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On invariant arithmetic statistically convergence and lacunary invariant arithmetic statistically convergence

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Abstract In this article, the concepts of invariant arithmetic convergence, lacunary invariant arithmetic convergence, invariant arithmetic statistically convergence and lacunary invariant arithmetic statistically convergence have been investigated. Finally, we give some relations between lacunary invariant arithmetic statistical convergence and invariant arithmetic statistical convergence.

1 Introduction

The idea of arithmetic convergence was firstly originated by Ruckle [1]. Then, it was further investigated by many authors, e.g. Yaying and Hazarika ([2], [3], [4], [5]).

A sequence $x = (x_m)$ is called arithmetically convergent if for each $\varepsilon > 0$, there is an integer n such that for every integer m we have $|x_m - x_{\langle m,n\rangle}| < \varepsilon$, where the symbol $\langle m, n\rangle$ denotes the greatest common divisior of two integers m and n. We denote the sequence space of all arithmetic convergent sequence by AC.

Statistical convergence of a real number sequence was firstly originated by Fast [7]. It became a notable topic in summability theory after the work of Fridy [8] and Šalát [9].

By a lacunary sequence we mean an increasing integer sequence $\theta = \{k_r\}$ such that $k_0 = 0$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r = (k_{r-1}, k_r]$, and ratio $\frac{k_r}{k_{r-1}}$ will be abbreviated by $q_{r.}$ The space of lacunary convergent sequence N_θ was introduced by Freedman [10]. Also, in [10] the connection between the strongly Cesàro summable sequences space and the strongly lacunary summable sequences space was established. Connor [11] gave the relationships between the concepts of strongly p-Cesàro convergence and statistical convergence of sequences. The notion of lacunary convergence has been investigated by Çolak [12], Fridy and Orhan ([13], [14]), Li [15] and many others in the recent years.

Several authors have studied invariant convergent sequences (see, [16], [17], [18], [19], [20], [21]).

Let σ be a one-to-one mapping of the set of positive integers into itself such that $\sigma^m(n) = \left(\sigma^{m-1}(n)\right), m = 1, 2, 3, ...$ A continuous linear functional Φ on l_∞, the space of real bounded sequences, is said to be an invariant mean or a σ mean, if and only if,

(1) $\Phi(x) \geq 0$, for all sequences $x = (x_n)$ with $x_n \geq 0$ for all n;

(2) $\Phi(e) = 1$, where $e = (1, 1, 1, ...)$;

(3) $\Phi\left(x_{\sigma(n)}\right) = \Phi(x)$ for all $x \in l_\infty$.

The mapping Φ are assumed to be one-to-one such that $\sigma^m(n) \neq n$ for all positive integers n and m, where $\sigma^m(n)$ denotes the mth iterate of the mapping σ at n. Thus, Φ extends the limit functional on c, the space of convergent sequences, in the sense that $\Phi(x) = \lim x$, for all $x \in c$. In case σ is translation mapping $\sigma(n) = n + 1$, the σ mean is often called a Banach limit and V_σ, the set of bounded sequences all of whose invariant means are equal, is the set of almost convergent sequences.

A set E of positive integers said to have uniform invariant density of zero if and only if the number of elements of E which lie in the set $\left\{\sigma(m), \sigma^2(m), ..., \sigma^n(m)\right\}$ is $o(n)$ as $n \to \infty$, uniformly in m.

By using uniform invariant density, following notions were given in [16] and [17].

The arithmetic statistically convergence and lacunary arithmetic statistically convergence was examined by Yaying and Hazarika [6].

A sequence $x = (x_m)$ is said to be arithmetic statistically convergent if for $\varepsilon > 0$, there is an integer n such that

$$\lim_{t\to\infty} \frac{1}{t}|\{m \leq t : |x_m - x_{\langle m,n\rangle}| \geq \varepsilon\}| = 0.$$

We shall use ASC to denote the set of all arithmetic statistical convergent sequences. Thus, for $\varepsilon > 0$ and integer n

$$ASC = \left\{ x = (x_m) : \lim_{t\to\infty} \frac{1}{t}|\{m \leq t : |x_m - x_{\langle m,n\rangle}| \geq \varepsilon\}| = 0 \right\}.$$

We shall write $ASC - \lim x_m = x_{\langle m,n\rangle}$ to denote the sequence (x_m) is arithmetic statistically convergent to $x_{\langle m,n\rangle}$.

A sequence $x = (x_m)$ is said to be lacunary arithmetic statistically convergent if for $\varepsilon > 0$ there is an integer n such that

$$\lim_{r\to\infty} \frac{1}{h_r}| \left\{ m \in I_r : |x_m - x_{\langle m,n\rangle}| \geq \varepsilon \right\} = 0.$$

We shall write

$$ASC_\theta = \left\{ x = (x_m) : \lim_{r\to\infty} \frac{1}{h_r}| \left\{ m \in I_r : |x_m - x_{\langle m,n\rangle}| \geq \varepsilon \right\} = 0 \right\}.$$

We will use $ASC_\theta - \lim x_m = x_{\langle m,n\rangle}$ to denote the sequence (x_m) is lacunary arithmetic statistically convergent to $x_{\langle m,n\rangle}$.

2 Main Results

Definition 2.1. A sequence $x = (x_p)$ is said to be invariant arithmetic convergent if for an integer n

$$\lim_{m\to\infty} \frac{1}{m} \sum_{p=1}^{m} x_{\sigma^p(s)} = x_{\langle p,n\rangle} \quad \text{uniformly in } s = 1, 2, \dots .$$

In this case we write $x_p \to x_{\langle p,n\rangle}\ (AV_\sigma)$ and the set of all invariant arithmetic convergent sequences will be demostrated by AV_σ.

Definition 2.2. A sequence $x = (x_p)$ is said to be strongly invariant arithmetic convergent if for an integer n

$$\lim_{m\to\infty} \frac{1}{m} \sum_{p=1}^{m} |x_{\sigma^p(s)} - x_{\langle p,n\rangle}| = 0 \quad \text{uniformly in } s = 1, 2, \dots .$$

In this case we write $x_p \to x_{\langle p,n\rangle}\ [AV_\sigma]$ to denote the sequence (x_p) is strongly invariant arithmetic convergent to $x_{\langle p,n\rangle}$ and the set of all invariant arithmetic convergent sequences will be demostrated by $[AV_\sigma]$.

Definition 2.3. A sequence $x = (x_p)$ is said to be invariant arithmetic statistically convergent if for every $\varepsilon > 0$, there is an integer n such that

$$\lim_{m\to\infty} \frac{1}{m}|\{p \leq m : |x_{\sigma^p(s)} - x_{\langle p,n\rangle}| \geq \varepsilon\}| = 0 \quad \text{uniformly in } s = 1, 2, \dots .$$

We shall use $AS_\sigma C$ to denote the set of all invariant arithmetic statistical convergent sequences. Thus, we define

$$AS_\sigma C = \left\{ x = (x_p) : \text{for some } x_{\langle p,n\rangle},\ AS_\sigma C - \lim x_p = x_{\langle p,n\rangle} \right\}.$$

In this case we write $AS_\sigma C - \lim x_p = x_{\langle p,n\rangle}$ or $x_p \to x_{\langle p,n\rangle}\ (AS_\sigma C)$.

Definition 2.4. A sequence $x = (x_p)$ is called to be lacunary invariant arithmetic statistical convergent if for every $\varepsilon > 0$, there is an integer n such that

$$\lim_{r\to\infty} \frac{1}{h_r} |\{p \in I_r : |x_{\sigma^p(s)} - x_{\langle p,n\rangle}| \geq \varepsilon\}| = 0 \quad \text{uniformly in } s = 1, 2, \dots .$$

We shall use $AS_{\sigma\theta}C$ to indicate the set of all lacunary invariant arithmetic statistical convergent sequences. Thus, we define

$$AS_{\sigma\theta}C = \left\{x = (x_p) : \text{for some } x_{\langle p,n\rangle},\ AS_{\sigma\theta}C - \lim x_p = x_{\langle p,n\rangle}\right\}.$$

In this case we write $AS_{\sigma\theta}C - \lim x_p = x_{\langle p,n\rangle}$ or $x_p \to x_{\langle p,n\rangle}\ (AS_{\sigma\theta}C)$.

Definition 2.5. A sequence $x = (x_p)$ is said to be strongly lacunary invariant arithmetic convergent if for an integer n

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{p\in I_r} |x_{\sigma^p(s)} - x_{\langle p,n\rangle}| = 0 \quad \text{uniformly in } s = 1, 2, \dots .$$

In this case we write $x_p \to x_{\langle p,n\rangle}\ (AL_{\sigma\theta})$ to denote the sequence (x_p) is strongly lacunary invariant arithmetic convergent to $x_{\langle p,n\rangle}$ and the set of all strongly lacunary invariant arithmetic convergent sequences will be denoted by $(AL_{\sigma\theta})$.

Now, we give some inclusion relations between $AL_{\sigma\theta}$-convergence and $AS_{\sigma\theta}C$-convergence and demonstrate that these are equivalent for bounded sequences. We also study relation between $AS_{\sigma\theta}C$-convergence and $AS_\sigma C$-convergence.

Theorem 2.6. *Let $\theta = \{k_r\}$ be a lacunary sequence.*

(i) $x_p \to x_{\langle p,n\rangle}\ (AL_{\sigma\theta})$ implies $x_p \to x_{\langle p,n\rangle}\ (AS_{\sigma\theta}C)$,

(ii) $x \in l_\infty$ and $x_p \to x_{\langle p,n\rangle}\ (AS_{\sigma\theta}C)$ imply $x_p \to x_{\langle p,n\rangle}\ (AL_{\sigma\theta})$,

(iii) $AS_{\sigma\theta}C \cap l_\infty = AL_{\sigma\theta}$.

Proof. (i) Let $\varepsilon > 0$ and $x_p \to x_{\langle p,n\rangle}\ (AL_{\sigma\theta})$. Then, we can write

$$\begin{aligned}\sum_{p\in I_r} |x_{\sigma^p(s)} - x_{\langle p,n\rangle}| &\geq \sum_{\substack{p\in I_r \\ |x_{\sigma^p(s)} - x_{\langle p,n\rangle}| \geq \varepsilon}} |x_{\sigma^p(s)} - x_{\langle p,n\rangle}| \\ &\geq \varepsilon . \left|\left\{p \in I_r : |x_{\sigma^p(s)} - x_{\langle p,n\rangle}| \geq \varepsilon\right\}\right|\end{aligned}$$

which gives the result.

(ii) Assume that $x_p \to x_{\langle p,n\rangle}\ (AS_{\sigma\theta}C)$ and $x \in l_\infty$. If $x \in l_\infty$, then, there exists a positive integer M such that

$$|x_{\sigma^p(s)} - x_{\langle p,n\rangle}| \leq M$$

for all p and s.

Given $\varepsilon > 0$, we get

$$\begin{aligned}\frac{1}{h_r} \sum_{p\in I_r} |x_{\sigma^p(s)} - x_{\langle p,n\rangle}| &= \frac{1}{h_r} \sum_{\substack{p\in I_r \\ |x_{\sigma^p(s)} - x_{\langle p,n\rangle}| \geq \varepsilon}} |x_{\sigma^p(s)} - x_{\langle p,n\rangle}| \\ &\quad + \frac{1}{h_r} \sum_{\substack{p\in I_r \\ |x_{\sigma^p(s)} - x_{\langle p,n\rangle}| < \varepsilon}} |x_{\sigma^p(s)} - x_{\langle p,n\rangle}| \\ &\leq \frac{M}{h_r} \left|\left\{p \in I_r : |x_{\sigma^p(s)} - x_{\langle p,n\rangle}| \geq \varepsilon\right\}\right| + \varepsilon\end{aligned}$$

from which the result follows.

Let $\theta = \{k_r\}$ be given and define x_p to be $1, 2, ..., \left[\sqrt{h_r}\right]$ for $p = \sigma^t(s)$, $t = k_{r-1}+1$, $k_{r-1}+2$, ..., $k_{r-1}+\left[\sqrt{h_r}\right]$; $s \geq 1$, and $x_p = 0$ otherwise (where [.] denotes the greatest integer function). Note that x is not bounded.

Further, for $0 < \varepsilon < 1$ we get

$$\frac{1}{h_r}\left|\{p \in I_r : |x_{\sigma^p(s)} - 0| \geq \varepsilon\}\right| = \frac{\left[\sqrt{h_r}\right]}{\sqrt{h_r}} \to 0 \text{ as } r \to \infty,$$

i.e. $x_p \to 0\,(AS_{\sigma\theta}C)$. But

$$\frac{1}{h_r}\sum_{p\in I_r}|x_{\sigma^p(s)} - 0| = \frac{1}{h_r}\left(\frac{\left[\sqrt{h_r}\right]\left(\left[\sqrt{h_r}\right]+1\right)}{2}\right) \to \frac{1}{2} \neq 0 \text{ as } r \to \infty,$$

hence, $x_p \nrightarrow 0\,(AL_{\sigma\theta})$. Thus, inclusion (i) is proper and this example denotes that the boundedness condition can not be omitted from the hypothesis (ii) .

(iii) This is an immediate consequence of (i) and (ii). □

We now give a lemma which will be used in the proof of Theorem 2.8.

Lemma 2.7. *Assume for given $\varepsilon_1 > 0$ and every $\varepsilon > 0$, there exists m_0 and s_0 and there is an integer n such that*

$$\frac{1}{m}\left|\{0 \leq p \leq m-1 : |x_{\sigma^p(s)} - x_{\langle p,n\rangle}| \geq \varepsilon\}\right| < \varepsilon_1$$

for all $m \geq m_0$ and $s \geq s_0$, then, $x = (x_p) \in AS_\sigma C$.

Proof. Let $\varepsilon_1 > 0$ be given. For each $\varepsilon > 0$, select m_0', s_0 such that

$$\frac{1}{m}\left|\{0 \leq p \leq m-1 : |x_{\sigma^p(s)} - x_{\langle p,n\rangle}| \geq \varepsilon\}\right| < \frac{\varepsilon_1}{2} \tag{2.1}$$

for all $m \geq m_0'$ and $s \geq s_0$. It is enough to prove that there exists m_0'' such that for $m \geq m_0''$ and $0 \leq s \leq s_0$,

$$\frac{1}{m}\left|\{0 \leq p \leq m-1 : |x_{\sigma^p(s)} - x_{\langle p,n\rangle}| \geq \varepsilon\}\right| < \varepsilon_1. \tag{2.2}$$

Since taking $m_0 = \max\{m_0', m_0''\}$, (2.2) will hold for $m \geq m_0$ and for all s, which gives the result.

Once s_0 has been choosen, $0 \leq s \leq s_0$, s_0 is fixed. So put

$$T = \left|\{0 \leq p \leq s_0 - 1 : |x_{\sigma^p(s)} - x_{\langle p,n\rangle}| \geq \varepsilon\}\right|.$$

Now taking $0 \leq s \leq s_0$ and $m \geq s_0$, by (2.1) we have

$$\begin{aligned}
&\tfrac{1}{m}\left|\{0 \leq p \leq m-1 : |x_{\sigma^p(s)} - x_{\langle p,n\rangle}| \geq \varepsilon\}\right| \\
&\leq \tfrac{1}{m}\left|\{0 \leq p \leq s_0-1 : |x_{\sigma^p(s)} - x_{\langle p,n\rangle}| \geq \varepsilon\}\right| \\
&+ \tfrac{1}{m}\left|\{s_0 \leq p \leq m-1 : |x_{\sigma^p(s)} - x_{\langle p,n\rangle}| \geq \varepsilon\}\right| \\
&\leq \tfrac{T}{m} + \tfrac{1}{m}\left|\{s_0 \leq p \leq m-1 : |x_{\sigma^p(s_0)} - x_{\langle p,n\rangle}| \geq \varepsilon\}\right| \leq \tfrac{T}{m} + \tfrac{\varepsilon_1}{2},
\end{aligned}$$

and taking m, sufficiently large, we can write $\frac{T}{m} + \frac{\varepsilon_1}{2} < \varepsilon_1$, which gives (2.2), and therefore, the result follows. □

Theorem 2.8. *$AS_{\sigma\theta}C = AS_\sigma C$ for every lacanary sequence θ.*

Proof. Let $x \in AS_{\sigma\theta}C$. Then, from Definition 2.4, given $\varepsilon_1 > 0$, there exist r_0 and $x_{\langle p,n\rangle}$ such that

$$\frac{1}{h_r}|\{0 \le p \le h_r - 1 : |x_{\sigma^p(s)} - x_{\langle p,n\rangle}| \ge \varepsilon\}| < \varepsilon_1$$

for $r \ge r_0$ and $s = k_{r-1} + 1 + u$, $u \ge 0$.

Let $m \ge h_r$, write $m = ih_r + q$, where $0 \le q \le h_r$, i is an integer. Since $m \ge h_r$, $i \ge 1$. Now

$$\begin{aligned} &\tfrac{1}{m}|\{0 \le p \le m - 1 : |x_{\sigma^p(s)} - x_{\langle p,n\rangle}| \ge \varepsilon\}| \\ &\le \tfrac{1}{m}|\{0 \le p \le (i+1)\,h_r - 1 : |x_{\sigma^p(s)} - x_{\langle p,n\rangle}| \ge \varepsilon\}| \\ &= \tfrac{1}{m}\sum_{j=0}^{i} |\{jh_r \le p \le (j+1)\,h_r - 1 : |x_{\sigma^p(s)} - x_{\langle p,n\rangle}| \ge \varepsilon\}| \\ &\le \tfrac{1}{m}(i+1)\,h_r\varepsilon_1 \le 2ih_r\tfrac{\varepsilon_1}{m},\ (i \ge 1) \end{aligned}$$

for $\frac{h_r}{m} \le 1$, and since $\frac{ih_r}{m} \le 1$,

$$\frac{1}{m}|\{0 \le p \le m - 1 : |x_{\sigma^p(s)} - x_{\langle p,n\rangle}| \ge \varepsilon\}| \ \le 2\varepsilon_1.$$

Then, by the Lemma 2.7, $AS_{\sigma\theta}C \subset AS_{\sigma}C$. It is easy to see that $AS_{\sigma}C \subset AS_{\sigma\theta}C$. □

By using the same techniques as in Theorem 2.8, we can prove the following theorem.

Theorem 2.9. $AL_{\sigma\theta} \Longleftrightarrow [AV_{\sigma}]$ *for every lacanary sequence* θ.

When $\sigma(s) = s + 1$, from Definitions 2.3 and 2.4 we get the definitions of arithmetic almost statistically convergence and lacunary almost arithmetic statistically convergence of a sequence. So, similar inclusions to Theorems 2.7 and 2.8 hold between strongly almost arithmetic convergent sequences and almost arithmetic statistical convergent sequences, which have not appeared anywhere by this time.

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