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# A NUMERICAL METHOD FOR SOLVING BRATU'S PROBLEM

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**Abstract** A brief survey of the properties and different treatments of the one-dimensional Bratu's problems is presented, and a numerical method is stated for solving this problem via Genocchi polynomial $G_n(x)$. In this paper, the obtained error was compared between the proposed technique, Chebyshev wavelets, and Legendre wavelets. Numerical examples are exhibited to verify the efficiency and accuracy of the stated technique.

## 1 Introduction

We consider the following classical Bratu's problem which is an elliptic nonlinear partial differential equation with homogeneous Dirichlet boundary conditions on the boundary. The problem is given by

$$\begin{aligned} &\nabla u + Ce^u = 0, \quad \text{in} \quad \Omega, \\ &u = 0 \quad \text{on} \quad \partial\omega, \\ &C > 0, \end{aligned}$$

where $\omega$ is a bounded domain with boundary $\partial\omega$. The problem is a nonlinear eigenvalue problem that is commonly utilized as a test problem for some numerical techniques. In the planar $1D$ case, the problem reduces to the following expression

$$\begin{aligned} u_{xx} + Ce^u = 0, \quad 0 \leq x \leq 1, \\ u(0) = u(1) = 0. \end{aligned} \tag{1.1}$$

The Bratu's problem appears in a large variety of application areas such as the fuel ignition model of thermal combustion, radiative heat transfer, thermal reaction, and the Chandrasekhar model of the expansion of the universe.
The planar $1D$ Bratu's problem has the following exact solution

$$u(x) = 2ln\frac{cosh\alpha}{cosh(\alpha(1-2x))}$$

where

$$\alpha = \frac{4}{\sqrt{2C}}\alpha.$$

The solution of this equation may be obtained by utilizing the modified Newton Raphson method [1].
The Laplace decomposition method was used for solving Bratu's problem by Syam and Hamdan [3]. Additionally, Aksoy and Pakdemirli [4] improved a perturbation solution for this problem. Wazwaz [5] shown the Adomian decomposition method for solving Bratu's problem. Also, the uses of spline method, wavelet technique, and Sinc-Galerkin method for solving the Bratu's problem have been utilized by [2].

In this paper, a new operational matrix of fractional order derivative based on Genocchi polynomials (GPs) is stated for solving of the Bratu's problem.
The outline of this study is as follow: In Section 2, some basic preliminaries is stated. Section 3 will give explanation of the problem. Also some numerical examples are stated in Section 4. Conclusion is stated in Section 5.

## 2 Some basic preliminaries

In various papers, genocchi numbers $G_n$ and $G_n(x)$ have been studied in [6]. The classical polynomial $G_n(x)$ is usually defined by the following form

$$\frac{2te^{xt}}{e^t+1} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!},\ (|t|<\pi), \tag{2.1}$$

where

$$\begin{aligned}
&G_n(x) = \sum_{k=0}^{n}\binom{n}{k} G_k x^{n-k},\\
&G_1 = 1, G_2 = 0, G_3 = 0, G_4 = 1, G_5 = 0, G_6 = -3, G_7 = 0,\\
&G_8 = 17, G_9 = 0, G_{10} = -155,\\
&G_{2n+1} = 0, n \in N, G_{12} = 2073,\\
&G_{2i} = 2iE_{2i-1}(0),\\
&E_{2i-1}(0) = \left(2^{1-2i}\sum_{j=1}^{2i-1}\left((-1)^{2i+j} j^{2i-1}\sum_{k=1}^{2i-j-1}\binom{k}{2i}\right)\right),\\
&G_1(x) = 1,\\
&G_2(x) = 2x-1,\\
&G_3(x) = 3x^2-3x,\\
&G_4(x) = 4x^3-6x^2+1,\\
&G_5(x) = 5x^4-10x^3+5x,\\
&G_6(x) = 6x^5-15x^4+15x^2-3,\\
&G_n(x+1)+G_n(x) = 2nx^{n-1}, n \ge 0,\\
&\frac{dG_n(x)}{dx} = nG_{n-1}(x), n \ge 1,\\
&\int_a^b G_n(x)dx = \frac{G_{n+1}(b)-G_{n+1}(a)}{n+1},\\
&\int_0^1 G_n(x)G_m(x)\,dx = \frac{2(-1)^n n!m!}{(n+m)!}G_{n+m},\ m,n \ge 1,
\end{aligned}$$

Now, one may have GPs in the following vector form:

$$G(x) = \Delta T(x),$$

$$\Delta = \begin{bmatrix} \binom{1}{0}g_1 & \binom{1}{1}g_0 & 0 & \dots & 0 & 0 & 0 \\ \binom{2}{0}g_2 & \binom{2}{1}g_1 & \binom{2}{2}g_0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \binom{n}{0}g_n & \binom{n}{1}g_{n-1} & \binom{n}{2}g_{n-2} & \dots & \binom{n}{n-2}g_2, & \binom{n}{n-1}g_1 & \binom{n}{n}g_0 \end{bmatrix},$$

$$T(x) = \begin{bmatrix} 1 \\ x \\ x^2 \\ \vdots \\ x^{n-1} \end{bmatrix}$$

## 3 Explanation of the problem

Now, the collocation method based on Genocchi operational matrix of derivatives to solve numerically Eq. (1.1) is stated.
Our strategy is utilizing GPs to approximate the solution $u(x)$ by $u_N(x)$ as given below.

$$u(x) \approx u_N(x) = \sum_{n=1}^{N} z_n G_n(x) = G(x)Z,$$

where

$$Z^T = [z_1, z_2, \dots, z_n],$$

$$G(x) = [G_1(x), G_2(x), \dots, G_N(x)],$$

$$M = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & 0 & 0, \\ 2 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & N-1 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & N & 0 \end{bmatrix} \tag{3.1}$$

and

$$G'(x)^T = MG^T(x), \Rightarrow G'(x) = G(x)M^T,$$

$$\vdots\vdots$$

$$G^{(k)}(x) = G(x)(M^T)^k, \tag{3.2}$$

then, the $k$-th derivative of $u_N(x)$ can be stated as

$$u_N^{(k)}(x) = G^{(k)}(x)Z = G(x)(M^T)^k Z, \tag{3.3}$$

for solving the problem, we can substitute Eq. (3.3) in Eq. (1.1). To obtain $u_N(x)$, one may use the collocation points $x_j = \frac{j-1}{N}$, $j = 1, 2, \dots, N-1$.
Now, we state the following lemma for error of this method.

**Lemma 3.1.** If $u(x) \in C^{n+1}[0,1]$ and $U = Span\{G_1(x), G_2(x), \dots, G_N(x)\}$, then $G(x)Z$ is the best approximation of $u(x)$ out of $U$ when

$$\|u(x) - G(x)Z\| \le \frac{h^{\frac{2n+3}{2}} R}{(n+1)!\sqrt{2n+3}}, \; x \in [x_i, x_{i+1}] \subset [0,1],$$

where $R = \max_{x\in[x_i,x_{i+1}]} |u^{(n+1)}(x)|$ and $h = x_{i+1} - x_i$.

*Proof.* See [6]. □

## 4 Some numerical examples

For illustrating the ability of this method, some examples are solved. The findings reveal that the method is very successful for obtaining the solution of the Bratu's problem.

**Example 4.1.** Consider the following Bratu's problem:

$$u'' - 2e^u = 0, \quad 0 < x < 1,$$
$$u(0) = u(1) = 0,$$
$$u_{exact}(x) = -2ln(cos(x)).$$

One may use the proposed method with $N = 6$ for solving this problem by Maple 16. we have

$$\begin{aligned} u_{apprx}(x) &= c_1 + c_2(2x - 1) + c_3(3x^2 - 3x) + c_4(4x^3 - 6x^2 + 1) \\ &+ c_5(5x^4 - 10x^3 + 5x) + c_6(6x^5 - 15x^4 + 15x^2 - 3), \end{aligned}$$

where

$$c_1 = 0.615626470386014267, c_2 = 0.777212897941271307,$$
$$c_3 = 0.682328165488200526, c_4 = 0.350693564139641808,$$
$$c_5 = 0.100664406155252193, c_6 = 0.063035712194794921,$$

The graphs of approximate and exact solution $u(x)$ are plotted in Fig. 1. Table 1 demonstrates the comparison between the absolute error of proposed technique, Chebyshev wavelets (with $k = 1$ and $M = 6$) [7], and Legendre wavelets [7].

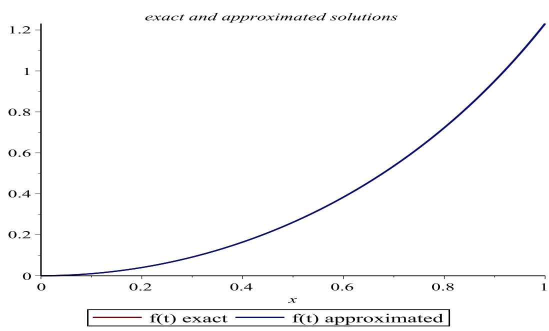


**Figure 1.** The graphs of approximated and exact solution $u(t)$ for Example 4.1

**Table 1.** Comparison of the error for proposed method, Chebyshev, and Legendre wavelets method for Example 4.1

| $x$ | error of $u$ for proposed method | error of Chebyshev wavelets | error of Legendre wavelets |
|---|---|---|---|
| 0.1 | $0.298900690959498800 \times 10^{3}$ | 0.0 | $9 \times 10^{-6}$ |
| 0.2 | $3.2 \times 10^{-18}$ | $4 \times 10^{-7}$ | $15 \times 10^{-6}$ |
| 0.3 | $0.169375917891259800 \times 10^{-3}$ | $15 \times 10^{-7}$ | $0.614 \times 10^{-4}$ |
| 0.4 | 0.000110721801935995000 | $6 \times 10^{-7}$ | $0.888 \times 10^{-4}$ |
| 0.5 | $8.0 \times 10^{-18}$ | $0.737 \times 10^{-4}$ | $0.5669 \times 10^{-3}$ |
| 0.6 | $1.0 \times 10^{-18}$ | $0.53673 \times 10^{-2}$ | $0.25577 \times 10^{-2}$ |
| 0.7 | $0.777427726952850000 \times 10^{-4}$ | $4 \times 10^{-5}$ | $0.92461 \times 10^{-2}$ |
| 0.8 | $4.0 \times 10^{-18}$ | $0.6398 \times 10^{-3}$ | $0.286198 \times 10^{-1}$ |
| 0.9 | $0.347014737643969000 \times 10^{-3}$ | $0.1528 \times 10^{-3}$ | $0.791248 \times 10^{-1}$ |
| 1.0 | $1.0 \times 10^{-17}$ | −− | −− |

**Example 4.2.** Presently, the following problem is considered as follows (see [8]):

$$u'' + 2e^u = 0, \;\; 0 < x < 1,$$
$$u(0) = u(1) = 0,$$

by utilizing the stated technique with $N = 7$, these results are obtained. The graph of approximated solution $u(x)$ is plotted in Fig. 2. Table 2 demonstrates comparison between the absolute error of exact and approximated solution.

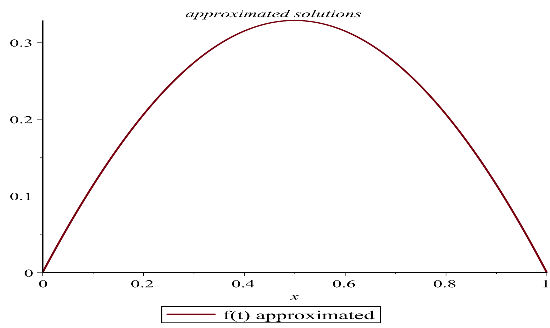


**Figure 2.** The graph of approximated $u(t)$ for Example 4.2

**Table 2.** The exact value and absolute errors of the this method for Example 4.2

| $x$ | exact of $u$ | error of $u$ |
|---|---|---|
| 0.125 | 0.13960278219 | 0.000292244606173129214 |
| 0.250 | 0.24333656779 | $1.5161 \times 10^{-18}$ |
| 0.375 | 0.30731941062 | 0.000143479839910824960 |
| 0.500 | 0.32895242134 | $2.985 \times 10^{-18}$ |
| 0.625 | 0.30731941062 | $0.163694837820151514 \times 10^{-3}$ |
| 0.750 | 0.24333656779 | $0.145613336750047240 \times 10^{-3}$ |
| 0.875 | 0.13960278219 | $1.92 \times 10^{-18}$ |

## 5 Conclusions

The aim of this sequel is to improve an effective and accurate technique for solving Bratu's problem. The GPs is utilized to obtain the approximate solution of this problem. Illustrative examples are included to show the validity of this GPs. Comparing with other methods, the results of numerical examples shows that this method is more accurate than others.

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## References

[1] B.I. Yun, *Solving nonlinear equations by a new derivative-free iterative method*, Appl. Math. Comput. **217(12)**, 5768-5773 (2011).

[2] J. Rashidinia, K. Maleknejad, N. Taheri, *Sinc-Galerkin method for numerical solution of the Bratu's problem*, Numer. Algorithms, **62**, 1-11, (2013).

[3] M.I. Syam, A. Hamdan, *An eflcient method for solving Bratu equations*, Appl. Math. Comput., **176**, 704-713, (2006).

[4] Y. Aksoy, M. Pakdemirli, *New perturbation iteration solutions for Bratu-type equations*, Comput. Math. Appl. **59**, 2802-2808, (2010).

[5] A.M. Wazwaz, *Adomian decomposition method for a reliable treatment of the Bratu-type equations*, Appl. Math. Comput. **166**, 652-663, (2005).

[6] A. Isah, C. Phang, *New operational matrix of derivative for solving non-linear fractional differential equations via Genocchi polynomials*, Journal of King Saud University-Science. (2017) DOI: 10.1016/j.jksus.2017.02.001.

[7] C. Yang, J. Hou, *Chebyshev wavelets method for solving Bratu's problem, Boundary Value Problems*, **142**, (2013).

[8] M. Zarebnia, Z, Sarvari, *New approach for numerical solution of the one-dimensional Bratu equation*, Thai J. Math., **11(3)**, 611-621, (2015).

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