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THE QUATERNIONIC RULED SURFACES IN TERMS OF ALTERNATIVE FRAME

Abdussamet Çalışkan

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Abstract In this paper, we investigate the quaternionic expression of the ruled surfaces drawn by the motion of the Alternative vectors. The distribution parameters, the pitches, and the angle of pitches of the ruled surfaces are calculated as quaternionic.

1 Introduction

The quaternion was discovered in 1843 by William Rowan Hamilton [18]. Quaternions arose historically from Hamilton's essays in the mid-nineteenth century to generalize complex numbers in some way that would apply to three-dimensional (3D) space. A feature of quaternions is closely related to 3D rotations, a fact apparent to Hamilton almost immediately but first published by Hamilton's contemporary Arthur Cayley in 1845 [1]. The technology did not penetrate the computer animation community until the landmark Siggraph 1985 paper of Ken Shoemake [11]. The importance of Shoemake's paper is that it took the concept of the orientation frame for moving 3D objects and cameras, which require precise orientation specification, exposed the deficiencies of the then-standard Euler-angle methods, and introduced quaternions to animators as a solution. Classical differential geometry is the study of local properties of curves and surfaces. Recently, Shaikh et.al initiated the study of surface curves differently, especially, rectifying, osculating and normal curves on a surface by considering isometry between two surfaces and investigated their invariance under such maps [2, 3, 4, 5]. The Serret-Frenet formulae for quaternionic curves in $\mathbb{IR}^3$ and $\mathbb{IR}^4$ were introduced by K. Bharathi and M. Nagaraj [9]. There are lots of studies that investigated quaternionic curves by using this study. One of them is Karadağ and Sivridağ's study whose they gave many characterizations for quaternionic inclined curves in $\mathbb{IR}^4$ [13]. Şenyurt *et al.* calculated curvature and torsion of spatial quaternionic involute curve according to the normal vector and the unit Darboux vector of Smarandache curve [15]. In [16], the authors investigated the ruled surface as spatial quaternionic. They quaternionally calculated the integral invariants of the ruled surface.

A surface is said to be "ruled" if it is generated by moving a straight line continuously in Euclidean space $\mathbb{IR}^3$. Ruled surfaces are one of the simplest objects in geometric modeling. One important fact about ruled surfaces is that they can be generated by straight lines. A practical application of ruled surfaces is that they are used in civil engineering. The result is that if engineers are planning to construct something with curvature, they can use a ruled surface since all the lines are straight. Among ruled surfaces, developable surfaces form an important subclass since they are useful in sheet metal design and processing [6, 10].

In this study, we investigate the ruled surfaces drawn by the motion of the Alternative vectors as quaternionic. We calculate integral invariants of the ruled surface with the theory of quaternion.

2 Preliminaries

In E^3, the standard inner product is given by

$$\langle x, x\rangle = x_1^2 + x_2^2 + x_3^2 \tag{2.1}$$

where $x = (x_1, x_2, x_3) \in E^3$. Let $\alpha : I \to E^3$ be a unit speed curve denote by $\{\vec{T}(s), \vec{N}(s), \vec{B}(s)\}$ the moving Frenet frame. $\vec{T}(s)$ is the tangent vector field, $\vec{N}(s)$ is the principal normal vector field and $\vec{B}(s)$ is the binormal vector field of the curve α, respectively. The Frenet formulas are given by [12]

$$\vec{T}'(s) = \kappa(s)\vec{N}(s), \ \vec{N}'(s) = -\kappa(s)\vec{T}(s) + \tau(s)\vec{B}(s), \ \vec{B}'(s) = -\tau(s)\vec{N}(s). \tag{2.2}$$

Here curvature and torsion of the curve α are defined with [12]

$$\kappa(s) = \|\alpha''(s)\|, \ \tau(s) = \frac{\langle \alpha'(s) \wedge \alpha''(s), \alpha'''(s)\rangle}{\|\alpha'(s) \wedge \alpha''(s)\|^2}. \tag{2.3}$$

The vector W is called the unit Darboux vector and defined by [17]

$$\vec{W} = \frac{\vec{w}}{\|\vec{w}\|} = \frac{1}{\sqrt{\kappa^2+\tau^2}}(\tau\vec{T} + \kappa\vec{B}). \tag{2.4}$$

The Darboux vector is perpendicular to the principal normal vector field $\vec{N}$. If $\vec{C}$ is taken as $\vec{C} = \vec{W} \wedge \vec{N}$, then $\{\vec{N}, \vec{C}, \vec{W}\}$ are another orthonormal moving frame along the curve α. This frame is called an alternative frame. The derivative formulae of the alternative frame are given by

$$\begin{bmatrix} \vec{N}' \\ \vec{C}' \\ \vec{W}' \end{bmatrix} = \begin{bmatrix} 0 & \beta & 0 \\ \beta & 0 & \gamma \\ 0 & -\gamma & 0 \end{bmatrix} \begin{bmatrix} \vec{N} \\ \vec{C} \\ \vec{W} \end{bmatrix} \tag{2.5}$$

where $\beta = \sqrt{\kappa^2+\tau^2}$ and $\gamma = \frac{\kappa^2}{\kappa^2+\tau^2}\left(\frac{\tau}{\kappa}\right)'$. The relationship between the Frenet and alternative vactors are

$$\begin{cases} \vec{C} = -\overline{\kappa}\vec{T} + \overline{\tau}\vec{B} \\ \vec{W} = \overline{\tau}\vec{T} + \overline{\kappa}\vec{B} \end{cases} \quad \text{or} \quad \begin{cases} \vec{T} = -\overline{\kappa}\vec{C} + \overline{\tau}\vec{W} \\ \vec{B} = \overline{\tau}\vec{C} + \overline{\kappa}\vec{W} \end{cases} \tag{2.6}$$

where principal normal vector $\vec{N}$ is same in both frames, $\overline{\kappa} = \frac{\kappa}{\beta}$ and $\overline{\tau} = \frac{\tau}{\beta}$ [14, 7].

Real quaternion is defined by the $1, e_1, e_2, e_3$. 1 is a real number, e_1, e_2, e_3 are vectors with the following properties:

$$\begin{cases} {e_1}^2 = {e_2}^2 = {e_3}^2 = e_1 \times e_2 \times e_3 = -1, e_1, e_2, e_3 \in \mathbb{R}^3 \\ e_1 \times e_2 = e_3, e_2 \times e_3 = e_1, e_3 \times e_1 = e_2. \end{cases} \tag{2.7}$$

The 4-dimensional real Euclidean space $\mathbb{R}^4$ is identified with the space of real quaternions

$$\mathbb{K} = \left\{q = d + ae_1 + be_2 + ce_3 | a, b, c, d \in \mathbb{R}, \vec{e}_1, e_2, e_3 \in \mathbb{R}^3\right\}$$

in [9, 8].

Let $q_1 = S_{q_1} + V_{q_1} = d_1 + a_1e_1 + b_1e_2 + c_1e_3$ and $q_2 = S_{q_2} + V_{q_2} = d_2 + a_2e_1 + b_2e_2 + c_2e_3$ be two quaternions in $\mathbb{K}$, the quaternion multiplication of q_1 and q_2 is given by

$$\begin{aligned} q_1 \times q_2 &= d_1d_2 - (a_1a_2 + b_1b_2 + c_1c_2) + (d_1a_2 + a_1d_2 + b_1c_2 - c_1b_2)e_1 \\ &+ (d_1b_2 + b_1d_2 + b_1a_2 - a_1b_2)e_2 + +(d_1c_2 + c_1d_2 + a_1b_2 - b_1a_2)e_3. \end{aligned}$$

The symmetric real-valued bilinear form h which is defined as

$$\begin{aligned} h &: \mathbb{K} \times \mathbb{K} \to \mathbb{R} \\ & h(q_1, q_2) = \frac{1}{2}(q_1 \times \bar{q}_2 + q_2 \times \bar{q}_1) \end{aligned} \tag{2.8}$$

is called quaternion inner product [9]. Let q be a real quaternion. Its conjugate is $\bar{q} = S_q - V_q$. The norm of a real quaternion is a real number in the form of

$$N(q) = \sqrt{h(q,q)} = \sqrt{d^2 + a^2 + b^2 + c^2}. \tag{2.9}$$

If $N(q) = 1$, q is called a unit quaternion. Invers of real quaternion is $q^{-1} = \dfrac{\bar{q}}{N(q)}\cdot$ Quaternion the division is noncommutative, and is defined by the (order-dependent) relations $r_1 = q_1 \times {q_2}^{-1}, r_2 = {q_2}^{-1} \times q_1$. Where r_1 is the right division, r_2 is the left division [8]. The three-dimensional real Euclidean space $\mathbb{R}^3$ is identified with the space of spatial quaternions

$$Q = \{q \in \mathbb{K} \mid q + \bar{q} = 0\}$$

in the obvious manner [9]. In this case, the elements of Q are $q = ae_1 + be_2 + ce_3$. As a result, the quaternion multiplication of the two spatial quaternions is [8]

$$q_1 \times q_2 = -\langle q_1, q_2\rangle + q_1 \wedge q_2. \tag{2.10}$$

Definition 2.1. Let $s \in I = [0, 1]$ be the arc parameter along the smooth curve

$$\begin{aligned} \alpha : [0,1] &\rightarrow Q \\ \alpha(s) &= \sum_{n=1}^{3} \alpha_i(s) e_i. \end{aligned}$$

This is called a spatial quaternionic curve [9].

Definition 2.2. A ruled surface in $\mathbb{R}^3$ is a surface that contains at least one 1-parameter family of straight lines. Thus a ruled surface has a parametrization in the form

$$\begin{aligned} \varphi : I \times \mathbb{R} &\rightarrow \mathbb{R}^3 \\ (s,v) &\rightarrow \overrightarrow{\varphi}(s,v) = \overrightarrow{\alpha}(s) + v\,\overrightarrow{x}(s). \end{aligned} \tag{2.11}$$

where we call α the anchor curve, X the generator vector of the ruled surface [12].

The quaternionic express of distribution parameter (drall) belonging to the ruled surface is given by [16]

$$P_x = \frac{h(\vec{x} \times \vec{x}', \alpha')}{\mathbf{N}(\vec{x}')^2} = \frac{1}{2}\frac{\left((\vec{x} \times \vec{x}') \times \overline{\alpha'} + \alpha' \times \overline{(\vec{x} \times \vec{x}')}\right)}{\mathbf{N}(\vec{x}')^2}. \tag{2.12}$$

The angle of pitch and the pitch of the closed quaternionic ruled surface, λ_x and L_x, are equal to the projection of the generator x on the Steiner rotation vector $\vec{D}$ and the Steiner translation vector $\vec{V}$ [16]

$$\lambda_x = h(\overrightarrow{D}, \overrightarrow{x}), \tag{2.13}$$

$$L_x = h(\overrightarrow{V}, \overrightarrow{x}). \tag{2.14}$$

3 The Quaternionic Ruled Surfaces in terms of Alternative Frame

The ruled surfaces drawn by the motion of the Alternative vectors are given by

$$\begin{aligned} \varphi_N(s,v) &= \vec{\alpha}(s) + v\vec{N}(s), \\ \varphi_C(s,v) &= \vec{\alpha}(s) + v\vec{C}(s), \\ \varphi_W(s,v) &= \vec{\alpha}(s) + v\vec{W}(s). \end{aligned}$$

Using the equation (2.12), the distribution parameter of the closed spatial quaternionic ruled surface drawn by the motion of the principal vectors $\vec{N}$ belonging to the spatial quaternionic curve α is

$$P_N = \frac{h(\vec{N} \times \vec{N}', \vec{\alpha}')}{\mathbf{N}(\vec{N}')^2}. \tag{3.1}$$

Considering the equation (2.5), we obtain

$$\begin{aligned}
h(\vec{N}\times\vec{N}',\vec{T}) &= h(\vec{N}\times\vec{N}',-\overline{\kappa}\vec{C}+\overline{\tau}\vec{W}) \\
&= \frac{1}{2}\Big((\vec{N}\times\vec{N}')\times\overline{(-\overline{\kappa}\vec{C}+\overline{\tau}\vec{W})}+(-\overline{\kappa}\vec{C}+\overline{\tau}\vec{W})\times(\overline{\vec{N}\times\vec{N}'})\Big) \\
&= \frac{1}{2}\Big(\tau\overline{\kappa}(\vec{T}\times\vec{C})-\tau\overline{\tau}(\vec{T}\times\vec{W})+\kappa\overline{\kappa}(\vec{B}\times\vec{C})-\kappa\overline{\tau}(\vec{B}\times\vec{W})+\tau\overline{\kappa}(\vec{C}\times\vec{T}) \\
&\quad +\kappa\overline{\kappa}(\vec{C}\times\vec{B})-\tau\overline{\tau}(\vec{W}\times\vec{T})-\kappa\overline{\tau}(\vec{W}\times\vec{B})\Big) \\
&= \frac{1}{2}\Big(\tau\overline{\kappa}(\overline{\kappa}-\overline{\tau}\vec{N})-\tau\overline{\tau}(-\overline{\tau}-\overline{\kappa}\vec{N})+\kappa\overline{\kappa}(-\overline{\tau}-\overline{\kappa}\vec{N})-\kappa\overline{\tau}(-\overline{\kappa}+\overline{\tau}\vec{N}) \\
&\quad +\tau\overline{\kappa}(\overline{\kappa}+\overline{\tau}\vec{N})+\kappa\overline{\kappa}(-\overline{\tau}+\overline{\kappa}\vec{N})-\tau\overline{\tau}(-\overline{\tau}+\overline{\kappa}\vec{N})-\kappa\overline{\tau}(-\overline{\kappa}-\overline{\tau}\vec{N})\Big) \\
&= \tau
\end{aligned}$$

and

$$\mathbf{N}(\vec{N}')^2 = h(\vec{N}',\vec{N}') = \frac{1}{2}\Big(\vec{N}'\times\overline{\vec{N}'}+\vec{N}'\times\overline{\vec{N}'}\Big) = \vec{N}'\times\overline{\vec{N}'} = \kappa^2+\tau^2.$$

If these values are substituted in equation (3.1), $P_N = \dfrac{\tau}{\kappa^2+\tau^2}$ is found.
By taking into consideration the equation (2.12), the distribution parameter of the closed spatial quaternionic ruled surface drawn by the motion of the vector $\vec{C}$ belonging to the spatial quaternionic curve α is

$$P_{\scriptscriptstyle C} = \frac{h(\vec{C}\times\vec{C}',\vec{\alpha}')}{\mathbf{N}(\vec{C}')^2}. \tag{3.2}$$

Considering the equation (2.5), we can write

$$\begin{aligned}
h(\vec{C}\times\vec{C}',\vec{T}) &= h(\vec{C}\times\vec{C}',-\overline{\kappa}\vec{C}+\overline{\tau}\vec{W}) \\
&= \frac{1}{2}\Big((\vec{C}\times\vec{C}')\times\overline{(-\overline{\kappa}\vec{C}+\overline{\tau}\vec{W})}+(-\overline{\kappa}\vec{C}+\overline{\tau}\vec{W})\times(\overline{\vec{C}\times\vec{C}'})\Big) \\
&= \frac{1}{2}\Big((\gamma\vec{N}-\beta\vec{W})\times(\overline{\kappa}\vec{C}-\overline{\tau}\vec{W})+(-\overline{\kappa}\vec{C}+\overline{\tau}\vec{W})\times(-\gamma\vec{N}+\beta\vec{W}\Big) \\
&= \frac{1}{2}\Big(\gamma\overline{\kappa}(\vec{N}\times\vec{C})-\gamma\overline{\tau}(\vec{N}\times\vec{W})-\beta\overline{\kappa}((\vec{W}\times\vec{C}))+\beta\overline{\tau}(\vec{W}\times\vec{W}) \\
&\quad +\gamma\overline{\kappa}((\vec{C}\times\vec{N}))-\gamma\overline{\tau}((\vec{W}\times\vec{N}))-\beta\overline{\kappa}(\vec{C}\times\vec{W})+\beta\overline{\tau}(\vec{W}\times\vec{W})\Big) \\
&= -\tau
\end{aligned}$$

and

$$\mathbf{N}(\vec{C}')^2 = h(\vec{C}',\vec{C}') = \frac{1}{2}\Big(\vec{C}'\times\overline{\vec{C}'}+\vec{C}'\times\overline{\vec{C}'}\Big) = \vec{C}'\times\overline{\vec{C}'} = \beta^2+\gamma^2.$$

If these values are substituted in equation (3.2), $P_C = \dfrac{-\tau}{\beta^2+\gamma^2}$ is found.
By considering the equation (2.12), the distribution parameter of the closed spatial quaternionic ruled surface drawn by the motion of the vector $\vec{W}$ belonging to the spatial quaternionic curve α is

$$P_{\scriptscriptstyle W} = \frac{h(\vec{W}\times\vec{W}',\vec{\alpha}')}{\mathbf{N}(\vec{W}')^2}. \tag{3.3}$$

Considering the equation (2.5), we can write

$$\begin{aligned} h(\vec{W}\times\vec{W}',\vec{T}) &= h(\vec{W}\times\vec{W}',-\overline{\kappa}\vec{C}+\overline{\tau}\vec{W}) \\ &= \frac{1}{2}\Big((\vec{W}\times\vec{W}')\times\overline{(-\overline{\kappa}\vec{C}+\overline{\tau}\vec{W})}+(-\overline{\kappa}\vec{C}+\overline{\tau}\vec{W})\times(\overline{\vec{W}\times\vec{W}'})\Big) \\ &= \frac{1}{2}\Big(\gamma\vec{N}\times(\overline{\kappa}\vec{C}-\overline{\tau}\vec{W})+(-\overline{\kappa}\vec{C}+\overline{\tau}\vec{W})\times(-\gamma N) \\ &= \frac{1}{2}\Big(\gamma\overline{\kappa}(\vec{N}\times\vec{C})-\gamma\overline{\tau}(\vec{N}\times\vec{W})+\gamma\overline{\kappa}(\vec{C}\times\vec{N})-\gamma\overline{\tau}((\vec{W}\times\vec{N})\Big) \\ &= 0. \end{aligned}$$

If these values are substituted in equation (3.3), $P_W = 0$ is found.

Corollary 3.1. *The distribution parameters of the closed spatial quaternionic ruled surfaces drawn by the Alternative vectors are*

$$\begin{cases} P_N = \dfrac{\tau}{\kappa^2+\tau^2}, \\ P_C = \dfrac{-\tau}{\beta^2+\gamma^2}, \\ P_W = 0. \end{cases}$$

Corollary 3.2. *The ruled surface drawn by the motion of the vector W is developable as quaternionic.*

According to the equation (2.14), the pitches of the closed spatial quaternionic ruled surfaces drawn by the motion of the Alternative vectors $\vec{N}$, $\vec{C}$, $\vec{W}$ belonging to the spatial quaternionic curve α are as follows:

$$\begin{aligned} L_{N} &= h(\oint d\vec{\alpha},\vec{N}) = h(\oint \vec{T}ds,\vec{N}) = \frac{1}{2}\Big(\vec{T}\oint ds\times\overline{\vec{N}}+\vec{N}\times\overline{\vec{T}\oint ds}\Big) \\ &= \frac{1}{2}\Big(\oint ds\big((-\overline{\kappa}\vec{C}+\overline{\tau}\vec{W})\times(-\vec{N})\big)+\oint ds\big(\vec{N}\times(\overline{\kappa}\vec{C}-\overline{\tau}\vec{W})\big)\Big) \\ &= \frac{1}{2}\Big(\oint ds(-\overline{\kappa}\vec{W}-\overline{\tau}\vec{C})+\oint ds(\overline{\kappa}\vec{W}+\overline{\tau}\vec{C})\Big) \\ &= 0. \end{aligned}$$

$$\begin{aligned} L_{C} &= h(\oint d\vec{\alpha},\vec{C}) = h(\oint \vec{T}ds,\vec{C}) = \frac{1}{2}\Big(\vec{T}\oint ds\times\overline{\vec{C}}+\vec{C}\times\overline{\vec{T}\oint ds}\Big) \\ &= \frac{1}{2}\Big(\big((-\overline{\kappa}\vec{C}+\overline{\tau}\vec{W})\times(-C)\big)\oint ds+\big(C\times(\overline{\kappa}\vec{C}-\overline{\tau}\vec{W})\big)\oint ds\Big) \\ &= \frac{1}{2}\Big((-\overline{\kappa}+\overline{\tau}\vec{N})\oint ds+(-\overline{\kappa}-\overline{\tau}\vec{N})\oint ds\Big) \\ &= -\frac{\kappa}{\beta}\oint ds. \end{aligned}$$

$$\begin{aligned} L_{W} &= h(\oint d\vec{\alpha},\vec{W}) = h(\oint \vec{T}ds,\vec{W}) = \frac{1}{2}\Big(\vec{T}\oint ds\times\overline{\vec{W}}+\vec{W}\times\overline{\vec{T}\oint ds}\Big) \\ &= \frac{1}{2}\Big(\big((-\overline{\kappa}\vec{C}+\overline{\tau}\vec{W})\times(-\vec{W})\big)\oint ds+\big(\vec{W}\times(\overline{\kappa}\vec{C}-\overline{\tau}\vec{W})\big)\oint ds\Big) \\ &= \frac{1}{2}\Big((\overline{\kappa}\vec{N}+\overline{\tau})\oint ds+(-\overline{\kappa}\vec{N}+\overline{\tau})\oint ds\Big) \\ &= \frac{\tau}{\beta}\oint ds. \end{aligned}$$

Corollary 3.3. *The pitches of the closed spatial quaternionic ruled surfaces drawn by the Alternative vectors are*

$$\begin{cases} L_N = 0, \\ L_C = -\dfrac{\kappa}{\beta}\oint ds, \\ L_W = \dfrac{\tau}{\beta}\oint ds. \end{cases}$$

According to the equation (2.13), the angles of pitches of the closed spatial quaternionic ruled surfaces drawn by the motion of the Alternative vectors $\vec{N}$, $\vec{C}$, $\vec{W}$ belonging to the spatial quaternionic curve α are as follows:

$$\begin{aligned}
\lambda_{N} &= h(\vec{D},\vec{N}) = \frac{1}{2}\Big(\vec{D}\times\overline{\vec{N}}+\vec{N}\times\overline{\vec{D}}\Big) \\
&= \frac{1}{2}\Big(-(\vec{T}\oint\tau ds+\vec{B}\oint\kappa ds)\times\vec{N}+\vec{N}\times(-\vec{T}\oint\tau ds-\vec{B}\oint\kappa ds)\Big) \\
&= \frac{1}{2}\Big(-(\vec{T}\times\vec{N})\oint\tau ds-(\vec{B}\times\vec{N})\oint\kappa ds-(\vec{N}\times\vec{T})\oint\tau ds-(\vec{N}\times\vec{B})\oint\kappa ds)\Big) \\
&= \frac{1}{2}\Big(-\big((-\overline{\kappa}\vec{C}+\overline{\tau}\vec{W})\times\vec{N}\big)\oint\tau ds-\big((\overline{\tau}\vec{C}+\overline{\kappa}\vec{W})\times\vec{N}\big)\oint\kappa ds \\
&\quad -\big(\vec{N}\times(-\overline{\kappa}\vec{C}+\overline{\tau}\vec{W})\big)\oint\tau ds-\big(\vec{N}\times(\overline{\tau}\vec{C}+\overline{\kappa}\vec{W})\big)\oint\kappa ds)\Big) \\
&= 0,
\end{aligned}$$

$$\begin{aligned}
\lambda_{C} &= h(\vec{D},\vec{C}) = \frac{1}{2}\Big(\vec{D}\times\overline{\vec{C}}+\vec{C}\times\overline{\vec{D}}\Big) \\
&= \frac{1}{2}\Big(-(\vec{T}\oint\tau ds+\vec{B}\oint\kappa ds)\times\vec{C}+\vec{C}\times(-\vec{T}\oint\tau ds-\vec{B}\oint\kappa ds)\Big) \\
&= \frac{1}{2}\Big(-(\vec{T}\times\vec{C})\oint\tau ds-(\vec{B}\times\vec{C})\oint\kappa ds-(\vec{C}\times\vec{T})\oint\tau ds-(\vec{C}\times\vec{B})\oint\kappa ds)\Big) \\
&= \frac{1}{2}\Big(-\big((-\overline{\kappa}\vec{C}+\overline{\tau}\vec{W})\times\vec{C}\big)\oint\tau ds-\big((\overline{\tau}\vec{C}+\overline{\kappa}\vec{W})\times\vec{C}\big)\oint\kappa ds \\
&\quad -\big(\vec{C}\times(-\overline{\kappa}\vec{C}+\overline{\tau}\vec{W})\big)\oint\tau ds-\big(\vec{C}\times(\overline{\tau}\vec{C}+\overline{\kappa}\vec{W})\big)\oint\kappa ds)\Big) \\
&= -\frac{\kappa}{\beta}\oint\tau ds+\frac{\tau}{\beta}\oint\kappa ds,
\end{aligned}$$

$$\begin{aligned}
\lambda_{W} &= h(\vec{D},\vec{W}) = \frac{1}{2}\Big(\vec{D}\times\overline{\vec{W}}+\vec{W}\times\overline{\vec{D}}\Big) \\
&= \frac{1}{2}\Big(-(\vec{T}\oint\tau ds+\vec{B}\oint\kappa ds)\times\vec{W}+\vec{W}\times(-\vec{T}\oint\tau ds-\vec{B}\oint\kappa ds)\Big) \\
&= \frac{1}{2}\Big(-(\vec{T}\times\vec{W})\oint\tau ds-(\vec{B}\times\vec{W})\oint\kappa ds-(\vec{W}\times\vec{T})\oint\tau ds-(\vec{W}\times\vec{B})\oint\kappa ds)\Big) \\
&= \frac{1}{2}\Big(-\big((-\overline{\kappa}\vec{C}+\overline{\tau}\vec{W})\times\vec{W}\big)\oint\tau ds-\big((\overline{\tau}\vec{C}+\overline{\kappa}\vec{W})\times\vec{W}\big)\oint\kappa ds \\
&\quad -\big(\vec{W}\times(-\overline{\kappa}\vec{C}+\overline{\tau}\vec{W})\big)\oint\tau ds-\big(\vec{W}\times(\overline{\tau}\vec{C}+\overline{\kappa}\vec{W})\big)\oint\kappa ds)\Big) \\
&= \frac{\kappa}{\beta}\oint\kappa ds+\frac{\tau}{\beta}\oint\tau ds.
\end{aligned}$$

Corollary 3.4. *The angle of pitches of the closed spatial quaternionic ruled surfaces drawn by* *the Alternative vectors are*

$$\begin{cases} \lambda_N = 0, \\ \lambda_C = -\dfrac{\kappa}{\beta}\displaystyle\oint \tau ds + \dfrac{\tau}{\beta}\displaystyle\oint \kappa ds, \\ \lambda_W = \dfrac{\kappa}{\beta}\displaystyle\oint \kappa ds + \dfrac{\tau}{\beta}\displaystyle\oint \tau ds. \end{cases}$$

References

[1] A. Cayley, On Certain Results Relating to Quaternions, *Phil. Mag.*, **26**, 141–145 (1845).

[2] A. A. Shaikh and P. R. Ghosh, Rectifying curves on a smooth surface immersed in the Euclidean space, *Indian J. Pure Appl. Math.*, **50**(4), 883–890 (2019).

[3] A. A. Shaikh and P. R. Ghosh, Rectifying and Osculating curves on a smooth surface, *Indian J. Pure Appl. Math.*, **51**(1), 67–75 (2020).

[4] A. A. Shaikh and P. R. Ghosh, Curves on a smooth surface with position vectors lie in the tangent plane, *Indian J. Pure Appl. Math.*, **51**(3), 1097–1104 (2020).

[5] A. A. Shaikh, M. L. Lone and P. R. Ghosh, Normal curves on a smooth immersed surface, *Indian J. Pure Appl. Math.*, **51**(4), 1343–1355 (2020).

[6] A. T. Ali, Non-lightlike ruled surfaces with constant curvatures in Minkowski 3-space, *Int. J. Geom. Methods Mod. Phys.*, **15**(4), 1850068 (2017).

[7] B. Uzunoğlu, İ. Gök and Y. Yaylı, A new approach on curves of constant precession, *Appl. Math. Comput.*, **275**, 317–232 (2016).

[8] H. H. Hacısalihoğlu, *Motion Geometry and Quaternions Theory* (in Turkish), Faculty of Sciences and Arts, University of Gazi Press (1983).

[9] K. Bharathi and M. Nagaraj, Quaternion Valued Function of a Real Variable Serret-Frenet Formulae, *Ind. J. P. Appl. Math.*, **18**, 507–511 (1987).

[10] K. Orbay, E. Kasap and İ. Aydemir, Mannheim Offsets of Ruled Surfaces, *Math. Probl. Eng.*, **2009**, (2009).

[11] K. Shoemake, Animating Rotation with Quaternion Curves, *Siggraph Computer Graphics*, **19**, 245–254 (1985).

[12] M. P. Do Carmo, *Differential Geometry of Curves and Surfaces* (Revised Second Edition), Prentice-Hall, Mineola, New York, (2016).

[13] M. Karadağ and A. İ. Sivridağ, Quaternion Valued Functions of a Single reel Variable and Inclined Curves, *Journal of Natural Science*, **13**(1-2), 23-36 (1997).

[14] O. Kaya, and M. Önder, New Partner Curves in The Euclidean 3-Space E^3, *Int. J. Geom.*, **6**(2), 41–50 (2017).

[15] S. Şenyurt, C. Cevahir and Y. Altun, On spatial quaternionic involute curve a new view, *Adv. Appl. Clifford Algebr.*, **27**(2), 1815–1824 (2016).

[16] S. Şenyurt and A. Çalışkan The Quaternionic Expression of Ruled Surfaces, *Filomat*, **32**(16), 5753–5766 (2018).

[17] W. Fenchel, On the differential geometry of closed space curves, *Bull. Amer. Math. Soc.*, **57**, 44–54 (1951).

[18] W. R. Hamilton, *Elements of quaternions*, Longmans, Green Company, (1866).

Author information

Abdussamet Çalışkan, Fatsa Vocational School, Accounting and Tax Applications
Ordu Universty, Ordu, TURKEY.
E-mail: `abdussamet65@gmail.com`

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