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Coefficient Estimates of Bi-Bazilevič Functions defined by $q-$ Ruscheweyh differential operator associated with Horadam polynomials

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Abstract In this paper, we introduce and investigate a new subclass of the function class Σ of bi-univalent functions of the Bazilevič type defined in the open unit disk, which are associated with the Horonam polynomial and satisfy some subordination conditions. Furthermore, we find estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in the new subclass introduced here. Several (known or new) consequences of the results are also pointed out.

1 Introduction, Definitions and Preliminaries

Let $\mathcal{A}$ denote the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk

$$\mathbb{U} = \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

Further, by $\mathcal{S}$ we shall denote the class of all functions $f(z)$ in $\mathcal{A}$ which are univalent in $\mathbb{U}$ and indeed normalized by

$$f(0) = f'(0) - 1 = 0.$$

Some of the important and well-investigated subclasses of the univalent function class $\mathcal{S}$ include (for example) the class $\mathcal{S}^*(\alpha)$ $(0 \leqq \alpha < 1)$ of starlike functions of order α in $\mathbb{U}$ and the class $\mathcal{K}(\alpha)$ $(0 \leqq \alpha < 1)$ of convex functions of order α in $\mathbb{U}$. It is well known that every function $f \in \mathcal{S}$ has an inverse f^{-1}, defined by

$$f^{-1}\big(f(z)\big) = z \qquad (z \in \mathbb{U})$$

and

$$f\big(f^{-1}(w)\big) = w \qquad \left(|w| < r_0(f);\, r_0(f) \geqq \frac{1}{4}\right),$$

where

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2 a_3 + a_4)w^4 + \cdots. \tag{1.2}$$

A function $f \in \mathcal{A}$ is said to be bi-univalent in $\mathbb{U}$ if both $f(z)$ and $f^{-1}(z)$ are univalent in $\mathbb{U}$. Let Σ denote the class of bi-univalent functions in $\mathbb{U}$ given by (1.1).

An analytic function f is subordinate to another analytic function g, written as follows:

$$f(z) \prec g(z) \qquad (z \in \mathbb{U}),$$

provided that there exists an analytic function (that is, Schwar'z function) $\omega(z)$ defined on $\mathbb{U}$ with

$$\omega(0) = 0 \qquad \text{and} \qquad |\omega(z)| < 1 \qquad (z \in \mathbb{U})$$

such that (see, for details, [16])

$$f(z) = g\big(\omega(z)\big) \qquad (z \in \mathbb{U}).$$

The Horadam polynomials $h_n(x)$ are defined by the following recurrence relation (see [9]):

$$h_n(x) = pxh_{n-1}(x) + qh_{n-2}(x) \quad (x \in \mathbb{R};\ n \in \mathbb{N} = \{1, 2, 3, \cdots\}) \tag{1.3}$$

with

$$h_1(x) = a \quad \text{and} \quad h_2(x) = bx,$$

for some real constants a, b, p and q. The characteristic equation of the recurrence relation (1.3) is given by

$$t^2 - pxt - q = 0.$$

This equation has the following two real roots:

$$\alpha = \frac{px + \sqrt{p^2x^2 + 4q}}{2} \qquad \text{and} \qquad \beta = \frac{px - \sqrt{p^2x^2 + 4q}}{2}.$$

Remark 1.1. By selecting the particular values of a, b, p and q, the Horadam polynomial $h_n(x)$ reduces to several known polynomials. Some of these special cases are recorded below.

1. Taking $a = b = p = q = 1$, we obtain the Fibonacci polynomials $F_n(x)$.

2. Taking $a = 2$ and $b = p = q = 1$, we get the Lucas polynomials $L_n(x)$.

3. Taking $a = q = 1$ and $b = p = 2$, we have the Pell polynomials $P_n(x)$.

4. Taking $a = b = p = 2$ and $q = 1$, we find the Pell-Lucas polynomials $Q_n(x)$.

5. Taking $a = b = 1$, $p = 2$ and $q = -1$, we obtain the Chebyshev polynomials $T_n(x)$ of the first kind.

6. Taking $a = 1$, $b = p = 2$ and $q = -1$, we have the Chebyshev polynomials $U_n(x)$ of the second kind.

These polynomials, the families of orthogonal polynomials and other special polynomials, as well as their extensions and generalizations, are potentially important in a variety of disciplines in many branches of science, especially in the mathematical, statistical and physical sciences. For more information associated with these polynomials, see [8, 9, 13, 14]. The generating function of the Horadam polynomials $h_n(x)$ is given as follows (see [10]):

$$H_n(x, z) = \sum_{n=1}^{\infty} h_n(x) z^{n-1} = \frac{a + (b - ap)xz}{1 - prx - qz^2}. \tag{1.4}$$

The study of operators plays an important rôle in Geometric Function Theory in Complex Analysis and its related fields. Many derivative and integral operators can be written in terms of convolution of certain analytic functions. It is observed that this formalism brings an ease in further mathematical exploration and also helps to better understand the geometric properties of such operators. The convolution or the Hadamard product of two functions $f, g \in \mathcal{A}$ is denoted by $f * g$ and is defined as follows:

$$(f * g)(z) := z + \sum_{n=2}^{\infty} a_n b_n z^n =: (g * f)(z), \tag{1.5}$$

where $f(z)$ is given by (1.1) and

$$g(z) = z + \sum_{n=2}^{\infty} b_n z^n.$$

In terms of the Hadamard product (or convolution). Now we recall here the notion of q-*operator* i.e. q-*difference operator* that play vital role in the theory of hypergeometric series, quantum physics and in the operator theory. The application of q-*calculus* was initiated by Jackson [11], recently Kanas and Răducanu [12] have used the fractional q-*calculus operators* in investigations of certain classes of functions which are analytic in $\mathbb{U}$.

Let $0 < q < 1$. For any non-negative integer n, the q-*integer number* n is defined by

$$[n]_q = \frac{1-q^n}{1-q} = 1 + q + \cdots + q^{n-1}, \qquad [0]_q = 0. \tag{1.6}$$

In general, we will denote

$$[x]_q = \frac{1-q^x}{1-q}$$

for a non-integer number x. Also the q-*number shifted factorial* is defined by

$$[n]_q! = [n]_q[n-1]_q ... [2]_q[1]_q, \qquad [0]_q! = 1. \tag{1.7}$$

Clearly,

$$\lim_{q\to 1^-} [n]_q = n \qquad \text{and} \qquad \lim_{q\to 1^-} [n]_q! = n!.$$

For $0 < q < 1$, the Jackson's q-*derivative operator* (or q-*difference operator*) of a function $f \in \mathcal{A}$ given by (1.1) defined as follows [11]:

$$\mathfrak{D}_q f(z) = \begin{cases} \dfrac{f(z) - f(qz)}{(1-q)z} & \text{for} \quad z \neq 0 \\ \\ f'(0) & \text{for} \quad z = 0 \end{cases}, \tag{1.8}$$

$\mathfrak{D}_q^0 f(z) = f(z)$, and $\mathfrak{D}_q^m f(z) = \mathfrak{D}_q(\mathfrak{D}_q^{m-1} f(z))$, $m \in \mathbb{N} = \{1, 2, \ldots\}$. From (1.8), we have

$$\mathfrak{D}_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1} \qquad (z \in \mathbb{U}), \tag{1.9}$$

where $[n]_q$ is given by (1.6). For a function $\psi(z) = z^n$, we obtain

$$\mathfrak{D}_q \psi(z) = \mathfrak{D}_q z^n = \frac{1-q^n}{1-q} z^{n-1} = [n]_q z^{n-1}$$

and

$$\lim_{q\to 1^-} \mathfrak{D}_q \psi(z) = \lim_{q\to 1^-} \left([n]_q z^{n-1}\right) = n z^{n-1} = \psi'(z),$$

where ψ' is the ordinary derivative.

Let $t \in \mathbb{R}$ and $n \in \mathbb{N}$. The q-*generalized Pochhammer symbol* is defined by

$$[t; n]_q = [t]_q[t+1]_q[t+2]_q ... [t+n-1]_q \tag{1.10}$$

and for $t > 0$ the q-*gamma function* is defined by

$$\Gamma_q(t+1) = [t]_q \Gamma_q(t) \qquad \text{and} \qquad \Gamma_q(1) = 1. \tag{1.11}$$

Using the q-*difference operator*, Kannas and Raducanu [12] defined the Ruscheweyh q-*differential operator* as below: For $f \in \mathcal{A}$,

$$\mathcal{R}_q^\delta f(z) = f(z) * F_{q,\delta+1}(z) \qquad (\delta > -1,\ z \in \mathbb{U}) \tag{1.12}$$

where

$$F_{q,\delta+1}(z) = z + \sum_{n=2}^{\infty} \frac{\Gamma_q(n+\delta)}{[n-1]_q!\,\Gamma_q(1+\delta)} z^n = z + \sum_{n=2}^{\infty} \frac{[\delta+1;n]_q}{[n-1]_q!} z^n. \tag{1.13}$$

Making use of (1.12) and (1.13), Aldweby and Darus[1] defined the $q-$analogue of Ruschewey operator $\mathcal{R}_q^\delta : \mathcal{A} \to \mathcal{A}$ as follows:

$$\begin{aligned} \mathcal{R}_q^\delta f(z) &= z + \sum_{n=2}^{\infty} \frac{\Gamma_q(n+\delta)}{[n-1]_q!\,\Gamma_q(1+\delta)} a_n z^n \qquad (z \in \mathbb{U}). \\ &= z + \sum_{n=2}^{\infty} \Lambda_n(q,\delta) a_n z^n \qquad (z \in \mathbb{U}). \end{aligned} \tag{1.14}$$

where

$$\Lambda_n := \Lambda_n(q,\delta) = \frac{\Gamma_q(n+\delta)}{[n-1]_q!\,\Gamma_q(1+\delta)}. \tag{1.15}$$

As $q \to 1^-$,we note that

$$\begin{aligned} \mathcal{R}_q^0 f(z) &= f(z), \\ \mathcal{R}_q^1 f(z) &= z\mathfrak{D}_q f(z) = zf'(z), \end{aligned}$$

Recently, especially after its revival by Srivastava *et al.* [20], there has been triggering interest in the study of the bi-univalent function class Σ leading to non-sharp coefficient estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ in (1.1). However, the coefficient problem for each of the following Taylor-Maclaurin coefficients:

$$|a_n| \qquad (n \in \mathbb{N} \setminus \{1,2\})$$

is still an open problem (see [2, 3, 4, 15, 22]). Motivated largely by (and following the work of) Srivastava *et al.* [20], many researchers (see, for example, [5, 7, 21]and references cited therein) have recently introduced and investigated several interesting subclasses of the bi-univalent function class Σ and they have found non-sharp estimates on the corresponding first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. Several authors have discussed various subfamilies of the well-known Bazilevič functions (see, for details, [6]; see also [19]) of type λ for different perspective. Motivated primarily by the recent work of Deniz [5] and Goyal and Goswami[7], we introduce here a new subfamily of Bazilevič type functions belonging to the function class Σdefined by $q-$ Ruscheweyh differential operator associated with Horadam polynomials. For this new subfamily of Bazilevič type functions, we find estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ and the Fekete-Szegö inequalities. Several closely-related function classes are also considered and relevant connections to earlier known results are pointed out.

Definition 1.2. A function $f \in \Sigma$ given by (1.1) is said to be in the class $\mathcal{B}_\Sigma(\beta,\vartheta;H_n(x))$ if the following conditions are satisfied:

$$e^{i\beta}\left(\frac{z^{1-\vartheta}\left(\mathcal{R}_q^\delta f(z)\right)'}{[\mathcal{R}_q^\delta f(z)]^{1-\vartheta}}\right) \prec (H_n(x;z)+1-a)\cos\beta + i\sin\beta \tag{1.16}$$

and

$$e^{i\beta}\left(\frac{w^{1-\vartheta}\left(\mathcal{R}_q^\delta g(w)\right)'}{[\mathcal{R}_q^\delta g(w)]^{1-\vartheta}}\right) \prec (H_n(x;w)+1-a)\cos\beta + i\sin\beta, \tag{1.17}$$

where

$$\beta \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right), \quad \vartheta \geqq 0 \qquad \text{and} \qquad z,w \in \mathbb{U}$$

and the function g is given by (1.2).

Example 1.3. For $\vartheta = 0$, we have

$$\mathcal{B}_{\Sigma}(\beta, 0; H_n(x)) =: \mathcal{S}_{\Sigma}(\beta; H_n(x)),$$

in which $\mathcal{S}_{\Sigma}(\beta; h)$ denotes the class of functions $f \in \Sigma$ given by (1.1) and satisfying the following conditions:

$$e^{i\beta}\left(\frac{z\left(\mathcal{R}_q^{\delta} f(z)\right)'}{\mathcal{R}_q^{\delta} f(z)}\right) \prec (H_n(x; z) + 1 - a)\cos\beta + i\sin\beta \tag{1.18}$$

and

$$e^{i\beta}\left(\frac{w\left(\mathcal{R}_q^{\delta} g(w)\right)'}{\mathcal{R}_q^{\delta} g(w)}\right) \prec (H_n(x; w) + 1 - a)\cos\beta + i\sin\beta \tag{1.19}$$

where $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $z, w \in \mathbb{U}$ and the function g is given by (1.2).

Example 1.4. For $\vartheta = 1$, we get

$$\mathcal{B}_{\Sigma}(\beta, 1; H_n(x)) =: \mathcal{G}_{\Sigma}(\beta; H_n(x)),$$

in which $\mathcal{G}_{\Sigma}(\beta; h)$ denotes the class of functions $f \in \Sigma$ given by (1.1) and satisfying the following conditions:

$$e^{i\beta}\left(\mathcal{R}_q^{\delta} f(z)\right)' \prec (H_n(x; z) + 1 - a)\cos\beta + i\sin\beta \tag{1.20}$$

and

$$e^{i\beta}\left(\mathcal{R}_q^{\delta} g(w)\right)' \prec (H_n(x; w) + 1 - a)\cos\beta + i\sin\beta \tag{1.21}$$

where $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, $z, w \in \mathbb{U}$ and the function g is given by (1.2).

We note here that, for $q = 1$ and $\delta = 0$, the classes $\mathcal{S}_{\Sigma}(\beta, H_n(x))$ and $\mathcal{G}_{\Sigma}(\beta, H_n(x))$ would reduce to the interesting subclasses given by Examples 1.5 and1.6 below.

Example 1.5. For $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we have

$$\mathcal{S}_{\Sigma}(\beta; H_n(x)) =: \mathcal{S}^*_{\Sigma}(\beta; H_n(x)),$$

in which $\mathcal{S}^*_{\Sigma}(\beta; H_n(x))$ denotes the class of functions $f \in \Sigma$ given by (1.1) and satisfying the following conditions:

$$e^{i\beta}\left(\frac{zf'(z)}{f(z)}\right) \prec (H_n(x; z) + 1 - a)\cos\beta + i\sin\beta$$

and

$$e^{i\beta}\left(\frac{wg'(w)}{g(w)}\right) \prec (H_n(x; w) + 1 - a)\cos\beta + i\sin\beta,$$

where $z, w \in \mathbb{U}$ and the function g is given by (1.2).

Example 1.6. For $\beta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we get

$$\mathcal{G}_{\Sigma}(\beta; H_n(x)) =: \mathcal{G}^*_{\Sigma}(\beta; H_n(x)),$$

in which $\mathcal{G}^*_{\Sigma}(\beta; H_n(x))$ denotes the class of functions $f \in \Sigma$ given by (1.1) and satisfying the following conditions:

$$e^{i\beta}\left(f'(z)\right) \prec (H_n(x; z) + 1 - a)\cos\beta + i\sin\beta$$

and

$$e^{i\beta}\left(g'(w)\right) \prec (H_n(x; z) + 1 - a)\cos\beta + i\sin\beta,$$

where $z, w \in \mathbb{U}$ and the function g is given by (1.2).

In order to derive our main results in Section 2 involving the estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions in the Bi-Bazilevič type subclass $\mathcal{B}_\Sigma(\beta,\vartheta;H_n)$ of the bi-univalent function class Σ, we shall need such coefficient inequalities as those asserted by the following lemmas :

Lemma 1.7. *(see [17]). If a function $p \in \mathcal{P}$ is given by*

$$p(z) = 1 + u_1 z + u_2 z^2 + \cdots \qquad (z \in \mathbb{U}),$$

then

$$|u_k| \leqq 2 \qquad (k \in \mathbb{N}),$$

where $\mathcal{P}$ is the family of all functions p, analytic in $\mathbb{U}$, for which

$$p(0) = 1 \qquad \text{and} \qquad \Re\big(p(z)\big) > 0 \qquad (z \in \mathbb{U}).$$

Mapping and many other properties and characteristics of various families of analytic, univalent and bi-univalent functions, including (for example) the Bi-Bazilevič functions associated with Horadam polynomial being considered here, are potentially useful in several problems in mathematical, physical and engineering sciences.

2 Main Result

We begin by finding the estimates on the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ in (1.1) for functions in the class $\mathcal{B}_\Sigma(\beta,\vartheta;H_n(x))$.

Theorem 2.1. *Let the function $f(z)$ given by (1.1) be in the class $\mathcal{B}_\Sigma(\beta,\vartheta;h)$, then*

$$|a_2| \leqq |bx|\sqrt{\frac{2|bx|\cos\beta}{|\left[(\vartheta-1)(\vartheta+2)\Lambda_2^2+2(\vartheta+2)\Lambda_3\right]b^2x^2\cos\beta-2[pbx^2+qa](\vartheta+1)^2|}} \tag{2.1}$$

and

$$|a_3| \leqq \frac{|bx|\cos\beta}{(\vartheta+2)\Lambda_3}+\frac{|bx|^2\cos^2\beta}{(\vartheta+1)^2\Lambda_2^2}, \tag{2.2}$$

where the coefficients Λ_n are given by (1.15).

Proof. Then there are two analytic functions $u, v : \mathbb{U} \longrightarrow \mathbb{U}$ given by

$$u(z) = u_1 z + u_2 z^2 + u_3 z^3 + \cdots \qquad (z \in \mathbb{U}) \tag{2.3}$$

and

$$v(w) = v_1 w + v_2 w^2 + v_3 w^3 + \cdots \qquad (w \in \mathbb{U}), \tag{2.4}$$

with

$$u(0) = v(0) = 0 \qquad \text{and} \qquad \max\{|u(z)|, |v(w)|\} < 1 \quad (z, w \in \mathbb{U}).$$

It is well-known that, if

$$\max\{|u(z)|, |v(w)|\} < 1 \qquad (z, w \in \mathbb{U}),$$

then

$$|u_j| \leqq 1 \qquad \text{and} \qquad |v_j| \leqq 1 \qquad (\forall\, j \in \mathbb{N}). \tag{2.5}$$

It follows from (1.16) and (1.17) that

$$e^{i\beta}\left(\frac{z^{1-\vartheta}\big(\mathcal{R}_q^\delta f(z)\big)'}{[\mathcal{R}_q^\delta f(z)]^{1-\vartheta}}\right) = \big((H_n(x;z)+1-a)\big)\cos\beta + i\sin\beta$$

and

$$e^{i\beta}\left(\frac{w^{1-\vartheta}\left(\mathcal{R}_q^\delta g(w)\right)'}{[\mathcal{R}_q^\delta g(w)]^{1-\vartheta}}\right) = ((H_n(x;w)+1-a)-a)\cos\beta + i\sin\beta,$$

Equivalently

$$\begin{aligned} e^{i\beta}\left(\frac{z^{1-\vartheta}\left(\mathcal{R}_q^\delta f(z)\right)'}{[\mathcal{R}_q^\delta f(z)]^{1-\vartheta}}\right) &= e^{i\beta}(\vartheta+1)\Lambda_2\, a_2 z \\ &+ e^{i\beta}\left[\left(2(\vartheta+2)\Lambda_3 + \frac{(\vartheta-1)(\vartheta+2)}{2}\Lambda_2^2\right)a_2^2 - (\vartheta+2)\Lambda_3\, a_3\right]z^2 \\ +\cdots. \end{aligned} \tag{2.6}$$

and

$$\begin{aligned} e^{i\beta}\left(\frac{w^{1-\vartheta}\left(\mathcal{R}_q^\delta g(w)\right)'}{[\mathcal{R}_q^\delta g(w)]^{1-\vartheta}}\right) &= -e^{i\beta}(\vartheta+1)\Lambda_2 a_2 w \\ &+ e^{i\beta}\left[\left(2(\vartheta+2)\Lambda_3 + \frac{(\vartheta-1)(\vartheta+2)}{2}\Lambda_2^2\right)a_2^2 - (\vartheta+2)\Lambda_3\, a_3\right]w^2 \\ +\cdots. \end{aligned} \tag{2.7}$$

Using (1.3) (1.4)and (2.3),(2.3)we get

$$\begin{aligned} &((H_n(x;z)+1-a))\cos\beta + i\sin\beta \\ &= \left(h_1(x) + h_2(x)u(z) + h_3(x)u^2(z) + \cdots\right)\cos\beta + i\sin\beta \\ &= \left(h_2(x)u_1 z + \left[h_2(x)u_2 + h_3(x)u_1^2\right]z^2 + \cdots\right)\cos\beta + i\sin\beta. \end{aligned} \tag{2.8}$$

Similarly,

$$\begin{aligned} &((H_n(x;w)+1-a)-a)\cos\beta + i\sin\beta \\ &= \left(h_1(x) + h_2(x)v(w) + h_3(x)v^2(w) + \cdots.\right)\cos\beta + i\sin\beta \\ &= \left(h_2(x)v_1 w + \left[h_2(x)v_2 + h_3(x)v_1^2\right]w^2 + \cdots\right)\cos\beta + i\sin\beta \\ &. \end{aligned} \tag{2.9}$$

Thus by (2.6) -(2.9) and Comparing coefficients we get

$$e^{i\beta}(\vartheta+1)\Lambda_2\, a_2 = h_2(x)u_1\cos\beta, \tag{2.10}$$

$$e^{i\beta}\left(\frac{(\vartheta-1)(\vartheta+2)}{2}\Lambda_2^2\, a_2^2 + (\vartheta+2)\Lambda_3\, a_3\right) = \left(h_2(x)u_2 + h_3(x)u_1^2\right)\cos\beta, \tag{2.11}$$

$$-e^{i\beta}(\vartheta+1)\Lambda_2 a_2 = h_2(x)v_1\cos\beta \tag{2.12}$$

and

$$e^{i\beta}\left[\left(2(\vartheta+2)\Lambda_3 + \frac{(\vartheta-1)(\vartheta+2)}{2}\Lambda_2^2\right)a_2^2 - (\vartheta+2)\Lambda_3\, a_3\right] = \left(h_2(x)v_2 + h_3(x)v_1^2\right)\cos\beta. \tag{2.13}$$

From (2.10) and (2.12), we find that

$$a_2 = \frac{u_1 e^{-i\beta}\cos\beta}{(\vartheta+1)\Lambda_2} = -\frac{v_1 e^{-i\beta}\cos\beta}{(\vartheta+1)\Lambda_2}, \tag{2.14}$$

which implies that

$$u_1 = -v_1 \tag{2.15}$$

and

$$2\left[(\vartheta+1)\Lambda_2\, a_2\right]^2 = h_2^2(u_1^2+v_1^2)\, e^{-2i\beta}\cos^2\beta.$$

$$a_2^2 = \frac{h_2^2(u_1^2+v_1^2)\, e^{-2i\beta}\cos^2\beta.}{2(\vartheta+1)^2\Lambda_2^2} \tag{2.16}$$

$$u_1^2+v_1^2 = \frac{2\left[(\vartheta+1)\Lambda_2\, a_2\right]^2}{h_2^2 e^{-2i\beta}\cos^2\beta}. \tag{2.17}$$

Upon adding (2.11)and (2.13), if we make use of (2.14) and (2.15), we obtain

$$e^{i\beta}\left[(\vartheta-1)(\vartheta+2)\Lambda_2^2 + 2(\vartheta+2)\Lambda_3\right] a_2^2 = h_2(u_2+v_2)\cos\beta + h_3(u_1^2+v_1^2)\cos\beta \tag{2.18}$$

which yields

$$a_2^2 = \frac{h_2^3(u_2+v_2)e^{-i\beta}\ \cos\beta}{\left[(\vartheta-1)(\vartheta+2)\Lambda_2^2+2(\vartheta+2)\Lambda_3\right] h_2^2\cos\beta - 2h_3(\vartheta+1)^2 e^{-i\beta}}. \tag{2.19}$$

$$|a_2|^2 \leq \frac{2b^3x^3\cos\beta}{\left[(\vartheta-1)(\vartheta+2)\Lambda_2^2+2(\vartheta+2)\Lambda_3\right] b^2x^2\cos\beta - 2[pbx^2+qa](\vartheta+1)^2}, \tag{2.20}$$

which easily yields the bound on $|a_2|$ as asserted in (2.1).

Next, in order to find the bound on $|a_3|$, by subtracting (2.13) from (2.11), we get

$$\begin{aligned} e^{i\beta}\left[2(\vartheta+2)\Lambda_3 a_3 - 2(\vartheta+2)\Lambda_3 a_2^2\right] &= h_2(u_2-v_2)\cos\beta + h_3(u_1^2-v_1^2)\cos\beta \\ a_3 &= \frac{h_2(u_2-v_2)e^{-i\beta}\cos\beta}{2(\vartheta+2)\Lambda_3} + a_2^2 \end{aligned} \tag{2.21}$$

It follows from (2.14), (2.15) and (2.21) that

$$a_3 = \frac{h_2(u_2-v_2)e^{-i\beta}\cos\beta}{2(\vartheta+2)\Lambda_3} + \frac{h_2^2(u_1^2+v_1^2)e^{-2i\beta}\cos^2\beta}{2(\vartheta+1)^2\Lambda_2^2}.$$

Applying Lemma 1.7, and (2.5) once again for the coefficients u_2 and v_2, we readily get

$$|a_3| \leqq \frac{|bx|\cos\beta}{(\vartheta+2)\Lambda_3} + \frac{|bx|^2\cos^2\beta}{(\vartheta+1)^2\Lambda_2^2}.$$

This completes the proof of Theorem 2.1. □

In the next theorem, we present the Fekete-Szegö inequality for $f \in \mathcal{B}_\Sigma(\beta,\vartheta;H_n(x))$.

Theorem 2.2. *For $\vartheta \geqq 0$, $\beta \in \left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, and $x,\mu \in \mathbb{R}$, let $f \in \mathcal{A}$ be in the class $\mathcal{B}_\Sigma(\beta,\vartheta;H_n(x))$. Then*

$$\left|a_3 - \mu a_2^2\right| \leqq = \begin{cases} \frac{|bx|cos\beta}{(\vartheta+2)\Lambda_3} & for \quad 0 \leqq |\psi(\mu,x)| \leqq \frac{cos\beta}{2(\vartheta+2)\Lambda_3} \\ |bx|\, cos\beta\, |\psi(\mu,x)| & for \quad 0 \leqq |\psi(\mu,x)| \leqq \frac{cos\beta}{2(\vartheta+2)\Lambda_3} \end{cases}, \tag{2.22}$$

where

$$\psi(\mu,x) = \frac{(1-\mu)\, h_2^2 e^{-i\beta}\ \cos\beta}{\left[(\vartheta-1)(\vartheta+2)\Lambda_2^2+2(\vartheta+2)\Lambda_3\right] h_2^2\cos\beta - 2h_3(\vartheta+1)^2 e^{-i\beta}}.$$

Proof. It follows from (2.21) that

$$\begin{aligned} a_3-\mu a_2^2 &= \frac{h_2(u_2-v_2)e^{-i\beta}\cos\beta}{2(\vartheta+2)\Lambda_3}+(1-\mu)\,a_2^2 \\ &= \frac{h_2(u_2-v_2)e^{-i\beta}\cos\beta}{2(\vartheta+2)\Lambda_3} \\ &\quad + \frac{(1-\mu)\,h_2^3(u_2+v_2)e^{-i\beta}\ \cos\beta}{\left[(\vartheta-1)(\vartheta+2)\Lambda_2^2+2(\vartheta+2)\Lambda_3\right]h_2^2\cos\beta-2h_3(\vartheta+1)^2e^{-i\beta}} \\ &= h_2(x)\left[\left(\psi(\mu,x)+\frac{e^{-i\beta}\cos\beta}{2(\vartheta+2)\Lambda_3}\right)u_2\right. \\ &\quad \left.+\left(\psi(\mu,x)-\frac{e^{-i\beta}\cos\beta}{2(\vartheta+2)\Lambda_3}\right)v_2\right], \end{aligned}$$

where

$$\psi(\mu,x)=\frac{(1-\mu)\,h_2^2e^{-i\beta}\ \cos\beta}{\left[(\vartheta-1)(\vartheta+2)\Lambda_2^2+2(\vartheta+2)\Lambda_3\right]h_2^2\cos\beta-2h_3(\vartheta+1)^2e^{-i\beta}}.$$

Thus, by using (1.3),in above equation we get (2.22). We have thus completed the proof of Theorem 2.2. □

Corresponding essentially to Examples 1.3 and 1.4, Theorem 2.1 yields the following corollaries.

Corollary 2.3. *Let the function $f(z)$ given by* (1.1) *be in the class $\mathcal{S}_\Sigma(\beta;H_n(x))$. Then*

$$|a_2|\leqq|bx|\sqrt{\frac{2|bx|\cos\beta}{|\left[4\Lambda_3-2\Lambda_2^2\right]b^2x^2\cos\beta-2[pbx^2+qa]|}} \tag{2.23}$$

and

$$|a_3|\leqq\frac{|bx|\cos\beta}{2\Lambda_3}+\frac{|bx|^2\cos^2\beta}{\Lambda_2^2}, \tag{2.24}$$

where the coefficients Λ_n are given by (1.15).

Corollary 2.4. *Let the function $f(z)$ given by* (1.1) *be in the class $\mathcal{G}_\Sigma(\beta;H_n(x))$, then*

$$|a_2|\leqq|bx|\sqrt{\frac{2|bx|\cos\beta}{|6\Lambda_3b^2x^2\cos\beta-8[pbx^2+qa]|}} \tag{2.25}$$

and

$$|a_3|\leqq\frac{|bx|\cos\beta}{3\Lambda_3}+\frac{|bx|^2\cos^2\beta}{4\Lambda_2^2}, \tag{2.26}$$

where the coefficients Λ_n are given by (1.15).

Concluding Remark: Suitably assuming $\delta=0$ and letting $q\to 1^-$, we can easily state the analogous results for the function class $\mathcal{S}^*_\Sigma(\beta;H_n(x))$ and $\mathcal{G}^*_\Sigma(\beta;H_n(x)$. By taking $\vartheta=0$ and $\vartheta=1$ in Theorem 2.2,we can deduce Fekete-Szegö inequality for $f\in\mathcal{S}_\Sigma(\beta,H_n(x))$ and $f\in\mathcal{G}_\Sigma(\beta,H_n(x))$. In their special cases when $\beta=0$, the results presented in this paper would lead to various other (new or known) results, some of which for the function class Σ were considered in earlier works (see, for example, [4, 5, 20, 21]). By selecting the particular values of a, b, p and q, the Horadam polynomial $h_n(x)$ as mentioned in Remark 1.1, our results reduces to several known polynomials. In closing, we indicate possible directions for future work. It would be of interest to replace the Horadam polynomial, by Gegenbauer polynomial. Also, it would be interesting to enlarge the class of study to meromorphic bi-univalent functions.

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