Palestine Journal of Mathematics

Vol. 11(2)(2022) , 38–44

Some Congruences for Ramanujan's General Partition function

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Communicated by V. Lokesha

MSC 2020 Classifications: Primary 11P83; Secondary 05A15, 05A17.

Keywords and phrases: Partition congruence, general partition function, q-identities.

Acknowledgement: *The authors would like to thank the anonymous referee for his valuable comments and suggestions.*

Abstract Let $p_r(n)$ be the Ramanujan's partition function in its general form where n and r denotes non-negative integer and non-zero integer respectively. Certain supplementary congruence for $p_r(n)$ with the application of theta function identities which are attributed to Ramanujan where are discussed here subsequently.

1 Introduction

In 1991, Bruce C Berndt [3, p. 34] discussed the general theta function developed by Ramanujan, which is denoted as $f(a,b)$, in which $|ab| < 1$ and mathematically it is represented as

$$f(a,b) := \sum_{k=-\infty}^{\infty} a^{k(k+1)/2} b^{k(k-1)/2}.$$

By using Jacobi's triple product identity [3, p. 35], the function $f(a,b)$ can be written as

$$f(a,b) := (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty.$$

Here and throughout the paper, we assume that $|q| < 1$ and employ the standard notation

$$(a;q)_\infty = \prod_{k=0}^{\infty}(1-aq^k).$$

A principal case of $f(a,b)$ is the Euler's pentagonal number theorem,

$$f(-q) := f(-q,-q^2) = \sum_{k=-\infty}^{\infty}(-1)^k q^{k(3k-1)/2} = (q;q)_\infty.$$

For convenience, we write $f_n := f(-q^n)$.
In 1918 Ramanujan [4, p. 192-193] set forth the discussion of the general partition function for any non-negative and non-zero integer represented as n and r and denoted by $p_r(n)$ as

$$\sum_{n=0}^{\infty} p_r(n) q^n = \frac{1}{(q;q)_\infty^r}, |q| < 1. \tag{1.1}$$

For value of r being 1, $p_1(n)$ represents the partition function which is unrestricted in nature and counts the number of unrestricted partition of any given non-negative integer n respectively. For simplification $p_1(n)$ can be denoted as $p(n)$ and Ramanujan had worked extensively on such function [9–11]. For example, we have Ramanujan's so called "most beautiful identity"

$$\sum_{n=0}^{\infty} p(5n+4)q^n = \frac{5f_5^5}{f_1^6},$$

which readily implies

$$p(5n+4) \equiv 0 \pmod 5.$$

Ramanujan [4] for asserted for a positive integer λ and for any prime $\overline{\nu}$ which can be represented as $6\lambda - 1$, satisfies

$$p_{-4}\left(n\overline{\nu} - \frac{\overline{\nu}+1}{6}\right) \equiv 0 \pmod{\overline{\nu}}.$$

After Ramanujan, the congruence properties of the partition function $p_r(n)$ are studied by Newman [7], Ramanathan [8, 9], Atkin [2], Andrews [1], Gandhi [5], Kiming and Olsson [6]. Recently, Saika and Chetry [13] and Srivatsava et al. [14] discussed the certain new properties of the general partition function $p_r(n)$ by considering r to be negative. In similar lines to the above mentioned references, we discussed an elaborate the study of infinite family of congruences modulo 25 for $p_r(n)$, where $r \in$ {-(25k+ν), -(125k+δ)}, where k is any non-negative integer with some restrictions, $\nu \in \{6, 8, 12, 14, 18, 24\}$ and $\delta \in \{6, 12, 18, 24\}$.
The important observations are:

Theorem 1.1. *Let* $r = -(25k+6)$ *and* $k \equiv -1 \pmod{25}$ *then we have*

$$p_r(25n+16) \equiv p_r(25n+22) \equiv 0 \pmod{25}, \tag{1.2}$$

and for $1 \le \nu \le 4$

$$p_r(125n+25\nu+6) \equiv 0 \pmod{25}. \tag{1.3}$$

Theorem 1.2. *Let* $r = -(25k+12)$ *and* $k \equiv -2 \pmod{25}$ *then we have for* $1 \le \nu \le 4$

$$p_r(125n+25\nu+12) \equiv 0 \pmod{25}. \tag{1.4}$$

Theorem 1.3. *Let* $r = -(25k+18)$ *and* $k \equiv -3 \pmod{25}$ *then we have*

$$p_r(25n+13) \equiv p_r(25n+18) \equiv p_r(25n+23) \equiv 0 \pmod{25}. \tag{1.5}$$

Theorem 1.4. *Let* $r = -(25k+24)$ *and* $k \equiv -4 \pmod{25}$ *then we have*

$$p_r(25n+24) \equiv 0 \pmod{25}. \tag{1.6}$$

Theorem 1.5. *Let* $r = -(25k+8)$ *then we have*

$$p_r(5n+3) \equiv 0 \pmod{25}. \tag{1.7}$$

Theorem 1.6. *Let* $r = -(25k+14)$ *then we have*

$$p_r(5n+4) \equiv 0 \pmod{25}. \tag{1.8}$$

Theorem 1.7. *Let* $r = -(125k+6)$ *and* $k \equiv -1 \pmod{25}$ *then we have*

$$p_r(125n+81) \equiv p_r(125n+106) \equiv 0 \pmod{25}, \tag{1.9}$$

and for $1 \le \nu \le 4$

$$p_r(625n+125\nu+31) \equiv 0 \pmod{25}. \tag{1.10}$$

Theorem 1.8. *Let* $r = -(125k+12)$ *and* $k \equiv -2 \pmod{25}$ *then we have for* $1 \le \nu \le 4$

$$p_r(625n+125\nu+62) \equiv 0 \pmod{25}. \tag{1.11}$$

Theorem 1.9. *Let* $r = -(125k+18)$ *then we have*

$$p_r(25n+13) \equiv p_r(25n+18) \equiv p_r(25n+23) \equiv 0 \pmod{25}. \tag{1.12}$$

Theorem 1.10. *Let* $r = -(125k+24)$ *then we have*

$$p_r(25n+9) \equiv 0 \pmod{25}. \tag{1.13}$$

2 Preliminaries

In this section, we collect some results in order to prove our main results.
From [3, p. 262. entry 10(iii)], we have

$$f_1 = f_{25}\left(\frac{1}{R(q^5)} - q - q^2R(q^5)\right), \tag{2.1}$$

where

$$R(q) = \frac{(q;q^5)_\infty(q^4;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}.$$

From [15], we have

$$\frac{f_1^6}{f_5^6} = \frac{1}{R(q)^5} - 11q - q^2R(q)^5. \tag{2.2}$$

With application of the congruence obtained from binomial theorem:

$$f_1^{25} \equiv f_5^5 \pmod{25}. \tag{2.3}$$

3 Proofs of Theorems 1.1-1.10

Proof of the Theorem 1.1. Setting $r = -(25k+6)$ in (1.1), we observe that

$$\sum_{n=0}^{\infty} p_{-(25k+6)}(n)q^n = f_1^{25k+6} = f_1^{25k}f_1^6, \tag{3.1}$$

Applying modulo 25 in (3.1) then using (2.3), we obtain

$$\sum_{n=0}^{\infty} p_{-(25k+6)}(n)q^n \equiv f_5^{5k}f_1^6 \pmod{25}. \tag{3.2}$$

With application of (2.1) in (3.2) and drawing out the q^{5n+1} terms from both the sides and then dividing the resulted identity by q and changing q^5 into q, we obtain

$$\sum_{n=0}^{\infty} p_{-(25k+6)}(5n+1)q^n \equiv f_1^{5k}f_5^6\left(\frac{19}{R(q)^5} + 16q + 6q^2R(q)^5\right) \pmod{25}. \tag{3.3}$$

Using (2.2) in (3.3), we obtain

$$\sum_{n=0}^{\infty} p_{-(25k+6)}(5n+1)q^n \equiv 19f_1^{5k+6} \equiv 19f_1^{5(k+1)}f_1 \pmod{25}. \tag{3.4}$$

From above identity if $k+1$ is multiples of 25, then we obtain

$$\sum_{n=0}^{\infty} p_{-(25k+6)}(5n+1)q^n \equiv 19f_5^{25m}f_1 \pmod{25}. \tag{3.5}$$

With application of (2.1) in (3.5) and drawing out the q^{5n+3} and q^{5n+4} terms from both the sides. We establish the desired result (1.2).
Now, with application of (2.1) in (3.5) and drawing out the q^{5n+1} terms from both the sides and then dividing the resulted identity by q and changing q^5 into q, we obtain

$$\sum_{n=0}^{\infty} p_{-(25k+6)}(25n+6)q^n \equiv 6f_5^{5m+1} \pmod{25}. \tag{3.6}$$

For $1 \leq \nu \leq 4$ draw out the common terms of $q^{5n+\nu}$ which occurs on both sides. We establish the desired result (1.3). □

We omit the proof of Theorem 1.2 - Theorem 1.4, since its follows the same line as Theorem 1.1 by fixing $r = -(25k+12), -(25k+18), -(25k+24)$ in equation (1.1).

Proof of the Theorem 1.5. Setting $r = -(25k+8)$ in (1.1), we observe

$$\sum_{n=0}^{\infty} p_{-(25k+8)}(n)q^n = f_1^{25k+8} = f_1^{25k} f_1^8. \tag{3.7}$$

Applying modulo 25 in (3.7) then using (2.3), we obtain

$$\sum_{n=0}^{\infty} p_{-(25k+8)}(n)q^n \equiv f_5^{5k} f_1^8 \pmod{25}. \tag{3.8}$$

Using (2.1) in (3.8) and drawing out the common terms of q^{5n+3} which occurs on both sides, the following results are established. □

We omit the proof of Theorem 1.6, since its follows the same line as Theorem 1.5 by fixing $r = -(25k+14)$ in equation (1.1).

Proof of the Theorem 1.7. Setting $r = -(125k+6)$ in (1.1), we observe

$$\sum_{n=0}^{\infty} p_{-(125k+6)}(n)q^n = f_1^{125k+6} = f_1^{125k} f_1^6. \tag{3.9}$$

Applying modulo 25 in (3.9) then using (2.3), we obtain

$$\sum_{n=0}^{\infty} p_{-(125k+6)}(n)q^n \equiv f_5^{25k} f_1^6 \pmod{25}. \tag{3.10}$$

Using (2.1) in (3.10) and drawing out the common terms of q^{5n+1} which occurs on both sides, then dividing the resulted identity by q and changing q^5 into q, we observe

$$\sum_{n=0}^{\infty} p_{-(125k+6)}(5n+1)q^n \equiv f_1^{25k} f_5^6 \left(\frac{19}{R(q)^5} + 16q + 6q^2 R(q)^5\right) \pmod{25}. \tag{3.11}$$

Using (2.2) in (3.11), we obtain

$$\sum_{n=0}^{\infty} p_{-(125k+6)}(5n+1)q^n \equiv 19 f_5^{5k} f_1^6 \pmod{25}. \tag{3.12}$$

By using (2.1) in (3.12) and drawing out the common terms of q^{5n+1} which occurs on both sides, then dividing the resulted identity by q and changing q^5 into q, together with (2.2), we obtain

$$\sum_{n=0}^{\infty} p_{-(125k+6)}(25n+6)q^n \equiv (19)^2 f_1^{5k+6} \equiv 11 f_1^{5(k+1)} f_1 \pmod{25}. \tag{3.13}$$

From above identity if $k+1$ is multiples of 25, then we obtain

$$\sum_{n=0}^{\infty} p_{-(125k+6)}(25n+6)q^n \equiv 11 f_5^{25m} f_1 \pmod{25}. \tag{3.14}$$

With application of (2.1) in (3.14) and drawing out the q^{5n+3} and q^{5n+4} terms from both the sides. We establish the desired result (1.9).
Now, with application of (2.1) in (3.14) and drawing out the q^{5n+1} terms from both the sides and then dividing the resulted identity by q and changing q^5 into q, we obtain

$$\sum_{n=0}^{\infty} p_{-(25k+6)}(125n+31)q^n \equiv 14 f_5^{5m+1} \pmod{25}. \tag{3.15}$$

For $1 \leq \nu \leq 4$ draw out the common terms of $q^{5n+\nu}$ which occurs on both sides . We establish the desired result (1.10). □

We omit the proof of Theorem 1.8, since its follows the same line as Theorem 1.7 by fixing $r = -(125k + 12)$ in equation (1.1).

Proof of the Theorem 1.9. Setting $r = -(125k + 18)$ in (1.1), we observe

$$\sum_{n=0}^{\infty} p_{-(125k+18)}(n)q^n = f_1^{125k+18} = f_1^{125k} f_1^{18}. \tag{3.16}$$

Applying modulo 25 in (3.16) then using (2.3), we obtain

$$\sum_{n=0}^{\infty} p_{-(125k+18)}(n)q^n \equiv f_5^{25k} f_1^{18} \pmod{25}. \tag{3.17}$$

Using (2.1) in (3.17) and drawing out the common terms of q^{5n+3} which occurs on both sides, then dividing the resulted identity by q^3 and changing q^5 into q, we observe

$$\begin{aligned}\sum_{n=0}^{\infty} p_{-(125k+18)}(5n+3)q^n \equiv & f_1^{25k} f_5^{18} \left(\frac{15}{R(q)^{15}} + \frac{5q}{R(q)^{10}} \right. \\ & \left. + 5q^5 R(q)^{10} + 10q^6 R(q)^{15} \right) \pmod{25},\end{aligned} \tag{3.18}$$

Using (2.2) in (3.18), we obtain

$$\sum_{n=0}^{\infty} p_{-(125k+18)}(5n+3)q^n \equiv 15 f_5^{5k} f_1^{18} \pmod{25}. \tag{3.19}$$

With application of (2.1) in (3.18) and drawing out the common terms of q^{5n+2}, q^{5n+3} and q^{5n+4} which occurs on both sides. We establish the desired result (1.12). □

We omit the proof of Theorem 1.10, since its follows the same line as Theorem 1.9 by fixing $r = -(125k + 24)$ in equation (1.1).

4 Corollaries

In this section, we obtain some Corollaries as direct consequences of our main theorems.

Corollary 4.1.

$$\sum_{n=0}^{\infty} p_{-(25k+6)}(5n+1)q^n = 19 \sum_{n=0}^{\infty} p_{-(5k+6)}(n)q^n \pmod{25}. \tag{4.1}$$

$$\sum_{n=0}^{\infty} p_{-(25k+12)}(5n+2)q^n = 4 \sum_{n=0}^{\infty} p_{-(5k+12)}(n)q^n \pmod{25}. \tag{4.2}$$

$$\sum_{n=0}^{\infty} p_{-(25k+18)}(5n+3)q^n = 15 \sum_{n=0}^{\infty} p_{-(5k+18)}(n)q^n \pmod{25}. \tag{4.3}$$

$$\sum_{n=0}^{\infty} p_{-(25k+24)}(5n+4)q^n = 5 \sum_{n=0}^{\infty} p_{-(5k+24)}(n)q^n \pmod{25}. \tag{4.4}$$

$$\sum_{n=0}^{\infty} p_{-(125k+6)}(5n+1)q^n = 19 \sum_{n=0}^{\infty} p_{-(25k+6)}(n)q^n \pmod{25}. \tag{4.5}$$

$$\sum_{n=0}^{\infty} p_{-(125k+6)}(25n+6)q^n = 11 \sum_{n=0}^{\infty} p_{-(5k+6)}(n)q^n \pmod{25}. \tag{4.6}$$

$$\sum_{n=0}^{\infty} p_{-(125k+12)}(5n+2)q^n = 4 \sum_{n=0}^{\infty} p_{-(25k+12)}(n)q^n \pmod{25}. \tag{4.7}$$

$$\sum_{n=0}^{\infty} p_{-(125k+12)}(25n+7)q^n = 16 \sum_{n=0}^{\infty} p_{-(5k+12)}(n)q^n \pmod{25}. \tag{4.8}$$

$$\sum_{n=0}^{\infty} p_{-(125k+18)}(5n+3)q^n = 15 \sum_{n=0}^{\infty} p_{-(25k+18)}(n)q^n \pmod{25}. \tag{4.9}$$

$$\sum_{n=0}^{\infty} p_{-(125k+24)}(5n+1)q^n = 5 \sum_{n=0}^{\infty} p_{-(25k+24)}(n)q^n \pmod{25}. \tag{4.10}$$

Proof. From the identity (3.4) we obtain (4.1). Similar proof hold for identity (4.2) - (4.4). From the identity (3.12) and (3.13) we obtain (4.5) and (4.6) respectively. Similar proof hold for identity (4.7) - (4.10). □

References

[1] G. E. Andrews, Survey of multipartitions: congruences and identities, *Dev. Math.*, **17**, 1—19 (2008).

[2] A. O. L. Atkin, Ramanujan congruences for $p_k(n)$, *Canad. J. Math.*,**20**, 67—78 (1968).

[3] B. C. Berndt, Ramanujan's Notebooks. Part III, *Springer*, New York, NY, USA (1991).

[4] B. C. Berndt, R. A. Rankin, Ramanujan: Letters and commentary, *American Mathematical Society*, Providence (1995).

[5] J. M. Gandhi, Congruences for $p_r(n)$ and Ramanujan's τ function, *Am. Math. Mon.*, **70**, 265—274 (1963).

[6] I. Kiming, J. B. Olsson, Congruences like Ramanujan's for powers of the partition function, *Arch. Math.* (Basel), **59(4)**, 348—360 (1992).

[7] M. Newmann, Congruence for the coefficients of modular forms and some new congruences for the partition function, *Canad. J. Math.*, **9**, 549—552 (1957).

[8] K. G. Ramanathan, Identities and congruences of the Ramanujan type, *Canad. J. Math.*, **2**, 168–178 (1950).

[9] K. G. Ramanathan, Ramanujan and the congruences properties of partitions, *Proc. Indian. Acad. Sci.* (Math. Sci.), **89**, 133–157 (1980).

[10] S. Ramanujan, Some properties of $p(n)$, the number of partition of n, *Proc. Camb. Philos. Soc.*, **19**, 207–210 (1919).

[11] S. Ramanujan, Congruence properties of partitions, *Proc. Lond. Math. Soc.*, **18**, (1920).

[12] S. Ramanujan, Congruence properties of partitions, *Math. Z.*, **9**, 147–153 (1921).

[13] N. Saikia, J. Chetry, Infinite family of congruences modulo 7 for Ramanujan's general partition function, *Ann. Math. Quebec.*, **42(1)**, 127–132 (2018).

[14] B. R. Srivatsa Kumar, Shruthi and D. Ranganatha, Some new congruences modulo 5 for the general partition function, *Russ Math.*, **64**, 73-–78 (2020).

[15] G. N. Watson, Theorems Stated by Ramanujan (VII): Theorems on Continued Fractions, *J. London Math. Soc.*, **s1-4**, 39–48 (1929).

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Received: June 29, 2020
Accepted: September 17, 2021