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DILATION THEOREM FOR p-APPROXIMATE SCHAUDER FRAMES FOR SEPARABLE BANACH SPACES

K. Mahesh Krishna and P. Sam Johnson

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Abstract : Famous Naimark-Han-Larson dilation theorem for frames in Hilbert spaces states that every frame for a separable Hilbert space $\mathcal{H}$ is the image of a Riesz basis under an orthogonal projection from a separable Hilbert space $\mathcal{H}_1$ which contains $\mathcal{H}$ isometrically. In this paper, we derive dilation result for p-approximate Schauder frames for separable Banach spaces. Our result contains Naimark-Han-Larson dilation theorem as a particular case.

1 Introduction

Let $\mathbb{K}$ be the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$ and $\mathcal{H}$ be a separable Hilbert space over $\mathbb{K}$. We start with the definitions of Riesz basis and frame for $\mathcal{H}$.

Definition 1.1. [3, 4] A sequence $\{\tau_n\}_n$ in $\mathcal{H}$ is said to be a Riesz basis for $\mathcal{H}$ if there exists an orthonormal basis $\{\omega_n\}_n$ for $\mathcal{H}$ and a bounded invertible linear operator $T : \mathcal{H} \to \mathcal{H}$ such that

$$T\omega_n = \tau_n, \quad \forall n \in \mathbb{N}.$$

Definition 1.2. [14] A sequence $\{\tau_n\}_n$ in $\mathcal{H}$ is said to be a frame for $\mathcal{H}$ if there exist $a, b > 0$ such that

$$a\|h\|^2 \leq \sum_{n=1}^{\infty} |\langle h, \tau_n\rangle|^2 \leq b\|h\|^2, \quad \forall h \in \mathcal{H}.$$

Theory of frames found its uses in sampling theory, filter banks, wireless communication, wavelet theory etc [29, 5, 9, 21]. It also motivates the study of framelets and multiframelets [17, 16, 27, 13]. Dilation theory usually tries to extend operator on Hilbert space to larger Hilbert space which are easier to handle as well as well-understood and study the original operator as a slice of it [26, 2, 30]. As long as frame theory for Hilbert spaces is considered, following theorem is known as Naimark-Han-Larson dilation theorem. This was proved independently by Han and Larson in 2000 [19] and by Kashin and Kukilova in 2002 [23]. We refer the reader to [12] for the history of this theorem.

Theorem 1.3. *[19, 23] (Naimark-Han-Larson dilation theorem) Let $\{\tau_n\}_n$ be a frame for $\mathcal{H}$. Then there exist a Hilbert space $\mathcal{H}_1$ which contains $\mathcal{H}$ isometrically and a Riesz basis $\{\omega_n\}_n$ for $\mathcal{H}_1$ such that*

$$\tau_n = P\omega_n, \quad \forall n \in \mathbb{N},$$

where P is the orthogonal projection from $\mathcal{H}_1$ onto $\mathcal{H}$.

Reason for names Han and Larson in Theorem 1.3 is clearly evident whereas that of Naimark is that Theorem 1.3 is a particular case of famous Naimark dilation theorem (see the introduction of the paper [12]). To the best of our knowledge, proofs of Theorem 1.3 can be found in [7], [19], [25] and [12]. By the way, proofs of dilation theorem in the finite dimensional case can be found in [18] and [8]. In this paper, we derive dilation theorem for p-approximate Schauder frames for separable Banach spaces (Theorem 2.13). Theorem 1.3 then becomes a particular case of Theorem 2.13.

2 Dilation theorem for p-approximate Schauder frames

Following theorem is the fundamental result in frame theory for Hilbert spaces which motivates the definition of frames for Banach spaces.

Theorem 2.1. *[14, 19] Let $\{\tau_n\}_n$ be a frame for $\mathcal{H}$. Then*

(i) *The map $S_\tau : \mathcal{H} \ni h \mapsto \sum_{n=1}^\infty \langle h, \tau_n\rangle \tau_n \in \mathcal{H}$ is a well-defined bounded linear, positive and invertible operator. Further,*

$$\text{(general Fourier expansion)} \quad h = \sum_{n=1}^\infty \langle h, S_\tau^{-1}\tau_n\rangle \tau_n = \sum_{n=1}^\infty \langle h, \tau_n\rangle S_\tau^{-1}\tau_n, \quad \forall h \in \mathcal{H}. \tag{2.1}$$

(ii) *The map $\theta_\tau : \mathcal{H} \ni h \mapsto \{\langle h, \tau_n\rangle\}_n \in \ell^2(\mathbb{N})$ is a well-defined bounded linear, injective operator.*

(iii) *Adjoint of θ_τ is given by $\theta_\tau^* : \ell^2(\mathbb{N}) \ni \{a_n\}_n \mapsto \sum_{n=1}^\infty a_n\tau_n \in \mathcal{H}$ which is surjective.*

(iv) *$S_\tau = \theta_\tau^*\theta_\tau$.*

(v) *$P_\tau := \theta_\tau S_\tau^{-1}\theta_\tau^* : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ is an orthogonal projection onto $\theta_\tau(\mathcal{H})$.*

Let $\mathcal{X}$ be a separable Banach space and $\mathcal{X}^*$ be its dual. General Fourier expansion in Equation (2.1) allows to define the notion of Schauder frame for $\mathcal{X}$.

Definition 2.2. [6] Let $\{\tau_n\}_n$ be a sequence in $\mathcal{X}$ and $\{f_n\}_n$ be a sequence in $\mathcal{X}^*$. The pair $(\{f_n\}_n, \{\tau_n\}_n)$ is said to be a Schauder frame for $\mathcal{X}$ if

$$x = \sum_{n=1}^\infty f_n(x)\tau_n, \quad \forall x \in \mathcal{X}.$$

Notion of Schauder frame has a very natural generalization which is stated as below.

Definition 2.3. [15, 31] Let $\{\tau_n\}_n$ be a sequence in $\mathcal{X}$ and $\{f_n\}_n$ be a sequence in $\mathcal{X}^*$. The pair $(\{f_n\}_n, \{\tau_n\}_n)$ is said to be an approximate Schauder frame (ASF) for $\mathcal{X}$ if

$$S_{f,\tau} : \mathcal{X} \ni x \mapsto S_{f,\tau}x := \sum_{n=1}^\infty f_n(x)\tau_n \in \mathcal{X}$$

is a well-defined bounded linear, invertible operator.

Recently, a particular case of Definition 2.3 was studied by same authors of this paper by defining p-approximate Schauder frames (p-ASFs).

Definition 2.4. [24] An ASF $(\{f_n\}_n, \{\tau_n\}_n)$ for $\mathcal{X}$ is said to be p-ASF, $p \in [1, \infty)$ if both the maps

$$\theta_f : \mathcal{X} \ni x \mapsto \theta_f x := \{f_n(x)\}_n \in \ell^p(\mathbb{N}),$$

$$\theta_\tau : \ell^p(\mathbb{N}) \ni \{a_n\}_n \mapsto \theta_\tau\{a_n\}_n := \sum_{n=1}^\infty a_n\tau_n \in \mathcal{X}$$

are well-defined bounded linear operators.

Remark 2.5. It is known that every p-approximate Schauder frame is an approximate Schauder frame and every Schauder frame is an approximate Schauder frame. We now give an example to show that the set of all p-approximate Schauder frames is strictly smaller than the set of all approximate Schauder frames. Let $\mathcal{X} = \mathbb{K}$. Define $\tau_n := \frac{1}{n^2}$, $f_n(x) = x, \forall x \in \mathbb{K}, \forall n \in \mathbb{N}$. Then $\sum_{n=1}^{\infty} f_n(x)\tau_n = \frac{\pi^2}{6}x, \forall x \in \mathbb{K}$. Therefore $(\{f_n\}_n, \{\tau_n\}_n)$ is an approximate Schauder frame for $\mathcal{X}$. Let $x \in \mathbb{K}$ be a non-zero element. Then for every $p \in [1, \infty)$,

$$\sum_{n=1}^{m} |f_n(x)|^p = m|x|^p \to \infty \quad \text{as} \quad m \to \infty.$$

Thus $\{f_n(x)\}_n \notin \ell^p(\mathbb{N})$ for any $p \in [1, \infty)$ and hence $(\{f_n\}_n, \{\tau_n\}_n)$ is not a p-ASF for any $p \in [1, \infty)$. It is noted that there is a bijection between the set of approximate Schauder frames and the set of all Schauder frames (for instance, Lemma 3.1 in [15]).

Advantage of p-ASF is that it gives a result similar to that of Theoerm 2.1.

Theorem 2.6. *[24] Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a p-ASF for $\mathcal{X}$. Then*

(i) *We have*

$$x = \sum_{n=1}^{\infty} (f_n S_{f,\tau}^{-1})(x)\tau_n = \sum_{n=1}^{\infty} f_n(x) S_{f,\tau}^{-1}\tau_n, \quad \forall x \in \mathcal{X}.$$

(ii) *The map $\theta_f : \mathcal{X} \ni x \mapsto \{f_n(x)\}_n \in \ell^p(\mathbb{N})$ is injective.*

(iii) *The map $\theta_\tau : \ell^p(\mathbb{N}) \ni \{a_n\}_n \mapsto \sum_{n=1}^{\infty} a_n\tau_n \in \mathcal{X}$ is surjective.*

(iv) $S_{f,\tau} = \theta_\tau \theta_f$.

(v) *$P_{f,\tau} := \theta_f S_{f,\tau}^{-1} \theta_\tau : \ell^p(\mathbb{N}) \to \ell^p(\mathbb{N})$ is a projection onto $\theta_f(\mathcal{X})$.*

In order to derive the dilation result we must have a notion of Riesz basis for Banach space. There are various characterizations for Riesz bases for Hilbert spaces (see Theorem 5.5.4 in [10], Theorem 7.13 in [20], and a recent generalization by Stoeva in [28]) but they use (implicitly or explicitly) inner product structures and orthonormal bases. These characterizations lead to the notion of p-Riesz basis for Banach spaces using a single sequence in the Banach space (see [1, 11]) but we do not consider that in this paper.
To define the notion of Riesz basis, which is compatible with Hilbert space situation, we first derive an operator-theoretic characterization for Riesz basis in Hilbert spaces, which does not use the inner product of Hilbert space. To do so, we need a result from Hilbert space frame theory.

Theorem 2.7. *[22] (Holub's theorem) A sequence $\{\tau_n\}_n$ in $\mathcal{H}$ is a frame for $\mathcal{H}$ if and only if there exists a surjective bounded linear operator $T : \ell^2(\mathbb{N}) \to \mathcal{H}$ such that $Te_n = \tau_n$, for all $n \in \mathbb{N}$, where $\{e_n\}_n$ is the standard orthonormal basis for $\ell^2(\mathbb{N})$.*

In the sequel, given a space $\mathcal{X}$, by $I_\mathcal{X}$ we mean the identity mapping on $\mathcal{X}$.

Theorem 2.8. *For sequence $\{\tau_n\}_n$ in $\mathcal{H}$, the following are equivalent.*

(i) *$\{\tau_n\}_n$ is a Riesz basis for $\mathcal{H}$.*

(ii) *$\{\tau_n\}_n$ is a frame for $\mathcal{H}$ and*

$$\theta_\tau S_\tau^{-1} \theta_\tau^* = I_{\ell^2(\mathbb{N})}. \tag{2.2}$$

Proof. (i) $\implies$ (ii). It is well-known that a Riesz basis is a frame (for instance, see Proposition 3.3.5 in [10]). Now there exist an orthonormal basis $\{\omega_n\}_n$ for $\mathcal{H}$ and bounded invertible operator $T : \mathcal{H} \to \mathcal{H}$ such that $T\omega_n = \tau_n$, for all $n \in \mathbb{N}$. We then have

$$\begin{aligned} S_\tau h &= \sum_{n=1}^{\infty} \langle h, \tau_n \rangle \tau_n = \sum_{n=1}^{\infty} \langle h, T\omega_n \rangle T\omega_n \\ &= T\left(\sum_{n=1}^{\infty} \langle T^*h, \omega_n \rangle \omega_n\right) = TT^*h, \quad \forall h \in \mathcal{H}. \end{aligned}$$

Therefore

$$\begin{aligned}
\theta_\tau S_\tau^{-1}\theta_\tau^*\{a_n\}_n &= \theta_\tau (TT^*)^{-1}\theta_\tau^*\{a_n\}_n = \theta_\tau (T^*)^{-1}T^{-1}\theta_\tau^*\{a_n\}_n \\
&= \theta_\tau (T^*)^{-1}T^{-1}\left(\sum_{n=1}^\infty a_n\tau_n\right) = \theta_\tau (T^*)^{-1}T^{-1}\left(\sum_{n=1}^\infty a_n T\omega_n\right) \\
&= \theta_\tau\left(\sum_{n=1}^\infty a_n (T^*)^{-1}\omega_n\right) = \sum_{k=1}^\infty \left\langle \sum_{n=1}^\infty a_n (T^*)^{-1}\omega_n, \tau_k\right\rangle e_k \\
&= \sum_{k=1}^\infty \left\langle \sum_{n=1}^\infty a_n (T^*)^{-1}\omega_n, T\omega_k\right\rangle e_k \\
&= \sum_{k=1}^\infty \left\langle \sum_{n=1}^\infty a_n \omega_n, \omega_k\right\rangle e_k = \{a_k\}_k, \quad \forall \{a_n\}_n \in \ell^2(\mathbb{N}).
\end{aligned}$$

(ii) $\implies$ (i). From Holub's theorem (Theorem 2.7), there exists a surjective bounded linear operator $T : \ell^2(\mathbb{N}) \to \mathcal{H}$ such that $Te_n = \tau_n$, for all $n \in \mathbb{N}$. Since all separable Hilbert spaces are isometrically isomorphic to one another and orthonormal bases map into orthonormal bases, without loss of generality we may assume that $\{e_n\}_n$ is an orthonormal basis for $\mathcal{H}$ and the domain of T is $\mathcal{H}$. It now reduces in showing T is invertible. Since T is already surjective, to show it is invertible, it suffices to show it is injective. Let $\{a_n\}_n \in \ell^2(\mathbb{N})$. Then $\{a_n\}_n = \theta_\tau(S_\tau^{-1}\theta_\tau^*\{a_n\}_n)$. Hence θ_τ is surjective. We now find

$$\theta_\tau h = \sum_{n=1}^\infty \langle h, \tau_n\rangle e_n = \sum_{n=1}^\infty \langle h, Te_n\rangle e_n = T^*h, \quad \forall h \in \mathcal{H}.$$

Therefore

$$\text{Ker }(T) = T^*(\mathcal{H})^\perp = \theta_\tau(\mathcal{H})^\perp = \mathcal{H}^\perp = \{0\}.$$

Hence T is injective.

□

Theorem 2.8 leads to the following definition of p-approximate Riesz basis.

Definition 2.9. A pair $(\{f_n\}_n, \{\tau_n\}_n)$ is said to be a p-approximate Riesz basis for $\mathcal{X}$ if it is a p-ASF for $\mathcal{X}$ and $\theta_f S_{f,\tau}^{-1}\theta_\tau = I_{\ell^p(\mathbb{N})}$.

Example 2.10. Let $\mathcal{X}$ be a Banach space which admits a Schauder basis $\{\omega_n\}_n$ and let $\{\zeta_n\}_n$ be the coordinate functionals associated with $\{e_n\}_n$. Let $U, V : \mathcal{X} \to \mathcal{X}$ be bounded linear operators such that VU is invertible. Define

$$f_n := \zeta_n U, \quad \tau_n := V\omega_n, \quad \forall n \in \mathbb{N}.$$

Then $(\{f_n\}_n, \{\tau_n\}_n)$ is an approximate Schauder frame for $\mathcal{X}$. If $VU = I_\mathcal{X}$, then $(\{f_n\}_n, \{\tau_n\}_n)$ is a Schauder frame for $\mathcal{X}$.

Example 2.11. Let $p \in [1, \infty)$ and $U : \mathcal{X} \to \ell^p(\mathbb{N})$, $V : \ell^p(\mathbb{N}) \to \mathcal{X}$ be bounded linear operators such that VU is invertible. Let $\{e_n\}_n$ denote the canonical Schauder basis for $\ell^p(\mathbb{N})$ and $\{\zeta_n\}_n$ denote the coordinate functionals associated with $\{e_n\}_n$ respectively. Define

$$f_n := \zeta_n U, \quad \tau_n := Ve_n, \quad \forall n \in \mathbb{N}.$$

Then $(\{f_n\}_n, \{\tau_n\}_n)$ is a p-ASF for $\mathcal{X}$.

Example 2.12. Let $p \in [1, \infty)$ and $U : \mathcal{X} \to \ell^p(\mathbb{N})$, $V : \ell^p(\mathbb{N}) \to \mathcal{X}$ be bounded invertible linear operators. Let $\{e_n\}_n$, $\{\zeta_n\}_n$, $\{f_n\}_n$, and $\{\tau_n\}_n$ be as in Example 2.11. Then $(\{f_n\}_n, \{\tau_n\}_n)$ is a p-approximate Riesz basis for $\mathcal{X}$.

We now derive the dilation theorem.

Theorem 2.13. *(Dilation theorem) Let $(\{f_n\}_n, \{\tau_n\}_n)$ be a p-ASF for $\mathcal{X}$. Then there exist a Banach space $\mathcal{X}_1$ which contains $\mathcal{X}$ isometrically and a p-approximate Riesz basis $(\{g_n\}_n, \{\omega_n\}_n)$ for $\mathcal{X}_1$ such that*

$$f_n = g_n P_{|\mathcal{X}}, \quad \tau_n = P\omega_n, \quad \forall n \in \mathbb{N},$$

where $P : \mathcal{X}_1 \to \mathcal{X}$ is an onto projection.

Proof. Let $\{e_n\}_n$ denote the standard Schauder basis for $\ell^p(\mathbb{N})$ and let $\{\zeta_n\}_n$ denote the coordinate functionals associated with $\{e_n\}_n$. Define

$$\mathcal{X}_1 := \mathcal{X} \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})(\ell^p(\mathbb{N})), \quad P : \mathcal{X}_1 \ni x \oplus y \mapsto x \oplus 0 \in \mathcal{X}_1$$

and

$$\omega_n := \tau_n \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})e_n \in \mathcal{X}_1, \qquad g_n := f_n \oplus \zeta_n(I_{\ell^p(\mathbb{N})} - P_{f,\tau}) \in \mathcal{X}_1^*, \quad \forall n \in \mathbb{N}.$$

Then clearly $\mathcal{X}_1$ contains $\mathcal{X}$ isometrically, $P : \mathcal{X}_1 \to \mathcal{X}$ is an onto projection and

$$\begin{aligned}
&(g_n P_{|\mathcal{X}})(x) = g_n(P_{|\mathcal{X}}x) = g_n(x) = (f_n \oplus \zeta_n(I_{\ell^p(\mathbb{N})} - P_{f,\tau}))(x \oplus 0) = f_n(x), \quad \forall x \in \mathcal{X},\\
&P\omega_n = P(\tau_n \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})e_n) = \tau_n, \quad \forall n \in \mathbb{N}.
\end{aligned}$$

Since the operator $I_{\ell^p(\mathbb{N})} - P_{f,\tau}$ is idempotent, it follows that $(I_{\ell^p(\mathbb{N})} - P_{f,\tau})(\ell^p(\mathbb{N}))$ is a closed subspace of $\ell^p(\mathbb{N})$ and hence a Banach space. Therefore $\mathcal{X}_1$ is a Banach space. Let $x \oplus y \in \mathcal{X}_1$ and we shall write $y = \{a_n\}_n \in \ell^p(\mathbb{N})$. We then see that

$$\begin{aligned}
&\sum_{n=1}^{\infty}(\zeta_n(I_{\ell^p(\mathbb{N})} - P_{f,\tau}))(y)\tau_n = \sum_{n=1}^{\infty}\zeta_n(y)\tau_n - \sum_{n=1}^{\infty}\zeta_n(P_{f,\tau}(y))\tau_n\\
&= \sum_{n=1}^{\infty}\zeta_n(\{a_k\}_k)\tau_n - \sum_{n=1}^{\infty}\zeta_n(\theta_f S_{f,\tau}^{-1}\theta_\tau(\{a_k\}_k))\tau_n\\
&= \sum_{n=1}^{\infty}a_n\tau_n - \sum_{n=1}^{\infty}\zeta_n\left(\theta_f S_{f,\tau}^{-1}\left(\sum_{k=1}^{\infty}a_k\tau_k\right)\right)\tau_n = \sum_{n=1}^{\infty}a_n\tau_n - \sum_{n=1}^{\infty}\zeta_n\left(\sum_{k=1}^{\infty}a_k\theta_f S_{f,\tau}^{-1}\tau_k\right)\tau_n\\
&= \sum_{n=1}^{\infty}a_n\tau_n - \sum_{n=1}^{\infty}\zeta_n\left(\sum_{k=1}^{\infty}a_k\sum_{r=1}^{\infty}f_r(S_{f,\tau}^{-1}\tau_k)e_r\right)\tau_n\\
&= \sum_{n=1}^{\infty}a_n\tau_n - \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}a_k\sum_{r=1}^{\infty}f_r(S_{f,\tau}^{-1}\tau_k)\zeta_n(e_r)\tau_n\\
&= \sum_{n=1}^{\infty}a_n\tau_n - \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}a_k f_n(S_{f,\tau}^{-1}\tau_k)\tau_n = \sum_{n=1}^{\infty}a_n\tau_n - \sum_{k=1}^{\infty}a_k\sum_{n=1}^{\infty}f_n(S_{f,\tau}^{-1}\tau_k)\tau_n\\
&= \sum_{n=1}^{\infty}a_n\tau_n - \sum_{k=1}^{\infty}a_k\tau_k = 0 \quad \text{and}
\end{aligned}$$

$$\sum_{n=1}^{\infty} f_n(x)(I_{\ell^p(\mathbb{N})} - P_{f,\tau})e_n = \sum_{n=1}^{\infty} f_n(x)e_n - \sum_{n=1}^{\infty} f_n(x)P_{f,\tau}e_n$$
$$= \sum_{n=1}^{\infty} f_n(x)e_n - \sum_{n=1}^{\infty} f_n(x)\theta_f S_{f,\tau}^{-1}\theta_\tau e_n = \sum_{n=1}^{\infty} f_n(x)e_n - \sum_{n=1}^{\infty} f_n(x)\theta_f S_{f,\tau}^{-1}\tau_n$$
$$= \sum_{n=1}^{\infty} f_n(x)e_n - \sum_{n=1}^{\infty} f_n(x)\sum_{k=1}^{\infty} f_k(S_{f,\tau}^{-1}\tau_n)e_k = \sum_{n=1}^{\infty} f_n(x)e_n - \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} f_n(x)f_k(S_{f,\tau}^{-1}\tau_n)e_k$$
$$= \sum_{n=1}^{\infty} f_n(x)e_n - \sum_{k=1}^{\infty}\sum_{n=1}^{\infty} f_n(x)f_k(S_{f,\tau}^{-1}\tau_n)e_k = \sum_{n=1}^{\infty} f_n(x)e_n - \sum_{k=1}^{\infty} f_k\left(\sum_{n=1}^{\infty} f_n(x)S_{f,\tau}^{-1}\tau_n\right)e_k$$
$$= \sum_{n=1}^{\infty} f_n(x)e_n - \sum_{k=1}^{\infty} f_k(x)e_k = 0.$$

By using previous two calculations, we get

$$S_{g,\omega}(x \oplus y) = \sum_{n=1}^{\infty} g_n(x \oplus y)\omega_n = \sum_{n=1}^{\infty}(f_n \oplus \zeta_n(I_{\ell^p(\mathbb{N})} - P_{f,\tau}))(x \oplus y)(\tau_n \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})e_n)$$
$$= \sum_{n=1}^{\infty}(f_n(x) + (\zeta_n(I_{\ell^p(\mathbb{N})} - P_{f,\tau}))(y))(\tau_n \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})e_n)$$
$$= \left(\sum_{n=1}^{\infty} f_n(x)\tau_n + \sum_{n=1}^{\infty}(\zeta_n(I_{\ell^p(\mathbb{N})} - P_{f,\tau}))(y)\tau_n\right) \oplus$$
$$\left(\sum_{n=1}^{\infty} f_n(x)(I_{\ell^p(\mathbb{N})} - P_{f,\tau})e_n + \sum_{n=1}^{\infty}(\zeta_n(I_{\ell^p(\mathbb{N})} - P_{f,\tau}))(y)(I_{\ell^p(\mathbb{N})} - P_{f,\tau})e_n\right)$$
$$= (S_{f,\tau}x + 0) \oplus \left(0 + (I_{\ell^p(\mathbb{N})} - P_{f,\tau})\sum_{n=1}^{\infty}\zeta_n((I_{\ell^p(\mathbb{N})} - P_{f,\tau})y)e_n\right)$$
$$= S_{f,\tau}x \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})(I_{\ell^p(\mathbb{N})} - P_{f,\tau})y = S_{f,\tau}x \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})y$$
$$= (S_{f,\tau} \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau}))(x \oplus y).$$

Since the operator $I_{\ell^p(\mathbb{N})} - P_{f,\tau}$ is idempotent, $I_{\ell^p(\mathbb{N})} - P_{f,\tau}$ becomes the identity operator on the space $(I_{\ell^p(\mathbb{N})} - P_{f,\tau})(\ell^p(\mathbb{N}))$. Hence we get that the operator $S_{g,\omega} = S_{f,\tau} \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})$ is bounded invertible from $\mathcal{X}_1$ onto itself. We next show that $(\{g_n\}_n, \{\omega_n\}_n)$ is a p-approximate Riesz basis for $\mathcal{X}_1$. For this, first we find θ_g and θ_ω. Consider

$$\theta_g(x \oplus y) = \{g_n(x \oplus y)\}_n = \{(f_n \oplus \zeta_n(I_{\ell^p(\mathbb{N})} - P_{f,\tau}))(x \oplus y)\}_n$$
$$= \{f_n(x) + \zeta_n((I_{\ell^p(\mathbb{N})} - P_{f,\tau})y)\}_n = \{f_n(x)\}_n + \{\zeta_n((I_{\ell^p(\mathbb{N})} - P_{f,\tau})y)\}_n$$
$$= \theta_f x + \sum_{n=1}^{\infty}\zeta_n((I_{\ell^p(\mathbb{N})} - P_{f,\tau})y)e_n = \theta_f x + (I_{\ell^p(\mathbb{N})} - P_{f,\tau})y, \quad \forall x \oplus y \in \mathcal{X}_1$$

and

$$\theta_\omega\{a_n\}_n = \sum_{n=1}^{\infty} a_n\omega_n = \sum_{n=1}^{\infty} a_n(\tau_n \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})e_n)$$
$$= \left(\sum_{n=1}^{\infty} a_n\tau_n\right) \oplus \left(\sum_{n=1}^{\infty} a_n(I_{\ell^p(\mathbb{N})} - P_{f,\tau})e_n\right)$$
$$= \theta_\tau\{a_n\}_n \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})\left(\sum_{n=1}^{\infty} a_n e_n\right)$$
$$= \theta_\tau\{a_n\}_n \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})\{a_n\}_n, \quad \forall\{a_n\}_n \in \ell^p(\mathbb{N}).$$

Therefore

$$\begin{aligned}P_{g,\omega}\{a_n\}_n &= \theta_g S_{g,\omega}^{-1}\theta_\omega\{a_n\}_n = \theta_g S_{g,\omega}^{-1}(\theta_\tau\{a_n\}_n \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})\{a_n\}_n)\\
&= \theta_g(S_{f,\tau}^{-1} \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau}))(\theta_\tau\{a_n\}_n \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})\{a_n\}_n)\\
&= \theta_g(S_{f,\tau}^{-1}\theta_\tau\{a_n\}_n \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})^2\{a_n\}_n)\\
&= \theta_g(S_{f,\tau}^{-1}\theta_\tau\{a_n\}_n \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})\{a_n\}_n)\\
&= \theta_f(S_{f,\tau}^{-1}\theta_\tau\{a_n\}_n) + (I_{\ell^p(\mathbb{N})} - P_{f,\tau})(I_{\ell^p(\mathbb{N})} - P_{f,\tau})\{a_n\}_n\\
&= P_{f,\tau}\{a_n\}_n + (I_{\ell^p(\mathbb{N})} - P_{f,\tau})\{a_n\}_n = \{a_n\}_n, \quad \forall \{a_n\}_n \in \ell^p(\mathbb{N}).\end{aligned}$$

□

Following dilation result of Han and Larson [19] is a particular case of Theorem 2.13.

Corollary 2.14. *[19, 23] Let $\{\tau_n\}_n$ be a frame for $\mathcal{H}$. Then there exist a Hilbert space $\mathcal{H}_1$ which contains $\mathcal{H}$ isometrically and a Riesz basis $\{\omega_n\}_n$ for $\mathcal{H}_1$ such that*

$$\tau_n = P\omega_n, \quad \forall n \in \mathbb{N},$$

where P is the orthogonal projection from $\mathcal{H}_1$ onto $\mathcal{H}$.

Proof. Let $\{\tau_n\}_n$ be a frame for $\mathcal{H}$. Define

$$f_n : \mathcal{H} \ni h \mapsto f_n(h) := \langle h, \tau_n\rangle \in \mathbb{K}, \quad \forall n \in \mathbb{N}.$$

Then $\theta_f = \theta_\tau$. Note that $(\{f_n\}_n, \{\tau_n\}_n)$ is a 2-approximate frame for $\mathcal{H}$. Theorem 2.13 now says that there exist a Banach space $\mathcal{X}_1$ which contains $\mathcal{H}$ isometrically and a 2-approximate Riesz basis $(\{g_n\}_n, \{\omega_n\}_n)$ for $\mathcal{X}_1 = \mathcal{H} \oplus (I_{\ell^2(\mathbb{N})} - P_\tau)(\ell^2(\mathbb{N}))$ such that

$$f_n = g_n P_{|\mathcal{H}}, \quad \tau_n = P\omega_n, \quad \forall n \in \mathbb{N},$$

where $P : \mathcal{X}_1 \to \mathcal{H}$ is an onto projection. Since $(I_{\ell^2(\mathbb{N})} - P_\tau)(\ell^2(\mathbb{N}))$ is a closed subspace of the Hilbert space $\ell^2(\mathbb{N})$, $\mathcal{X}_1$ now becomes a Hilbert space. From the definition of the operator P we get that it is an orthogonal projection. Now to prove Theorem 1.3, we are left with proving $\{\omega_n\}_n$ is a Riesz basis for $\mathcal{X}_1$. To show $\{\omega_n\}_n$ is a Riesz basis for $\mathcal{X}_1$, we use Theorem 2.8. Since $\{\tau_n\}_n$ is a frame for $\mathcal{H}$ there exist $a, b > 0$ such that

$$a\|h\|^2 \leq \sum_{n=1}^\infty |\langle h, \tau_n\rangle|^2 \leq b\|h\|^2, \quad \forall h \in \mathcal{H}.$$

Let $h \oplus (I_{\ell^2(\mathbb{N})} - P_{f,\tau})\{a_k\}_k \in \mathcal{X}_1$. Then by noting $b \geq 1$, we get

$$\begin{aligned}&\sum_{n=1}^\infty |\langle h \oplus (I_{\ell^2(\mathbb{N})} - P_\tau)\{a_k\}_k, \omega_n\rangle|^2 = \sum_{n=1}^\infty |\langle h \oplus (I_{\ell^2(\mathbb{N})} - P_\tau)\{a_k\}_k, \tau_n \oplus (I_{\ell^2(\mathbb{N})} - P_\tau)e_n\rangle|^2\\
&= \sum_{n=1}^\infty |\langle h, \tau_n\rangle|^2 + \sum_{n=1}^\infty |\langle (I_{\ell^2(\mathbb{N})} - P_\tau)\{a_k\}_k, (I_{\ell^2(\mathbb{N})} - P_\tau)e_n\rangle|^2\\
&= \sum_{n=1}^\infty |\langle h, \tau_n\rangle|^2 + \sum_{n=1}^\infty |\langle (I_{\ell^2(\mathbb{N})} - P_\tau)(I_{\ell^2(\mathbb{N})} - P_\tau)\{a_k\}_k, e_n\rangle|^2\\
&= \sum_{n=1}^\infty |\langle h, \tau_n\rangle|^2 + \sum_{n=1}^\infty |\langle (I_{\ell^2(\mathbb{N})} - P_\tau)\{a_k\}_k, e_n\rangle|^2\\
&= \sum_{n=1}^\infty |\langle h, \tau_n\rangle|^2 + \|(I_{\ell^2(\mathbb{N})} - P_\tau)\{a_k\}_k\|^2\\
&\leq b\|h\|^2 + \|(I_{\ell^2(\mathbb{N})} - P_\tau)\{a_k\}_k\|^2\\
&\leq b(\|h\|^2 + \|(I_{\ell^2(\mathbb{N})} - P_\tau)\{a_k\}_k\|^2)\\
&= b\|h \oplus (I_{\ell^2(\mathbb{N})} - P_\tau)\{a_k\}_k\|^2.\end{aligned}$$

Previous calculation tells that $\{\omega_n\}_n$ is a Bessel sequence for $\mathcal{X}_1$. Hence $S_\omega : \mathcal{X}_1 \ni x\oplus\{a_k\}_k \mapsto \sum_{n=1}^\infty \langle x\oplus\{a_k\}_k, \omega_n\rangle\omega_n \in \mathcal{X}_1$ is a well-defined bounded linear operator. Next we claim that

$$g_n(x\oplus\{a_k\}_k) = \langle x\oplus\{a_k\}_k, \omega_n\rangle, \quad \forall\, x\oplus\{a_k\}_k \in \mathcal{X}_1, \forall n\in\mathbb{N}. \tag{2.3}$$

Consider

$$\begin{aligned}
g_n(x\oplus\{a_k\}_k) &= (f_n\oplus\zeta_n(I_{\ell^2(\mathbb{N})}-P_\tau))(x\oplus\{a_k\}_k)\\
&= f_n(x)+\zeta_n((I_{\ell^2(\mathbb{N})}-P_\tau)\{a_k\}_k) = f_n(x)+\zeta_n(\{a_k\}_k)-\zeta_n(P_\tau\{a_k\}_k)\\
&= f_n(x)+\zeta_n(\{a_k\}_k)-\zeta_n(\theta_\tau S_\tau^{-1}\theta_\tau^*\{a_k\}_k)\\
&= f_n(x)+a_n-\zeta_n\left(\theta_\tau S_\tau^{-1}\left(\sum_{k=1}^\infty a_k\tau_k\right)\right)\\
&= f_n(x)+a_n-\zeta_n\left(\sum_{k=1}^\infty a_k\theta_\tau S_\tau^{-1}\tau_k\right)\\
&= f_n(x)+a_n-\zeta_n\left(\sum_{k=1}^\infty a_k\sum_{r=1}^\infty\langle S_\tau^{-1}\tau_k,\tau_r\rangle e_r\right)\\
&= f_n(x)+a_n-\sum_{k=1}^\infty a_k\langle S_\tau^{-1}\tau_k,\tau_n\rangle = \langle x,\tau_n\rangle+a_n-\sum_{k=1}^\infty a_k\langle S_\tau^{-1}\tau_k,\tau_n\rangle
\end{aligned}$$

and

$$\begin{aligned}
\langle x\oplus\{a_k\}_k,\omega_n\rangle &= \langle x\oplus\{a_k\}_k, \tau_n\oplus(I_{\ell^2(\mathbb{N})}-P_\tau)e_n\rangle\\
&= \langle x,\tau_n\rangle+\langle\{a_k\}_k,(I_{\ell^2(\mathbb{N})}-P_\tau)e_n\rangle = \langle x,\tau_n\rangle+\langle\{a_k\}_k,e_n\rangle+\langle\{a_k\}_k,P_\tau e_n\rangle\\
&= \langle x,\tau_n\rangle+a_n-\left\langle\{a_k\}_k,\theta_\tau S_\tau^{-1}\theta_\tau^* e_n\right\rangle = \langle x,\tau_n\rangle+a_n-\left\langle\{a_k\}_k,\theta_\tau S_\tau^{-1}\tau_n\right\rangle\\
&= \langle x,\tau_n\rangle+a_n-\langle\{a_k\}_k,\{\langle S_\tau^{-1}\tau_n,\tau_k\rangle\}_k\rangle = \langle x,\tau_n\rangle+a_n-\sum_{k=1}^\infty a_k\overline{\langle S_\tau^{-1}\tau_n,\tau_k\rangle}\\
&= \langle x,\tau_n\rangle+a_n-\sum_{k=1}^\infty a_k\langle\tau_k,S_\tau^{-1}\tau_n\rangle = \langle x,\tau_n\rangle+a_n-\sum_{k=1}^\infty a_k\langle S_\tau^{-1}\tau_k,\tau_n\rangle.
\end{aligned}$$

Thus Equation (2.3) holds. Therefore for all $x\oplus\{a_k\}_k\in\mathcal{X}_1$,

$$S_{g,\omega}(x\oplus\{a_k\}_k) = \sum_{n=1}^\infty g_n(x\oplus\{a_k\}_k)\omega_n = \sum_{n=1}^\infty\langle x\oplus\{a_k\}_k,\omega_n\rangle\omega_n = S_\omega(x\oplus\{a_k\}_k).$$

Since $S_{g,\omega}$ is invertible, S_ω becomes invertible. Clearly S_ω is positive. Therefore

$$\frac{1}{\|S_\omega\|^{-1}}\|g\|^2 \le \langle S_\omega g,g\rangle \le \|S_\omega\|\|g\|^2, \quad \forall g\in\mathcal{X}_1.$$

Hence

$$\frac{1}{\|S_\omega\|^{-1}}\|g\|^2 \le \sum_{n=1}^\infty|\langle g,\omega_n\rangle|^2 \le \|S_\omega\|\|g\|^2, \quad \forall g\in\mathcal{X}_1.$$

That is, $\{\omega_n\}_n$ is a frame for $\mathcal{X}_1$.

Finally we show Equation (2.2) in Theorem 2.8 for the frame $\{\omega_n\}_n$. Consider

$$\begin{aligned}
\theta_\omega S_\omega^{-1}\theta_\omega^*\{a_n\}_n &= \theta_\omega S_\omega^{-1}\left(\sum_{n=1}^\infty a_n\omega_n\right) = \theta_\omega\left(\sum_{n=1}^\infty a_n S_\omega^{-1}\omega_n\right)\\
&= \sum_{k=1}^\infty \left\langle \sum_{n=1}^\infty a_n S_\omega^{-1}\omega_n, \omega_k\right\rangle = \sum_{k=1}^\infty\sum_{n=1}^\infty a_n\langle S_\omega^{-1}\omega_n, \omega_k\rangle\\
&= \sum_{k=1}^\infty\sum_{n=1}^\infty a_n\langle (S_\tau^{-1}\oplus(I_{\ell^2(\mathbb{N})}-P_\tau))(\tau_n\oplus(I_{\ell^2(\mathbb{N})}-P_\tau)e_n), \tau_k\oplus(I_{\ell^2(\mathbb{N})}-P_\tau)e_k\rangle\\
&= \sum_{k=1}^\infty\sum_{n=1}^\infty a_n\langle (S_\tau^{-1}\tau_n\oplus(I_{\ell^2(\mathbb{N})}-P_\tau)^2)e_n, \tau_k\oplus(I_{\ell^2(\mathbb{N})}-P_\tau)e_k\rangle\\
&= \sum_{k=1}^\infty\sum_{n=1}^\infty a_n(\langle S_\tau^{-1}\tau_n, \tau_k\rangle + \langle (I_{\ell^2(\mathbb{N})}-P_\tau)e_n, (I_{\ell^2(\mathbb{N})}-P_\tau)e_k\rangle)\\
&= \sum_{k=1}^\infty \left\langle \sum_{n=1}^\infty a_n S_\tau^{-1}\tau_n, \tau_k\right\rangle + \sum_{k=1}^\infty\sum_{n=1}^\infty a_n\langle (I_{\ell^2(\mathbb{N})}-P_{f,\tau})e_n, (I_{\ell^2(\mathbb{N})}-P_\tau)e_k\rangle\\
&= P_\tau\{a_n\}_n + \sum_{k=1}^\infty\sum_{n=1}^\infty a_n\langle (I_{\ell^2(\mathbb{N})}-P_\tau)e_n, e_k\rangle\\
&= P_\tau\{a_n\}_n + \sum_{k=1}^\infty\sum_{n=1}^\infty a_n\langle e_n, e_k\rangle - \sum_{k=1}^\infty\sum_{n=1}^\infty a_n\langle P_\tau e_n, e_k\rangle\\
&= P_\tau\{a_n\}_n + \sum_{k=1}^\infty a_k e_k - \sum_{k=1}^\infty\sum_{n=1}^\infty a_n\langle \theta_\tau S_\tau^{-1}\theta_\tau^* e_n, e_k\rangle\\
&= P_\tau\{a_n\}_n + \sum_{k=1}^\infty a_k e_k - \sum_{k=1}^\infty\sum_{n=1}^\infty a_n\langle S_\tau^{-1}\tau_n, \theta_\tau^* e_k\rangle\\
&= P_\tau\{a_n\}_n + \sum_{k=1}^\infty a_k e_k - \sum_{k=1}^\infty\sum_{n=1}^\infty a_n\langle S_\tau^{-1}\tau_n, \tau_k\rangle\\
&= P_\tau\{a_n\}_n + \sum_{k=1}^\infty a_k e_k - P_\tau\{a_n\}_n = \{a_n\}_n, \quad \forall\{a_n\}_n\in\ell^2(\mathbb{N}).
\end{aligned}$$

Thus $\{\omega_n\}_n$ is a Riesz basis for $\mathcal{X}_1$ which completes the proof. □

We now illustrate Theorem 2.13 with an example.

Example 2.15. Let $p \in [1,\infty)$. Let $\{e_n\}_n$ denote the canonical Schauder basis for $\ell^p(\mathbb{N})$ and $\{\zeta_n\}_n$ denote the coordinate functionals associated with $\{e_n\}_n$ respectively. Define

$$R : \ell^p(\mathbb{N}) \ni (x_n)_{n=1}^\infty \mapsto (0, x_1, x_2, \dots) \in \ell^p(\mathbb{N}),$$
$$L : \ell^p(\mathbb{N}) \ni (x_n)_{n=1}^\infty \mapsto (x_2, x_3, x_4, \dots) \in \ell^p(\mathbb{N}).$$

Then $LR = I_{\ell^p(\mathbb{N})}$. Example 2.12 says that $(\{f_n := \zeta_n R\}_n, \{\tau_n := Le_n\}_n)$ is a p-ASF for $\ell^p(\mathbb{N})$. Note that $\theta_f = R$ and $\theta_\tau = L$. Therefore $S_{f,\tau} = LR = I_{\ell^p(\mathbb{N})}$ and $P_{f,\tau} = RL$. Then

$$\begin{aligned}
(I_{\ell^p(\mathbb{N})} - P_{f,\tau})(x_n)_{n=1}^\infty &= (x_n)_{n=1}^\infty - RL(x_n)_{n=1}^\infty\\
&= (x_n)_{n=1}^\infty - (0, x_2, x_3, \dots) = (x_1, 0, 0, \dots), \quad \forall (x_n)_{n=1}^\infty \in \ell^p(\mathbb{N})
\end{aligned}$$

which says that $(I_{\ell^p(\mathbb{N})} - P_{f,\tau})(\ell^p(\mathbb{N})) \cong \mathbb{K}$. Using Theorem 2.13,

$$\mathcal{X}_1 = \ell^p(\mathbb{N}) \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})(\ell^p(\mathbb{N})) \cong \ell^p(\mathbb{N}) \oplus \mathbb{K} \cong \ell^p(\mathbb{N}\cup\{0\})$$
$$P : \ell^p(\mathbb{N}\cup\{0\}) \ni (x_n)_{n=0}^\infty \mapsto (x_n)_{n=1}^\infty \in \ell^p(\mathbb{N}),$$

$$\begin{aligned}
\omega_1 &= \tau_1 \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})\tau_1 = Le_1 \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})Le_1 = 0 \oplus 0,\\
\omega_2 &= \tau_2 \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})\tau_2 = Le_n \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})Le_2\\
&= e_1 \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})e_1 = e_1 \oplus RLe_1 = e_1 \oplus 0,\\
\omega_n &= \tau_n \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})\tau_n = Le_n \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})Le_n\\
&= e_{n-1} \oplus (I_{\ell^p(\mathbb{N})} - P_{f,\tau})e_{n-1} = e_{n-1} \oplus RLe_{n-1} = e_{n-1} \oplus e_{n-1}, \quad \forall n \geq 3,\\
g_n &= f_n \oplus \zeta_n(I_{\ell^p(\mathbb{N})} - P_{f,\tau}) = \zeta_n R \oplus \zeta_n RL = \zeta_n R(I_{\ell^p(\mathbb{N})} \oplus L), \quad \forall n \in \mathbb{N}
\end{aligned}$$

and $(\{g_n\}_n, \{\omega_n\}_n)$ is a p-approximate Riesz basis for $\ell^p(\mathbb{N})$.

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Author information

K. Mahesh Krishna and P. Sam Johnson, Department of Mathematical and Computational Sciences
National Institute of Technology Karnataka (NITK), Surathkal, Mangaluru 575 025, India.
E-mail: kmaheshak@gmail.com, sam@nitk.edu.in

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