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# ON RINGS WHOSE CYCLICS ARE UTUMI MODULES

Soumitra Das and Ardeline M. Buhphang

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**Abstract** A right $R$-module $M$ is called a *Utumi module* (*U-module*, in short) if for any two submodules $A$ and $B$ of $M$ with $A \cong B$ and $A \cap B = 0$, there exist two summands $K$ and $L$ of $M$ such that $A$ essential in $K$, $B$ essential in $L$ and $K \oplus L$ is a direct summand of $M$. Rings over which every cyclic right module is Utumi, are called right *CU-rings*. In this work we have characterized commutative (right duo) CU-rings. Also, it is shown that a right nonsingular right CU-ring is a direct product of a semisimple and a reduced square-free ring; moreover, the ring itself satisfy the internal cancellation property. Our approach is independent of the results obtained by Ibrahim, Kosan, Qyung and Yousif recently.

## 1 Introduction

Throughout this article, rings are associative with unity, and modules are unital right modules. Let $R$ be a ring. If $N$ is a submodule of $M$ (denoted by $N \leq M$) then $N \leq^{ess} M$ (respectively, $N \leq^{\oplus} M$) denotes that $N$ is essential in $M$ ($N$ is a direct summand of $M$).

For undefined terms, we refer to [3] [16], [17] and [21].

An $R$-module $M$ is called a *Utumi-module* if for any two submodules $A$ and $B$ of $M$ with $A \cong B$ and $A \cap B = 0$, there exist two summands $K$ and $L$ of $M$ such that $A \leq^{ess} K$, $B \leq^{ess} L$ and $K \oplus L \leq^{\oplus} M$ (see [11, Definition 2.1]). The ring $R$ is called *right Utumi* if the right module $R_R$ is Utumi. The class of Utumi-modules provides a positive answer for an open question due to Crawley and Jónsson "whether the finite exchange property always implies the full exchange property" (see [11, Theorem 5.2]). Furthermore, this class of modules properly contains the class of quasi-continuous modules (a module $M$ is known as *quasi-continuous* if every submodule of $M$ is essential in a direct summand of $M$ and direct sum of two direct summands of $M$ intersecting trivially is again a direct summand of $M$). Rings all of whose (finitely generated, free, cyclic) $R$-modules are quasi-continuous have been studied by several authors (see, for instance, [13]). This motivated us to study rings all of whose (finitely generated, cyclic, free) $R$-modules are Utumi. Also, it is worth mentioning that in [5] rings all of whose (finitely generated, free, cyclic) $R$-modules are dual-Utumi were studied.

Recall that following Azumaya a module $M$ is said to be *injective relative to the module* $N$ or $N$*-injective* if, for any submodule $A$ of $N$, every homomorphism $A \longrightarrow M$ can be extended to a homomorphism $N \longrightarrow M$. A module is *injective* if it is injective with respect to any module. A module $M$ is *quasi-injective* if it is $M$-injective. A module $M$ is *pseudo-*$N$*-injective* if, for any submodule $K$ of $N$, every monomorphism $K \longrightarrow M$ can be extended to a homomorphism from $N$ into $M$. A module $M$ is *pseudo-injective* or *automorphism-invariant* if it is pseudo-$M$-injective. A module is *square-free* if it does not contain a direct sum of two non-zero isomorphic submodules. A submodule $K$ of a module $M$ is *essential* in $M$ if $K \cap L \neq 0$ for any non-zero submodule $L$ of $M$. A submodule $K$ of the module $M$ is *closed* in $M$ if $K = L$ for every submodule $L$ of $M$ which is an essential extension of the module $K$. We denote by $Z(M)$ the *singular submodule* of the right $R$-module $M$, i.e., $Z(M)$ consists of all elements $m \in M$ such that right annihialtor of $m$ is an essential right ideal of the ring $R$. A module $M$ is *nonsingular* if $Z(M) = 0$. Consider the following conditions on an $R$-module $M$:

$(C1 - condition)$ For every submodule $A$ of $M$, there is a direct summand $K \leq M$ such that $A \leq^{ess} K$.

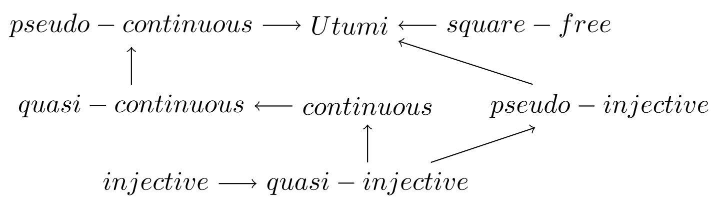
$(C2 - condition)$ If $A \leq M$ such that $A$ is isomorphic to a summand of $M$, then $A$ is a summand of $M$.

$(C3 - condition)$ If $M_1$ and $M_2$ are summands of $M$ with $M_1 \cap M_2 = 0$, then $M_1 \oplus M_2$ is a summand of $M$.

$(C4 - condition)$ If $M = M_1 \oplus M_2$, and $f : M_1 \longrightarrow M_2$ is a homomorphism with $ker(f) \leq^{\oplus} M_1$, then $Im(f) \leq^{\oplus} M_2$.

A module $M$ is called an *extending* or a *CS-module* if it satisfies condition $C1$. A module $M$ is called *continuous* if it satisfies both $(C1)$ and $(C2)$ conditions; $M$ is called quasi-continuous if it satisfies both $(C1)$ and $(C3)$ conditions. A module $M$ is called *pseudo-continuous* if it satisfies both $(C1)$ and $(C4)$ conditions (see [2], [6]). Note that every Utumi-module is a $C4$-module (see [11, Lemma 2.8]).

We have the following implications for the above mentioned module theoretic properties which are of interest to us:




The paper is organized as follows. In Section 2, we study various rings whose modules are Utumi and review the known results on the subclass (e.g. injective modules) of the class of Utumi modules and prove similar results for Utumi modules. Section 3 is devoted to those rings over which all cyclic modules are Utumi. A characterization for the commutative (duo) rings over which all cyclic modules are Utumi is obtained in Corollary 3.15, which tells us that precisely these are the arithmetical rings, that is, the distributive law $A \cap (B + C) = (A \cap B) + (A \cap C)$ holds for any three ideals $A$, $B$, $C$ of such a ring $R$. In Theorem 3.20, we prove that if a ring $R$ is right (respectively, left) nonsingular and if every cyclic right (respectively, left) $R$-module is Utumi, then $R$ can be decomposed as $S \oplus T$ such that $S$ is semisimple and $T$ is a reduced square-free ring.

## 2 From injective modules to Utumi modules

Let us consider the following:

**Question 2.1.** *: Let $n \geq 1$. What is the structure of a ring $R$ over which every $n$-generated right $R$-module is Utumi* ?

In order to answer the above question we make a few observations about the class of Utumi modules and note them below.

**Lemma 2.2.** (cf. [11, Proposition 3.6]) Let $N$ and $M$ be $R$-modules and $f : N \longrightarrow M$ be an $R$-monomorphism. If $N \oplus M$ is a Utumi-module, then $Im(f)$ is a quasi-injective direct summand of $M$.

*Proof.* Let $L = N \oplus M$ be a Utumi-module. By [11, Lemma 2.8], $Im(f)$ is a direct summand of $M$. It follows that $N \oplus N$ is isomorphic to a direct summand of $L$ and hence $N \oplus N$ is also a Utumi-module by [11, Proposition 3.2]. Thus, $Im(f)$ is quasi-injective by [11, Corollary 3.7]. □

The following remark was included in [6, Remark 2.31].

**Remark 2.3.** It is not difficult to show that a ring $R$ is semisimple iff the direct sum of any two C4-modules is a C4-module, $R$ is right hereditary iff every factor module of an injective right

$R$-module is a C4-module, $R$ is regular iff every 2-generated submodule of a projective right $R$-module is a C4-module iff every two generated right ideal of $R$ is a C4-module, and $R$ is right noetherian iff every direct sum of injective right $R$-modules is a C4-module.

Now we shall incorporate the above remark for Utumi-modules.

Recall that a ring $R$ is called *right V-ring* if every simple right $R$-module is injective. A ring $R$ is called *right hereditary* if every submodule of a projective right $R$-module is projective, equivalently, if every factor module of an injective right $R$-module is injective.

**Theorem 2.4.** *Let $R$ be a ring.*

*(i) $R$ is a right V-ring if and only if every finitely cogenerated right $R$-module is Utumi.*

*(ii) $R$ is a right hereditary ring if and only if every factor module of an injective right $R$-module is a Utumi-module.*

*Proof.* (1) Let $M$ be a simple right $R$-module. Since $M \oplus E(M)$ is finitely cogenerated (see [16, Exercise 19.(7)]) it must be a Utumi-module by our assumption. Then $M$ is injective by Lemma 2.2. Thus $R$ is a right V-ring. The converse is clear.
(2) Let $M$ be injective and $K \leq M$. Then $\frac{M}{K} \oplus E\left(\frac{M}{K}\right)$ is a homomorphic image of $M \oplus E\left(\frac{M}{K}\right)$. Hence $\frac{M}{K} \leq^{\oplus} E\left(\frac{M}{K}\right)$ by Lemma 2.2. Therefore, $\frac{M}{K} = E\left(\frac{M}{K}\right)$ as required. The converse is clear. □

A module $M$ is called $\Pi$-*quasi-injective* if every direct product $M^I$ of copies of $M$ is quasi-injective.

**Proposition 2.5.** *The following statements are equivalent:*

*(i) Every factor ring of $R$ is right hereditary.*

*(ii) Every factor module of a $\Pi$-quasi-injective right R-module is quasi-injective.*

*(iii) Every factor module of a $\Pi$-quasi-injective right R-module is a Utumi-module.*

*Proof.* The proof is analogous to that of [2, Proposition 2.15] (also refer to [22, Theorem 6]). □

It is noted that the class of rings $R$ for which every free right (respectively, left) $R$-module is Utumi, is exactly that of quasi-Frobenius rings.

**Theorem 2.6.** *Let $R$ be a ring. The following statements are equivalent:*

*(i) $R$ is a quasi-Frobenius ring;*

*(ii) Every free $R$-module is a Utumi-module;*

*(iii) Every projective $R$-module is a Utumi-module;*

*(iv) Every flat $R$-module is a Utumi-module.*

*Proof.* (1) $\implies$ (2) $\implies$ (3) are clear.
(3) $\implies$ (1) Let $M$ be a projective $R$-module. Then $M \oplus M$ is a Utumi-module, and so $M$ is quasi-injective by Lemma 2.2. Hence $R$ is quasi-Frobenius by [4, Corollary 2.3].
(4) $\implies$ (1) Let $M$ be a projective module. Then by hypothesis $M$ is a Utumi-module. Hence by preceding argument $R$ is quasi-Frobenius.
(1) $\implies$ (4) Suppose $R$ is quasi-Frobenius, then by [2, Theorem 2.28 and Corollary 2.32] $R$ is right perfect. Hence the result follows from the well-known fact due to Bass, that every flat modules over a perfect ring are projective (see, [3, Theorem 28.4]). □

A ring is called *right* (respectively, *left*) *uniserial* if the lattice of right (respectively, left) ideals is linearly ordered.

**Theorem 2.7.** *Let $R$ be a ring. The following statements are equivalent:*

*(i) $R$ is uniserial;*

*(ii) Every quasi-projective $R$-module is a Utumi-module.*

*Proof.* We only need to show that (2) $\Longrightarrow$ (1). To see, (2) $\Longrightarrow$ (1) let $M$ be a quasi-projective $R$-module. Then $M \oplus M$ is quasi-projective by [21, 18.1 and 18.2(2)] and hence a Utumi-module. Then $M$ is quasi-injective by Lemma 2.2. Thus $R$ is uniserial by [4, Proposition 2.8]. □

A module $M$ is called *(countably)* $\Sigma$*-(quasi-)injective* if every (countable) direct sum of copies of $M$ is (quasi-)injective. Analogously, $M$ is called a *(countably)* $\Sigma-$*Utumi-module* if every (countable) direct sum of copies of $M$ is a Utumi-module. A result of Faith and Walker (see [3, Theorem 25.8]) asserts that a ring $R$ is right noetherian if and only if every injective right R-module is Σ–injective, equivalently, as shown by Fuller [3], if every quasi-injective right $R$-module is Σ-quasi-injective.

**Theorem 2.8.** *The following conditions are equivalent for a ring $R$:*

*(i) $R$ is right Noetherian;*

*(ii) Every direct sum of injective right $R$-modules is a Utumi-module;*

*(iii) Every countable direct sum of injective right $R$-modules is a Utumi-module;*

*(iv) Every injective right $R$-module is a countably Σ-Utumi-module;*

*(v) Every quasi-injective right $R$-module is a countably Σ-Utumi-module.*

*Proof.* (1) $\Longrightarrow$ (2) is well-known. (2) $\Longrightarrow$ (3), (3) $\Longrightarrow$ (4) and (4) $\Longrightarrow$ (5) are clear.
(5) $\Longrightarrow$ (1) We show that every countable direct sum of injective right $R$-module is injective. Let $L = \oplus_{i\geq 1} M_i$, where each $M_i$ is an injective right $R$-module. Suppose further that $M_0 = E(L)$, $K = \Pi_{i\geq 0} M_i$ and $I = \mathbb{N} \cup \{0\}$. By (5) the $R$-module $K^I$ is Utumi. Clearly, for each $i \geq 0$, $K \cong M_i \oplus T_i$, where $T_i = \Pi_{j\neq i} M_j$. It follows that $L \oplus E(L)$ is a Utumi-module as it is isomorphic to a direct summand of $K^I$. The proof is now complete by Lemma 2.2. □

We shall now include a complete answer to the Question 2.1.

**Proposition 2.9.** *The following are equivalent for any ring $R$:*

*(i) $R$ is a semisimple ring;*

*(ii) Every right (respectively, left) $R$-module is Utumi;*

*(iii) Every finitely generated (respectively, left) $R$-module is Utumi;*

*(iv) Every 2-generated right (respectively, left) $R$-module is Utumi.*

*(v) Every direct sum of two cyclic right (respectively, left) $R$-module is Utumi.*

*(vi) Every direct sum of two Utumi right (respectively, left) $R$-module is Utumi.*

*Proof.* We need to show that (5) $\Longrightarrow$ (1) and (6) $\Longrightarrow$ (1). Assume that (5) hold. Then $C \oplus C$ is Utumi for every cyclic right $R$-module $C$. Hence $C$ is a quasi-injective right $R$-module. Therefore the right uniform dimension of $R$ is finite by ([16, Lemma 6.43 and Corollary 6.45]). On the other hand, for all $x \in R$, $xR \oplus R$ is Utumi which implies that $xR$ is a direct summand of $R$. That means $R$ is a (von Neumann) regular ring with finite uniform dimension, proving that $R$ is semisimple. Now let us assume that (6) holds. If $K$ is either a semisimple or a uniform right $R$-module. Then by (6), $K \oplus E(K)$ is Utumi and so $K$ is an injective $R$-module. Thus every uniform $R$-module is injective and this in turn implies that every uniform $R$-module is simple and injective. Also $R$ is a Noetherian ring because semisimple $R$-modules are injective. This shows that $R$ is semisimple. □

## 3 Rings over which all cyclic modules are Utumi

From Proposition 2.9, it is clear that if every $n$-generated $R$-module is Utumi, then every $(n-1)$-generated $R$-module is Utumi. Hence taking this fact into account, the following question arise naturally.

**Question 3.1.** *: What is the structure of a ring $R$ over which every cyclic right $R$-module is Utumi ?*

Before we try to address Question 3.1, we make a key observation for Utumi-modules. For any module $M_R$, we denote $\mathrm{L}(M_R)$ for *the lattice of all submodules of* $M_R$.

**Proposition 3.2.** *(i) Suppose that $M$ and $N$ are $R$-modules with a lattice isomorphism $L(M_R) \overset{\theta}{\cong} L(N_R)$ such that the $R$-modules $\theta(K_1)$ and $\theta(K_2)$ are isomorphic whenever $K_1 \cong K_2$. Then $M_R$ is Utumi-module if and only if $N_R$ is so.*

*(ii) Being Utumi-module is a Morita-invariant property.*

*Proof.* (1) This has a routine argument.
(2) This is obtained by (1) and [3, Proposition 21.7].

□

Now we introduce the following definition in the context of Question 3.1.

**Definition 3.3.** (cf. [9, Definition 2.1]) A ring $R$ is called a right *CU-ring* if every cyclic right $R$-module is a Utumi-module.

**Remarks and Examples 3.4.** *(i) In [18, Proposition 1.1], Stephenson proved that a module $M$ is distributive iff every homomorphic image of $M$ is square-free iff $Hom(\frac{A}{A\cap B}, \frac{B}{A\cap B}) = 0$ for all submodules $A$ and $B$ of $M$. Thus every right distributive ring $R$ is right CU-ring.*

*(ii) If $R$ is a semiprime right duo ring, then $R_R$ is square-free. For $xR \cong yR$ with $xR \cap yR = 0$ then $xRy = 0$ which implies that $yRy = 0$. Hence $y = 0$ by our assumption on $R$.*

*(iii) By (2), any strongly regular ($\equiv$ reduced and regular) ring is a right (left) CU-ring. For example, any direct product of division rings is a right CU-ring.*

Although Proposition 3.2 shows that Utumi condition is a Morita-invariant property for modules, in the following Theorem we observe that the right fully Utumi condition (i.e., all factors are Utumi) is not a Morita-invariant property for rings. However, Proposition 3.7 shows that if $R$ is right CU then so is $eRe$ for any full idempotent $e \in R$.

**Theorem 3.5.** *The following conditions on a ring $R$ are equivalent:*

*(i) $R$ is a semisimple ring;*

*(ii) The matrix ring $\mathbb{M}_n(R)$ is a right CU-ring for some $n \geq 2$;*

*(iii) The matrix ring $\mathbb{M}_2(R)$ is a right CU-ring.*

*Proof.* The result follows from Proposition 3.2 and the fact that under the natural Morita equivalences between $R$ and $\mathbb{M}_n(R)$, every $n$-generated $R$-module corresponds to cyclic modules.

□

**Corollary 3.6.** *If $R$ is a right CU-ring, then $R_R^{(n)} \ncong R_R$ for every $n \geq 2$.*

*Proof.* This is obtained by Theorem 3.5 and the fact that semisimple rings have invariant basis number property.

□

For the next result we use usual arguments (see, for instance, [12, Lemma 3.14.]).

**Proposition 3.7.** *Let $e$ be an idempotent of $R$ with $ReR = R$. If $R$ is a right CU-ring, then so is $eRe$.*

*Proof.* Write $S = eRe$ and $P = (eR)_R$. Then it is well-known that $Hom_R(P, -) : Mod - R \longrightarrow Mod - S$ and $- \otimes_S P : Mod - S \longrightarrow Mod - R$ are naturally equivalent functors and defines a Morita equivalence between $Mod - R$ and $Mod - S$. Since the Morita equivalence property is preserved for Utumi-modules (Proposition 3.2), every factor module of the right $S$-module $Hom_R({}_SP_R, R)$ is a Utumi-module. But, $Hom_R({}_SP_R, R) \cong (Re)_S = [(1-e)Re \oplus eRe]_S$. So every factor module of $S_S$ is a Utumi-module. □

**Lemma 3.8.** *Let $R$ be a ring and $M$ be a right $R$-module. If $I$ is an ideal with $MI = 0$.*

*(i) $M_R$ is Utumi if and only if $M_{R/I}$ is Utumi.*

*(ii) If $R$ is a right CU-ring then $R/I$ is a right CU-ring for each ideal $I$ of $R$.*

*Proof.* These have routine arguments. □

**Corollary 3.9.** *Let $\{S_i\}_{i\in\mathbb{N}}$ be an infinite family of rings and $R = \Pi_{i\in\mathbb{N}}\mathbb{M}_{n_i}(S_i)$. If $R$ is a right CU-ring then the set $\{n_i \mid n_i \geq 2\}$ is finite. Consequently, an infinite direct product of right CU-rings need not be a right FU-ring.*

*Proof.* Let $S = \Pi_{n_i\geq 2}\mathbb{M}_{n_i}(S_i)$. For each $n_i \geq 2$, let $e_i$ be the matrix in $\mathbb{M}_{n_i}(S_i)$ whose $(1,1)$ and $(2,2)$-entries are 1 and all other entries are zero, and let $e = (e_i) \in S$. Then $e^2 = e$ and $SeS = S$. By Lemma 3.8, $S$ is a right CU-ring. So, by Proposition 3.7, $eSe \cong \Pi_{n_i\geq 2}\mathbb{M}_2(S_i) \cong \mathbb{M}_2(\Pi_{n_i\geq 2}S_i)$ is a right CU-ring. By Theorem 3.5, we deduce that $\Pi_{n_i\geq 2}S_i$ is a semisimple ring. Thus the set $\{n_i \mid n_i \geq 2\}$ must be finite. The last statement is now clear. □

**Lemma 3.10.** *Let $R_1$, $R_2$ be rings and $M_i$ be $R_i$-module ($i = 1, 2$). Then the $R_1 \oplus R_2$-module $M_1 \oplus M_2$ is Utumi if and only if each $M_i$ is a Utumi $R_i$-module.*

*Proof.* ($\Rightarrow$) By Lemma 3.8.
($\Leftarrow$) Let $T = R_1 \oplus R_2$, $e_1 = (1, 0)$, $e_2 = (0, 1)$ and $X, Y$ be $T$-submodules of $M := M_1 \oplus M_2$ such that $X \cong Y$ with $X \cap Y = 0$. Replace $M_i$ with $Me_i$. Then for each $i = 1, 2$ we have $Xe_i \cong Ye_i$ as $R_i$-modules and $Xe_i \cap Ye_i = 0$. Thus by hypothesis, there are direct summands $E_i, V_i$ of $M_i$ such that $E_i \oplus V_i$ is direct summand $M_i$ and $Xe_i$ , $Ye_i$ are essential in $E_i$ , $V_i$ respectively. Let $E = E_1 + E_2$ and $V = V_1 + V_2$, then $X = Xe_1 + Xe_2$ is essential in $E_T$ and $Y = Ye_1 + Ye_2$ is essential in $V_T$ such that $E + V = (E_1 + V_1) + (E_2 + V_2)$ is a direct summand of $M_T$. This proves that $M_T$ is Utumi. □

**Corollary 3.11.** *Let $R = R_1 \oplus \cdots \oplus R_n$ where $R_i$, $i = 1, ..., n$ are rings. Then $R$ is a right CU-ring iff each $R_i$, $i = 1, ..., n$ is a right CU-ring.*

*Proof.* By Lemma 3.10 and the fact that $M_1 \oplus M_2$ is a cyclic $R_1 \oplus R_2$-module if and only if each $M_i$ is a cyclic $R_i$-module. □

In the following we show that for any ring $R$ the triangular ring $T_n(R)$ ($n \geq 2$) is not a right Utumi ring.

**Proposition 3.12.** *Let $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ be a formal triangular matrix ring where $A$, $B$ are rings and ${}_AM_B$ is a bimodule. If there exists a nonzero right $B$-monomorphism $f : M \to B$, then $R$ is not a right Utumi ring.*

*Proof.* Write $R = I_1 \oplus I_2$, where $I_1 = \begin{pmatrix} A & M \\ 0 & 0 \end{pmatrix}$ is an ideal in $R$ and $I_2 = \begin{pmatrix} 0 & 0 \\ 0 & B \end{pmatrix}$ is a right ideal of $R$. Note that $Hom_R(I_1, I_2) = 0$. Now, if $R$ is a right Utumi ring, then by [11, Proposition 3.1], $I_1$ and $I_2$ are relatively pseudo-injective right $R$-modules. Hence if $f : M \to B$ is a right $B$-monomorphism (equivalently, there is a corresponding $R$-monomorphism $\begin{pmatrix} 0 & M \\ 0 & 0 \end{pmatrix}$ $\to I_2$), then we must have a non-zero $\bar{f} \in Hom_R(I_1, I_2)$, a contradiction. □

**Corollary 3.13.** *Let $n \geq 2$. For any ring $R$ the triangular ring $T_n(R)$ is not a right Utumi ring.*

Recall that a submodule $K$ of $M$ is called *fully invariant* if for every $f \in End_R(M)$, $f(K) \leq K$; a module $M$ is called a *duo module*, if every submodule of $M$ is fully invariant. The ring $R$ is called a *right duo ring* if the right $R$-module $R$ is a duo module. Note that a ring $R$ is a right duo ring if and only if every right ideal of $R$ is a two-sided ideal.

**Proposition 3.14.** *Let $M$ be a quasi-projective duo module. Then every homomorphic image of $M$ is Utumi if and only if $M$ is distributive.*

*Proof.* We apply Remarks and Examples 3.4(1). Assume that every homomorphic image of $M$ is Utumi. Then by [11, Theorem 3.13], for any $K \lneq M$, we have $\frac{M}{K} = \frac{Q}{K} \oplus \frac{T}{K}$ where $\frac{Q}{K}$ is quasi-injective and $\frac{T}{K}$ is a square-free module. Moreover, $\frac{Q}{K} = \frac{A}{K} \oplus \frac{B}{K} \oplus \frac{D}{K}$ such that $\frac{A}{K} \cong^{\theta} \frac{B}{K}$ and $\frac{D}{K}$ is isomorphic to a summand of $\frac{A}{K} \oplus \frac{B}{K}$. Thus there exists $\bar{\theta} \in End\left(\frac{M}{K}\right)$ such that $\bar{\theta} \mid_{\frac{A}{K}} = \theta$. Now $M$ being quasi-projective, there exists $f \in End(M)$ such that $\bar{\theta}(m + K) = f(m) + K$. Thus $\theta(a + K) = f(a) + K \leq A + K = A$ for all $a \in A$. It follows that $\theta\left(\frac{A}{K}\right) \leq \frac{A}{K} \cap \frac{B}{K} = 0$. Therefore, $\frac{M}{K} = \frac{T}{K}$ is square-free. □

**Corollary 3.15.** *Let $R$ be a right duo ring. Then $R$ is a right CU-ring if and only if $R$ is a right distributive ring.*

*Proof.* This is obtained by Proposition 3.14.

□

**Lemma 3.16.** *Let $R$ be a ring, $M_R$ be quasi-injective and $N$ is fully invariant submodule of $M$.*

*(i) If End$_R(M)$ is a right CU-ring, then the ring End$_R(N)$ is right CU.*

*(ii) If $M_R$ is nonsingular and $N$ is essential submodule of $M$, then End$_R(M) \cong$ End$_R(N)$.*

*Proof.* Note that the mapping $f \to f|_N$ is a surjective ring homomorphism from $\text{End}_R(M)$ to $\text{End}_R(N)$, and is an isomorphism if $M_R$ is nonsingular and $N$ is essential. Hence (1) follows from Lemma 3.8(2). □

**Proposition 3.17.** *If $R$ is a right self-injective right CU ring, then every homogenous component of Soc$(R_R)$ has a finite length.*

*Proof.* Let $M$ be a homogenous component of $\text{Soc}(R_R)$. By Lemma 3.16, $T := \text{End}_R(M)$ is a right FU-ring. Now if length$(M_R)$ is infinite then $M \cong M^{(2)}$. Hence $T \cong \mathbb{M}_2(T)$ and so $T$ must be a semisimple ring by Theorem 3.5. This follows that $M_R$ is finitely generated, contradiction. Therefore length$(M_R)$ is finite. □

Recall that a ring $R$ is called *directly finite* if $ab = 1$ in $R$ implies $ba = 1$ for all $a, b \in R$. An idempotent $e$ in a regular ring $R$ is called an *abelian idempotent* if the ring $eRe$ is abelian, and is called a *directly finite idempotent* if the ring $eRe$ is directly finite. An idempotent $e$ in a (von Neumann) regular right self-injective ring is called a *faithful idempotent* if 0 is the only central idempotent orthogonal to $e$. A (von Neumann) regular right self-injective ring is: of Type $I_f$ if it contains a faithful abelian idempotent and is directly finite; of Type $II_f$ if it contains a faithful directly finite idempotent but contains no nonzero abelian idempotents and is directly finite; and purely infinite if it contains no nonzero directly finite central idempotents (see [8, pp. 111–115]). Two $R$-modules $M$ and $N$ are called *orthogonal* to each other, if they do not contain nonzero isomorphic submodules.

**Proposition 3.18.** *Let $R$ be a regular right self-injective ring.*

*(i)* (See the proof of [12, Theorem 4.6] and [13, Lemma 6.23]) If $\mathrm{Soc}(R_R) = 0$ then $R$ is a right CU-ring if and only if $R$ is strongly regular.

(ii) If $\mathrm{Soc}(R_R)$ is an essential right ideal of $R_R$, then $R$ is a right CU-ring if and only if $R \cong R_1 \times R_2$ such that $R_1$ is a semisimple ring and $R_2$ is a direct product of division rings.

*Proof.* (1) One direction is obtained by Remarks 3.4 (3). Conversely, let $R$ be a right CU-ring. By [8, Theorem 10.21] $R$ is a direct product of directly finite and purely infinite rings. Corollary 3.6 and [8, Theorem 10.16] show that purely infinite rings are not right CU. Thus $R$ must be directly finite. Now by [8, Theorem 10.22] $R$ is a direct product of rings of types $I_f$ and $II_f$. However if $R$ is of type $II_f$ then by [8, Theorem 10.16], $R \cong M^{(2)}$ for some $R$-module $M$. This in turn implies that $\mathrm{Soc}(R_R) \neq 0$ by Theorem 3.5. Therefore $R$ is of type $I_f$. Hence by [8, Theorem 10.24], $R$ is a direct product of matrix rings over strongly regular rings. Since $\mathrm{Soc}(R_R) = 0$ the size of each matrix ring in the product must be 1, proving that $R$ is strongly regular.
(2) ($\Rightarrow$). By Proposition 3.17, we may assume that $\mathrm{Soc}(R_R) \cong \bigoplus_i S^{(n_i)}$ where each $S_i$ is a simple right $R$-module, $n_i \geq 1$ and $S_i \ncong S_j$ for all $i \neq j$. Now applying Lemma 3.16(2), we have $R = \Pi_i \mathbb{M}_{n_i}(D_i)$ where each $D_i \cong \mathrm{End}_R(S_i)$ is a division ring. The result is now obtained by Corollary 3.9.
($\Leftarrow$) By Corollary 3.11 and Remarks and Examples 3.4.

□

**Lemma 3.19.** *If $R = R_1 \times R_2$ then $R_1$ and $R_2$ are orthogonal as $R$-modules.*

*Proof.* Consider $e_1 = (1, 0)$, $e_2 = (0, 1)$ and $X_1 \overset{f}{\cong} X_2$ where each $X_i$'s are $R$-submodules of $R_i$. Then $f(X_1) = f(X_1)e_1 \subseteq X_2 e_1 = 0$.

□

**Theorem 3.20.** *Let $R$ be a right nonsingular, right CU-ring. Then $R = S \times T$, where $S$ is a semisimple ring and $T$ is a reduced square-free ring.*

*Proof.* Following [11, Proposition 3.22], we have a ring decomposition $R = A \oplus B$, where $A$ is a right self-injective ring and $B$ is a square-free ring. Since $A$ is a right nonsingular self-injective ring, it is a (von Neumann) regular ring. Now $A$ being injective, $\mathrm{Soc}(A_A)$ is essential in a direct summand $D$ (say) of $A_A$. Write $A = D \oplus H$ for some right ideal $H$ of $A$. Then it is easy to verify that $D$ and $H$ are ideals in $A$ (notice that $D$ and $H$ are orthogonal $A$-modules, so there are no nonzero homomorphisms between them) and $A \cong D \times H$ as rings. Now by Proposition 3.18, $A = S \oplus T_1$ where $S$ is a semisimple ring and $T_1$ is a strongly regular (hence square-free) ring. Let $T = T_1 \times B$. By Lemma 3.19 and [11, Lemma 2.17], $T$ is a square-free ring. Hence any closed right ideal of $T$ is an ideal (see [7, Theorem 6(i)]) and so $T$ (note that $Z(R_R) = 0 \implies Z(T_T) = 0$) is a reduced ring by [19, Lemma 2.4].

□

**Corollary 3.21.** *A (von Neumann) regular ring is right (respectively, left) CU-ring if and only if it is a direct product of a semisimple and a reduced ring.*

*Proof.* This follows from Theorem 3.20 and the fact that regular reduced rings are CU. □

Recall that an $R$-module $M$ is said to have *internal cancellation property* (*IC* for short) if, given internal decompositions

$$M = A \oplus X = B \oplus Y$$

where $A \cong B$, then $X \cong Y$. If $R_R$ has IC, $R$ is said to be a *right IC ring* (see [14, 1.4]). In the following we let $\mathrm{Z}_2(R) = B$ when $\mathrm{Z}(R/\mathrm{Z}(R_R)) = B/\mathrm{Z}(R_R)$.

**Proposition 3.22.** (cf. [11, Proposition 4.8] and [10, Proposition 4.9].) Every square-free right CU-ring is a right IC ring.

*Proof.* Let $R$ be a square-free right CU-ring. Since every Utumi module is a $C4$-module, every cyclic right $R$-module is $C4$. Hence $R_R$ is $C3$ by [1, Proposition 4.8]. Therefore, by [10, Proposition 4.9], $R_R$ satisfy the internal cancellation property. □

**Corollary 3.23.** *Let $R$ be a right CU-ring. If either $Z_2(R) \subseteq J(R)$ or $R$ is a right self-injective ring. Then $R$ is a right IC ring.*

*Proof.* It is well-known that the internal cancellation property is closed under direct products (see [14, Paragraph before Proposition 5.1]). Thus if $R$ is a ring that is isomorphic to a direct product of a semisimple ring and a square-free right CU-ring then $R$ is a right IC ring by Proposition 3.22. Now let $R$ be a right CU-ring with the above hypothesis. To show that $R$ is right IC, we apply [14, Proposition 5.2(1)] and show that $R/I$ is a right IC ring for some suitable ideal $I \subseteq \mathrm{J}(R)$. Note that for any ring $R$, the ring $R/\mathrm{Z}_2(R)$ is a right nonsingular ring and if $R$ is a right self-injective ring then $R/\mathrm{J}(R)$ is a regular ring. Therefore, the result is obtained by Theorem 3.20 and Corollary 3.21. □

**Corollary 3.24.** *Let $R$ be a right nonsingular, right CU-ring. Then $R$ is a right $U^*$ ring (i.e., all right ideals of $R$ are Utumi).*

*Proof.* Follows from Theorem 3.20, [10, Theorem 3.2] and Lemma 3.19. □

**Remark 3.25.** (i) The converse of Corollary 3.24 is not true in general. For this, consider $R = \mathbb{Z}[x]$. Then $R$ being a commutative integral domain (reduced $\equiv$ nonsingular), $R$ is uniform and hence a $U^*$-ring. But $R$ is not a CU-ring by Corollary 3.15 and [20, Proposition 4.31]. This example shows that the property of being "Fully-Utumi" on a ring $R$ does not extend to the polynomial ring $R[x]$.

(ii) A right nonsingular, right CU-ring need not be a right *a-ring* (rings having the property that every right ideal is automorphism-invariant, see [15]). For instance, the ring of integers $\mathbb{Z}$ being a distributive ring is an CU-ring by Corollary 3.15. But $\mathbb{Z}$ is not an $a$-ring, since the injective hull of $\mathbb{Z}$ is $\mathbb{Q}$ and $\mathbb{Z}$ is not invariant under every non-zero element of $\mathbb{Q}$. However, $\mathbb{Z}$ is a $U^*$-ring.

(iii) A (right nonsingular) right $a$-ring need not be a right CU-ring. For this, note that any commutative self-injective ring $R$ is an $a$-ring, however $R$ need not necessarily be a distributive ring. For instance,
$(i)$ Let $R = \Pi_p\ (\mathbb{Z}/p\mathbb{Z})$, where $p$ runs over all distinct prime integers. Then the injective hull of $R$ is $R$ itself and hence $R$ is a commutative (von Neumann) regular, $a$-ring. But $R$ cannot be an CU-ring by Corollary 3.15 and [18, Remark on Page 295].
$(ii)$ Consider the group ring $FG$, where $F$ is a field of order 2 and $G$ is the direct product of two groups of order 2. Then $FG$ is a self-injective, local ring which is not uniserial and hence can not be distributive (see [20, 3.23]).

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## Author information

Soumitra Das and Ardeline M. Buhphang, Department of Mathematics, KPR Institute of Engineering and Technology, Coimbatore-641407 and Department of Mathematics, North-Eastern Hill University, Shillong-793022, Meghalaya, India.
E-mail: soumitrad330@gmail.com

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