 2\}$. This completes the proof since from [4] it is known that X is a $R - T_1-$space if and only if $\mathrm{cl}_i(\{x\}) = \{x\}$ holds for all $x \in X$and $i \in \{1, 2\}$. Also, each $R - T_1-$space is $S - T_1-$space. □

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