Palestine Journal of Mathematics

Vol. 11(2)(2022) , 395–405

HANKEL DETERMINANT FOR A SUBCLASS OF ANALYTIC FUNCTIONS ASSOCIATED WITH GENERALIZED STRUVE FUNCTION OF ORDER p BOUNDED BY CONICAL REGIONS

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Communicated by R. K. Raina

MSC 2010 Classifications: Primary 30C45; Secondary 30C45, 33E99.

Keywords and phrases: Analytic function, univalent function, Struve function, Hankel determinant, subordination.

The authors would like to express their profound gratitude to the Editor and the Referees for their valuable comments and suggestions that had helped the quality of this work.

Abstract In this work, new subclasses of analytic and univalent functions are defined using a generalized Struve function of order p. The upper estimates for the second Hankel determinants of the classes are established. Results obtained generalize some earlier known results.

1 Introduction

Let $\mathbb{C}$ denotes the set of complex numbers such that $p, b, c \in \mathbb{C}$, $k = p + \frac{b+2}{2} \neq 0, -1, -2, \cdots$ and $z \in U = \{z \in c : |z| < 1\}$ be a complex variable.

In the usual notation, we let A denote the class of functions $f(z)$ which are analytic on the unit disk and of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

and is normalized by the condition $f(0) = f'(0) - 1 = 0$. Also, let S be the subclass of A consisting of univalent functions in U.

A function $f \in A$ is said to be in the class S^* of starlike functions if it satisfies the following condition:

$$Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in U. \tag{1.2}$$

Similarly, a function $f \in A$ is said to be in the class C of convex functions if it satifies the following condition:

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in U. \tag{1.3}$$

For the two functions $f(z)$ of the form (1.1) and $h(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$ which are analytic in U, the convolution (or Hadamard product) of f and h is denoted by $(f * h)(z)$ and given by

$$(f * h)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n. \tag{1.4}$$

It is well known that

$$H_p(z) = z + \sum_{n=0}^{\infty} \frac{(-1)^n}{\Gamma(n + \frac{3}{2})\Gamma(p + n + \frac{3}{2})} \left(\frac{z}{2}\right)^{2n+p+1}, \quad z \in \mathbb{C} \tag{1.5}$$

and

$$w_{p,b,c}(z) = z + \sum_{n=0}^{\infty} \frac{(-1)^n c^n}{\Gamma(n + \frac{3}{2})\Gamma(p + n + \frac{b+2}{2})} \left(\frac{z}{2}\right)^{2n+p+1}, \quad z \in \mathbb{C} \tag{1.6}$$

are particular solutions of certain second-order nonhomogeneous differential equations and are respectively called Struve and generalized Struve functions. Let

$$u_{p,b,c}(z) = 2^p\sqrt{\pi}\Gamma\left(p+\frac{b+2}{2}\right)z^{\frac{-p-1}{2}}w_{p,b,c}(\sqrt{z}). \tag{1.7}$$

By utilizing the Pochhammer symbol, $(k)_n = \frac{\Gamma(k+n)}{\Gamma(k)} = k(k+1)\cdots(k+n-1)$, one can write the following form of $u_{p,b,c}(z)$:

$$u_{p,b,c}(z) = \sum_{n=0}^{\infty}\frac{(\frac{-c}{4})^n}{(\frac{3}{2})_n(k)_n}z^n = b_0 + b_1z + b_2z^2 + \cdots, \tag{1.8}$$

$k = p + \frac{b+2}{2} \neq 0, -1, -2, \cdots$, $b_n = \frac{(-1)^n c^n \Gamma(\frac{3}{2})\Gamma(k)}{4^n\Gamma(n+\frac{3}{2})\Gamma(n+k)}$ for $n \geq 0$, $b_0 = 1$.

We note that the function $u_{p,b,c}$ is analytic in $\mathbb{C}$, and satisfies the condition $u_{p,b,c}(0) = 1$. For more information on Struve function (1.5) and its generalized form (1.6), see [[13],[18],[19],[20]].

Also, let

$$g_{p,b,c}(z) = zu_{p,b,c}(z) = z + \sum_{n=2}^{\infty}\frac{(\frac{-c}{4})^{n-1}}{(\frac{3}{2})_{n-1}(k)_{n-1}}z^n \tag{1.9}$$

Orhan and Yagmur [13] investigated the geometric properties of the functions given by (1.9) and they include univalency, starlikeness, and convexity properties of the functions.

Following (1.4) and by making use of (1.1) and (1.9), Raza and Yagmur [16] defined the function

$$T_{p,b,c}(z) = (f * g_{p,b,c})(z) = z + \sum_{n=2}^{\infty}\frac{(\frac{-c}{4})^{n-1}}{(\frac{3}{2})_{n-1}(k)_{n-1}}a_nz^n. \tag{1.10}$$

For the purpose of this present work and for convenience, we shall let

$$T_{p,b,c}(z) = \vartheta = \left\{G : G(z) = z + \sum_{n=2}^{\infty}\frac{(\frac{-c}{4})^{n-1}}{(\frac{3}{2})_{n-1}(k)_{n-1}}a_nz^n, f \in A\right\} \tag{1.11}$$

ϑ is the class of generalized Struve function.

We claim that

$$G(z) = z - \frac{c}{6k_1}a_2z^2 + \frac{c^2}{20k_2}a_3z^3 - \frac{c^3}{56k_3}a_4z^4 + \frac{c^4}{144k_4}a_5z^5 - \cdots \tag{1.12}$$

where

$$(k)_1 = k_1,\ (k)_2 = k_2, \ldots, (k)_n = \frac{\Gamma(k+n)}{\Gamma(k)} = k_n;\ \forall n \in \mathbb{N}.$$

The conic region Ω_k was introduced and studied by Kanas and Wisniowska [5] and it was given by

$$\Omega_k = \{u + iv : u^2 > k^2(u-1)^2 + k^2v^2\},\ k \geq 0.$$

The above region represents the right half plane for $k = 0$, a hyperbola for $0 < k < 1$, a parabola for $k = 1$ and an ellipse for $k > 1$.

Now, let P denote the class of functions such that $p(0) = 1$ and $Rep(z) > 0$ for $z \in U$. The class of functions in P are called Carathéodory functions (see [6]).

Also, let $P(p_k)$, $0 \leq k < \infty$ denote the class of functions p, such that $p \in P$, and $p \prec p_k$ in U; where the function p_k maps the unit disk conformally onto the region $\Omega_l : 1 \in \Omega_k$. For functions that play the role of extremal functions for these comic regions and the variant definition of Ω_k, one may see Ramachandran et al. [15] and Oladipo [12] respectively.

If the functions $f(z)$ of the form (1.1) and $h(z) = z + \sum_{n=2}^{\infty} b_kz^k$ are analytic on U, then f is said to be subordinate to h , written as

$$f(z) \prec h(z), \quad z \in U$$

if there exits a Schwarz function $w(z)$ analytic on U with $w(0) = 0$ and

$$|w(z)| = |z| < 1, \quad z \in U$$

such that

$$f(z) = h(w(z)), \quad z \in U.$$

This is known as subordination principle and the details can be found in [17], [9] and [2]. In particular, when $h \in S$,

$$f \prec h \iff f(0) = h(0) \quad \text{and} \quad f(U) \subset h(U).$$

As consequences of the definitions of Carathéodory functions and subordination principles, equations (1.2) and (1.3) can be written equivalently as follows:

$$p(z) = \frac{zf'(z)}{f(z)} \prec p_k(z) \tag{1.13}$$

and

$$p(z) = 1 + \frac{zf''(z)}{f'(z)} \prec p_k(z). \tag{1.14}$$

Therefore, by virtue of (1.13) and (1.14) and the properties of the domains, we have

$$Re(p(z)) > Re(p_k(z)) > \frac{k}{k+1}.$$

The q^{th} Hankel determinant for $q \geq 1$ and $n \geq 1$ is stated by Nooman and Thomas [10] as

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q+1} \\ a_{n+1} & \cdots & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2(q-1)} \end{vmatrix}. \tag{1.15}$$

Many authors including but not limited to [11, 3, 7] have considered the determinant given by (1.15). It is very obvious from $H_q(n)$ that $H_2(1)$ is the Fekete-Szegö functional. For $f \in S$, and μ a real number, Fekete and Szegö further generalized estimate $|a_3 - \mu a_2^2|$.

We now use the concept of starlike, convex and conic region to give the following definitions:

Definition 1.1. A function $f \in A$ is said to be in the class $\mathcal{XS}^*$ if the following subordinate hold

$$\frac{zG'(z)}{G(z)} \prec p_k(z), \quad z \in U. \tag{1.16}$$

Definition 1.2. A function $f \in A$ is said to be in the class $\mathcal{XC}$ if the following subordination hold

$$1 + \frac{zG''(z)}{G'(z)} \prec p_k(z), \quad z \in U. \tag{1.17}$$

Thus, it follows that $f \in \mathcal{XC} \iff zf' \in \mathcal{XS}^*$.

Our focus in this paper is to determine Hankel coefficient estimates for the functions in the classes $\mathcal{XS}^*$ and $\mathcal{XC}$.

2 Preliminaries and Definitions

Some basic results which are relevant to our main results shall be stated as lemmas to set a good background for the works in this paper.

Arising from the definition of class P, of all functions p analytic in U for which $Re(p(z)) > 0$ and for $z \in U$, we let

$$p(z) = 1 + c_1 z + c_2 z^2 + \cdots. \tag{2.1}$$

Lemma 2.1. *[14] If $p \in P$, then $|c_k| \leq 2$ for each k.*

Lemma 2.2. *[1] Let $p \in P$. Then*

$$\left|c_2 - \sigma\frac{c_1^2}{2}\right| \leq \begin{cases} 2(1-\sigma), & if \quad \sigma \leq 0 \\ 2, & if \quad 0 \leq \sigma \leq 2 \\ 2(\sigma-1), & if \quad \sigma \geq 2 \end{cases}$$

Lemma 2.3. *[8] Let the function $p \in P$ be given by the power series (2.1), then*

$$2c_2 = c_1^2 + x(4-c_1^2) \tag{2.2}$$

for some x, $|x| \leq 1$ and

$$4c_3 = c_1^3 + 2(4-c_1^2)c_1x - c_1(4-c_1^2)x^2 + 2(4-c_1^2)(1-|x|^2)z \tag{2.3}$$

for some z, $|z| \leq 1$.

Lemma 2.4. *[4] Let $0 \leq k < \infty$, be fixed and p_k be the Riemann map of $\mathbb{D} \subset \mathbb{C}$ onto Ω_k, satisfying $p_k(0) = 1$ and $Re(p_k'(0)) \geq 0$. If $p_k(z) = 1 + P_1z + P_2z^2 + \cdots$, then*

$$P_k(z) = \begin{cases} \frac{2A^2}{1-K^2} & for \quad 0 < k < 1, \\ \frac{8}{\pi^2} & for \quad k = 1, \\ \frac{\pi^2}{4(k^2-1)K^2(k)(1+k)\sqrt{k}} & for \quad k > 1. \end{cases} \tag{2.4}$$

Lemma 2.5. *For $n \in \mathbb{N}$, $k = p + \frac{b+2}{2} \neq 0, -1, -2. \cdots$ and $p, b \in \mathbb{C}$. If $(k)_1 = k_1$, $(k)_2 = k_2, \cdots, (k)_n = \frac{\Gamma(k+n)}{\Gamma(k)} \equiv k(k+1)\cdots(k+n-1)$, then the following assertions are true:*

(i) If $p = -1$ and $b = 2$, then $k_1 = 1$,

(ii) If $p = -1$ and $b = 2$, then $k_2 = 2$,

(iii) If $p = -1$ and $b = 2$, then $k_3 = 6$.

Proof. (i) Let the assumption of the Lemma 2.5 holds. Then $k_1 = (k)_1 = k$. So that by definition,

$$k = p + \frac{b+2}{2} = \frac{2p+b+2}{2} = 1$$

when $p = -1, b = 2$.

(ii) Let the assumption of the Lemma 2.5 holds. Then $k_2 = (k)_2 = k(k+1)$. So that by definition

$$k(k+1) = \left(\frac{2p+b+2}{2}\right)\left(\frac{2p+b+2}{2}+1\right) = \left(\frac{2p+b+2}{2}\right)\left(\frac{2p+b+4}{2}\right) = 2,$$

when $p = -1, b = 2$.

(iii) Let the assumption of the Lemma 2.5 holds. Then
$k_3 = (k)_3 = k(k+1)(k+2)$. So that by definition

$$\begin{aligned} k(k+1)(k+2) &= \left(\frac{2p+b+2}{2}\right)\left(\frac{2p+b+4}{2}\right)\left(\frac{2p+b+2}{2}+2\right) \\ &= \left(\frac{2p+b+2}{2}\right)\left(\frac{2p+b+4}{2}\right)\left(\frac{2p+b+6}{2}\right) \\ &= 6, \end{aligned}$$

when $p = -1, b = 2$. □

In what follows, we shall state and proof the main results in this paper. We are motivated by the results in [15].

3 Main Results

Theorem 3.1. *If $f \in \mathcal{XS}^*$, then*

$$|a_2a_4 - a_3^2| \leq \psi\nabla^2 + \sigma\nabla + \frac{100k_2^2}{c^4}P_1^2$$

$\psi = \lambda - \mu - \frac{7k_3}{16}P_1^2 + \frac{25k_2^2}{4c^4}P_1^2$, $\sigma = 4\mu + \left(\frac{7k_3}{4} - \frac{50k_2^2}{c^4}P_1^2\right)$ *and*
$\nabla = \frac{\left(\frac{200k_2^2 - 7k_3c^4}{2c^4}\right)P_1^2 - 8\mu}{4\lambda - 4\mu - \left(\frac{7k_3c^4 - 100k_2^2}{4c^4}\right)}P_1^2.$

Proof. Let $f \in \mathcal{XS}^*$, then

$$z\frac{G'(z)}{G(z)} \prec p_k(z) \tag{3.1}$$

where

$$p_k(z) = 1 + P_1z + P_2z^2 + \cdots$$

By virtue of subordination relation (3.1), it can be seen that the function $p(z)$ given by

$$p(z) = \frac{1 + p_k^{-1}(q(z))}{1 - p_k^{-1}(q(z))} = 1 + c_1z + c_2z^2 + \cdots, \quad q(0) = 0, \ |q(z)| < 1$$

is analytic and has positive real part in U. We also have that

$$z\frac{G'(z)}{G(z)} = p_k\left(\frac{p(z) - 1}{p(z) + 1}\right), \quad z \in U \tag{3.2}$$

is analytic and has positive real part in U.
From (3.2), we have that

$$zG'(z) = G(z)p_k\left(\frac{p(z) - 1}{p(z) + 1}\right), \quad z \in U. \tag{3.3}$$

By simple calculation, we get

$$1 - \frac{c}{3k_1}a_2z + \frac{3c^2}{20k_2}a_3z^2 - \frac{c^3}{14k_3}a_4z^3 + \frac{5c^4}{144k_4}a_5z^4 - \cdots$$

$$= \left(1 - \frac{c}{6k_1}a_2z + \frac{c^2}{20k_2}a_3z^2 - \frac{c^3}{56k_3}a_4z^3 + \frac{c^4}{144k_4}a_5z^4 - \cdots\right) \times \left[1 + \frac{p_1c_1}{2}z + \left(\frac{p_1c_2}{2} - \frac{p_1c_1^2}{4} + \frac{p_2c_1^2}{4}\right)z^2\right]. \tag{3.4}$$

In order to determine a_2, a_3 and a_4, we equate the like terms in (3.4) as follows: for a_2, we have

$$-\frac{c}{3k_1}a_2z = \left(\frac{3k_1P_1c_1 - 2ca_2}{6k_1}\right)z, \tag{3.5}$$

for a_3, we have

$$\frac{3c^2}{20k_2}a_3z^2 = \frac{c^2}{20k_2}a_3z^2 - \frac{cP_1c_1}{12k_1}a_2z^2 + \left(\frac{P_1c_2}{2} - \frac{P_1c_1^2}{4} + \frac{P_2c_1^2}{4}\right)z^2 \tag{3.6}$$

and for a_4, we have

$$-\frac{c^3}{14k_3}a_4z^3 = \frac{c^2P_1c_1}{40k_2}a_3z^3 - \frac{c^3}{56k_3}a_4z^3 - \frac{c}{6k_1}a_2\left(\frac{P_1c_2}{2} - \frac{P_1c_1^2}{4} + \frac{P_2c_1^2}{4}\right)z^3. \tag{3.7}$$

A little computation on (3.5), (3.6) and (3.7) yields the following:

$$\begin{aligned} a_2 &= -\frac{3k_1P_1c_1}{c} \\ a_3 &= \frac{5}{2c^2}(P_1^2 - P_1 + P_2)k_2c_1^2 + \frac{5}{c^2}k_2P_1c_2 \\ a_4 &= \frac{7k_3}{6c^3k_1}\left[4P_2 - 2P_3 - 2P_1 - P_1^3 + 3P_1^2 - 3P_1P_2\right]c_1^3 \\ &\quad - \frac{7k_3}{6c^2k_1}\left[4P_2 - 4P_1 + 3P_1^2\right]c_1c_2 - \frac{7k_3}{6k_1}P_1c_3. \end{aligned}$$

For the purpose of brevity, we let

$$A(P) = P_1^2 - P_1 + P_2,\; B(P) = 4P_2 - 2P_3 - 2P_1 - P_1^3 + 3P_1^2 - 3P_1P_2$$

and $C(P) = 4P_2 - 4P_1 + 3P_1^2$, so that

$$a_2 = -\frac{3k_1P_1c_1}{c} \tag{3.8}$$

$$a_3 = \frac{5}{2c^2}A(P) + \frac{5}{c^2}k_2P_1c_2 \tag{3.9}$$

$$a_4 = \frac{7k_3}{6c^3k_1}B(P)c_1^3 - \frac{7k_3}{6c^2k_1}C(P)c_1c_2 - \frac{7k_3}{6k_1}P_1c_3. \tag{3.10}$$

Now, from (3.8), (3.9) and (3.10), we have that

$$a_2a_4 = -\frac{7k_3}{4c^3}B(P)P_1c_1^4 + \frac{7k_3}{4c^2}C(P)P_1c_1^2c_2 + \frac{7k_3}{4}P_1^2c_1c_3 \tag{3.11}$$

$$a_3^2 = \frac{25}{4c^4}A(P)^2k_2^2c_1^4 + \frac{25}{c^4}\left(A(P)k_2^2P_1c_1^2c_2 + k_2^2P_1^2c_2^2\right). \tag{3.12}$$

Therefore,

$$\begin{aligned} a_2a_4 - a_3^2 &= \left(-\frac{7k_3}{4c^3}B(P)P_1 - \frac{25}{4c^4}A(P)^2k_2^2\right)c_1^4 \\ &\quad + \left(\frac{7k_3}{4c^2}C(P)P_1 - \frac{25}{4c^4}A(P)P_1k_2^2\right)c_1^2c_2 - \frac{25}{4c^4}P_1^2k_2^2c_2^2 + \frac{7k_3}{4}P_1^2c_1c_3. \end{aligned} \tag{3.13}$$

Substituting for c_2 and c_3 from Lemma 2.3 into (3.13) and letting $c_1 = t$ we get

$$\begin{aligned} a_2a_4 - a_3^2 =& \Bigg[\frac{-7k_3}{4c^3}B(P)P_1 - \frac{25}{4c^4}A(P)^2k_2^2 + \frac{7k_3}{8c^2}C(P)P_1 - \frac{25}{2c^4}A(P)P_1k_2^2 \\ &\qquad + \left(\frac{7k_3c^4 - 100k_2^2}{16c^4}\right)P_1^2\Bigg]t^4 \\ &+ \left[\frac{7k_3}{8c^2}C(P)P_1 - \frac{25}{2c^4}A(P)P_1k_2^2 + \left(\frac{7k_3}{8} - \frac{25k_2^2}{2c^4}\right)P_1^2\right]t^2(4-t^2)x \\ &- \frac{7k_3}{16}P_1^2(4-t^2)t^2x^2 - \frac{25k_2^2}{4c^4}P_1^2(4-t^2)^2x^2 \\ &+ \frac{7k_3}{8}P_1^2t(4-t^2)(1-|x|^2)z. \end{aligned} \tag{3.14}$$

Since $|t| = |c_1| \leq 2$ by making use of Lemma 2.1, we may assume without restriction that $0 \leq t \leq 2$. Then using the triangle inequality with $\rho = |x|$, we obtain

$$\begin{aligned} |a_2a_4 - a_3^2| \leq& \lambda t^4 + \mu t^2(4-t^2)\rho + \frac{7k_3}{16}P_1^2t^2(4-t^2)\rho^2 \\ &+ \frac{25k_2^2}{4c^4}P_1^2(4-t^2)^2\rho^2 + \frac{7k_3}{8}P_1^2t(4-t^2)(1-\rho^2) \\ =& F(t,\rho,k_n), \quad (n = 2,3) \end{aligned} \tag{3.15}$$

where

$$\lambda = \frac{-7k_3}{4c^3}B(P)P_1 - \frac{25}{4c^4}A(P)^2k_2^2 + \frac{7k_3}{8c^2}C(P)P_1 - \frac{25}{2c^4}A(P)P_1k_2^2 + \left(\frac{7k_3c^4 - 100k_2^2}{16c^4}\right)P_1^2$$

and

$$\mu = \frac{7k_3}{8c^2}C(P)P_1 - \frac{25}{2c^4}A(P)P_1k_2^2 + \left(\frac{7k_3}{8} - \frac{25k_2^2}{2c^4}\right)P_1^2.$$

Then

$$\frac{\partial F}{\partial \rho} = \mu t^2(4-t^2) + \frac{7k_3}{8}P_1^2t^2(4-t^2)\rho + \frac{25A(P)k_2^2}{2c^4}P_1^2t^2(4-t^2)\rho - \frac{7k_3}{4}P_1^2t(4-t^2)\rho.$$

Clearly, $\frac{\partial F}{\partial \rho} > 0$ which shows that $F(t, k_n, \rho)$ is an increasing function on the interval $[0, 1]$. This implies that the maximum occurs at $\rho = 1$. Therefore

$$\max F(t, k_n, \rho) = F(t, k_n, 1) = H(t, k_n).$$

Now,

$$\begin{aligned} F(t, k_n, 1) = H(t, k_n) &= \lambda t^4 + \mu t^2(4-t^2) + \frac{7k_3}{16}P_1^2t^2(4-t^2) + \frac{25k_2^2}{4c^4}P_1^2(4-t^2)^2 \\ &= \left(\lambda - \mu - \frac{7k_3}{16}P_1^2 + \frac{25k_2^2}{4c^4}P_1^2\right)t^4 + \left[4\mu + \left(\frac{7k_3}{4} - \frac{50k_2^2}{c^4}\right)P_1^2\right]t^2 + \frac{100k_2^2}{c^4}P_1^2. \end{aligned} \quad (3.16)$$

Now,

$$H(t, k_n) = \psi t^4 + \sigma t^2 + \frac{100k_2^2}{c^4}P_1^2, \quad (3.17)$$

where

$$\psi = \lambda - \mu - \frac{7k_3}{16}P_1^2 + \frac{25k_2^2}{4c^4}P_1^2$$
$$\sigma = 4\mu + \left(\frac{7k_3}{4} - \frac{50k_2^2}{c^4}\right)P_1^2.$$

$$H_t = 4\psi t^3 + 2\sigma t \quad (3.18)$$
$$H_{tt} = 12\psi t^2 + 2\sigma < 0. \quad (3.19)$$

For optimum value of $H(t, k_n)$, we consider $H_t = 0$ so that

$$t^2 = \frac{\left(\frac{200k_2^2 - 7k_3c^4}{2c^4}\right)P_1^2 - 8\mu}{4\lambda - 4\mu - \left[\frac{7k_3c^4 - 100k_2^2}{4c^4}\right]P_1^2} = \nabla. \quad (3.20)$$

Substituting the value of t^2 from (3.20) in (3.19), it is possible to show that

$$H_{tt} = 12\left[\lambda - \mu - \frac{7k_3P_1^2}{4} + \frac{257k_2^2P_1^2}{4c^4}\right]\nabla + 2\left[4\mu + \left(\frac{7k_3}{4} - \frac{50k_2^2}{c^4}\right)P_1^2\right].$$

Therefore, by the second derivative test, $H(t, k_n)$ has maximum value at t, where t^2 is given by (3.20). Substituting the obtained value of t^2 in the expression (3.16), which gives the maximum value of $H(t, k_n)$ as

$$|a_2a_4 - a_3^2| \leq \psi\nabla^2 + \sigma\nabla + \frac{100k_2^2}{c^4}P_!^2.$$

□

Theorem 3.2. *If $f \in \mathcal{XC}$, then*

$$|a_2a_4 - a_3^2| \le \psi_1 \nabla_1^2 + \sigma_1 \nabla_1 + \frac{100k_2^2}{9c^4}P_1^2$$

$\psi_1 = \lambda_2 - \mu_2 - \left(\frac{7k_3}{128} + \frac{25k_2^2}{36c^4}\right)P_1^2$, $\sigma_1 = 4\mu_2 + \left(\frac{7k_3}{32} - \frac{50k_2^2}{9c^4}\right)P_1^2$ *and*
$\nabla_1 = \frac{\left(\frac{1600k_2^2 - 63k_3C^4}{144c^4}\right)P_1^2 - 8\mu_2}{4\lambda_2 - 4\mu_2 - \left[\frac{63k_3c^4 - 800k_2^2}{288c^4}\right]}P_1^2.$

Proof. Going by the definition of the class $\mathcal{S}^*$ and $\mathcal{C}$, it follows that the function $f \in \mathcal{XC}$ if and only if $zf' \in \mathcal{XS}^*$. Therefore by replacing a_n by na_n, in (3.8), (3.9) and (3.10), we obtain

$$a_2 = -\frac{3k_1P_1c_1}{2c} \tag{3.21}$$

$$a_3 = \frac{5}{6c^2}A(P)k_2c_1^2 + \frac{5}{3c^2}k_2P_1c_2 \tag{3.22}$$

$$a_4 = \frac{7k_3}{24c^3k_1}B(P)c_1^3 - \frac{7k_3}{24c^2k_1}C(P)c_1c_2 - \frac{7k_3}{24k_1}P_1c_3. \tag{3.23}$$

where $A(P)$, $B(P)$ and $C(P)$ are as defined earlier under the proof of Theorem 3.1.

From (3.21), (3.22) and (3.23) we have that

$$\begin{aligned} a_2a_4 - a_3^2 = &\left(-\frac{7k_3}{32c^3}B(P)P_1 - \frac{25}{36c^4}A(P)^2k_2^2\right)c_1^4 \\ &+ \left(\frac{7k_3}{32c^2}C(P)P_1 - \frac{25}{9c^4}A(P)P_1k_2^2\right)c_1^2c_2 - \frac{25}{9c^4}P_1^2k_2^2c_2^2 + \frac{7k_3}{32}P_1^2c_1c_3. \end{aligned} \tag{3.24}$$

Substituting for c_2 and c_3 from Lemma 2.3 into (3.24) and letting $c_1 = t$ we get

$$\begin{aligned} a_2a_4 - a_3^2 = &\Bigg[\frac{-7k_3}{32c^3}B(P)P_1 - \frac{25}{36c^4}A(P)^2k_2^2 + \frac{7k_3}{64c^2}C(P)P_1 - \frac{25}{18c^4}A(P)P_1k_2^2 \\ &+ \left(\frac{63k_3c^4 - 800k_2^2}{1152c^4}\right)P_1^2\Bigg]t^4 \\ &+ \left[\frac{7k_3}{64c^2}C(P)P_1 - \frac{25}{18c^4}A(P)P_1k_2^2 + \left(\frac{7k_3}{64} - \frac{25k_2^2}{18c^4}\right)P_1^2\right]t^2(4-t^2)x \\ &- \frac{7k_3}{128}P_1^2(4-t^2)t^2x^2 - \frac{25k_2^2}{36c^4}P_1^2(4-t^2)^2x^2 \\ &+ \frac{7k_3}{64}P_1^2t(4-t^2)(1-|x|^2)z. \end{aligned} \tag{3.25}$$

Since $|t| = |c_1| \le 2$ by making use of Lemma 2.1, we may assume without restriction that $0 \le t \le 2$. Then using the triangle inequality with $\rho = |x|$, we obtain

$$\begin{aligned} |a_2a_4 - a_3^2| \le &\lambda_2t^4 + \mu_2t^2(4-t^2)\rho + \frac{7k_3}{128}P_1^2t^2(4-t^2)\rho^2 \\ &+ \frac{25k_2^2}{36c^4}P_1^2(4-t^2)^2\rho^2 + \frac{7k_3}{64}P_1^2t(4-t^2)(1-\rho^2) \\ &= F_2(t,\rho,k_n), \quad (n = 2,3) \end{aligned} \tag{3.26}$$

where

$$\begin{aligned} \lambda_2 = &\frac{-7k_3}{32c^3}B(P)P_1 - \frac{25}{36c^4}A(P)^2k_2^2 + \frac{7k_3}{64c^2}C(P)P_1 \\ &- \frac{25}{18c^4}A(P)P_1k_2^2 + \left(\frac{63k_3c^4 - 800k_2^2}{1152c^4}\right)P_1^2 \end{aligned}$$

and

$$\mu_2 = \frac{7k_3}{64c^2}C(P)P_1 - \frac{25}{18c^4}A(P)P_1k_2^2 + \left(\frac{7k_2}{64} - \frac{25k_2^2}{18c^4}\right)P_1^2.$$

Then

$$\frac{\partial F_2}{\partial \rho} = \mu_2 t^2(4-t^2) + \frac{7k_3}{64}P_1^2t^2(4-t^2)\rho + \frac{25k_2^2}{18c^4}P_1^2(4-t^2)\rho - \frac{7k_3}{32}P_1^2t(4-t^2).$$

Clearly, $\frac{\partial F_2}{\partial \rho} > 0$ which shows that $F_2(t, k_n, \rho)$ is an increasing function on the interval $[0,1]$. This implies that the maximum occurs at $\rho = 1$. Therefore

$$\max F_2(t, k_n, \rho) = F_2(t, k_n, 1) = H_2(t, k_n).$$

Now,

$$\begin{aligned} F_2(t, k_n, 1) &= H_2(t, k_n) \\ &= \lambda_2 t^4 + \mu_2 t^2(4-t^2) + \frac{7k_3}{128}P_1^2t^2(4-t^2) + \frac{25k_2^2}{36c^4}P_1^2(4-t^2)^2 \\ &= \left(\lambda_2 - \mu_2 - \frac{7k_3}{128}P_1^2 + \frac{25k_2^2}{36c^4}P_1^2\right)t^4 + \left[4\mu_2 + \left(\frac{7k_3}{32} - \frac{50k_2^2}{9c^4}\right)P_1^2\right]t^2 \\ &\quad + \frac{100k_2^2}{9c^4}P_1^2. \end{aligned} \tag{3.27}$$

Now,

$$H_2(t, k_n) = \psi_1 t^4 + \sigma_1 t^2 + \frac{100k_2^2}{9c^4}P_1^2, \quad (n = 2, 3) \tag{3.28}$$

where

$$\psi_1 = \lambda_2 - \mu_2 - \frac{7k_3}{128}P_1^2 + \frac{25k_2^2}{36c^4}P_1^2$$
$$\sigma_1 = 4\mu_2 + \left(\frac{7k_3}{32} - \frac{50k_2^2}{9c^4}\right)P_1^2.$$

$$(H_2)_t = 4\psi_1 t^3 + 2\sigma_1 t \tag{3.29}$$
$$(H_2)_{tt} = 12\psi_1 t^2 + 2\sigma_1 < 0. \tag{3.30}$$

For optimum value of $H_2(t, k_n)$, we consider $(H_2)_t = 0$. So that

$$t^2 = \frac{\left(\frac{1600k_2^2 - 63k_3c^4}{144c^4}\right)P_1^2 - 8\mu_2}{4\lambda_2 - 4\mu_2 - \left[\frac{63k_3c^4 - 800k_2^2}{288c^4}\right]P_1^2} = \nabla_1. \tag{3.31}$$

Substituting the value of t^2 from (3.31) in (3.30), it is possible to show that

$$(H_2)_{tt} = 12\psi_1\nabla_1 + 2\sigma_1$$
$$= \left[12\lambda_2 - 12\mu_2 - \left(\frac{21k_3}{62} - \frac{25k_2^2}{3c^4}\right)P_1^2\right]\nabla_1 + \left[8\mu_2 + \left(\frac{7k_3}{16} - \frac{100k_2^2}{9c^4}\right)P_1^2\right] < 0.$$

Therefore, by the second derivative test, $H_2(t, k_n)$ has maximum value at t, where t^2 is given by (3.31). Substituting the obtained value of t^2 in the expression (3.28), which gives the maximum value of $H_2(t, k_n)$ as

$$|a_2a_4 - a_3^2| \le \psi_1\nabla_1^2 + \sigma_1\nabla_1 + \frac{100k_2^2}{9c^4}P_!^2.$$

□

4 Applications

In this section, we shall exhibit some interesting consequences of our results as applications.

By making use of Lemma 2.5 and letting $c = 2$ in Theorem 3.1 and Theorem 3.2 we have the following:

Corollary 4.1. *If $f \in \mathcal{ES}^*$, then*

$$|a_2a_4 - a_3^2| \leq \psi\nabla^2 + \sigma\nabla + 25P_1^2$$

$\psi = \lambda - \mu - \frac{17}{16}P_1^2$, $\sigma = 4\mu - 2P_1^2$ *and* $\nabla = \frac{4p_1^2 - 8\mu}{4\lambda - 4\mu - \frac{17}{4}P_1^2}$.

Which corresponds to the result in Ramachandran et al. [[15], Theorem 1] when $\psi = \eta$, $\sigma = \vartheta$ and $\nabla = B$.

Corollary 4.2. *If $f \in \mathcal{EC}$, then*

$$|a_2a_4 - a_3^2| \leq \psi_1\nabla_1^2 + \sigma_1\nabla_1 + \frac{25}{9}P_1^2$$

$\psi_1 = \lambda_2 - \mu_2 - \frac{89}{576}P_1^2$, $\sigma_1 = 4\mu_2 - \frac{11}{144}P_1^2$ *and* $\nabla_1 = \frac{\frac{11}{72}P_1^2 - 8\mu_2}{4\lambda_2 - 4\mu_2 - \frac{89}{144}P_1^2}$.

Which corresponds to the result in Ramachandran et al. [[15], Theorem 2] when $\psi = \eta$, $\sigma = \vartheta$ and $\nabla_1 = B_1$.

5 Conclusion

Generalized Struve function has been used to define new subclasses of analytic and univalent functions and the upper bounds for the second Hankel determinants are obtained. The upper estimates obtained are the best possible. Some earlier known results, which are special cases of the results obtained are pointed out as applications.

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Received: January 9, 2021
Accepted: August 30, 2021