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Derivations and Centralizers in Rings

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Abstract Let R be an associative ring with identity e and let F, D and T be additive maps from R into itself. The main aim of this article is to obtain some identities involving at the most 4 terms satisfied by F, D and T on the semiprime ring R which is m-torsion free for some m so that F, D and T become a Jordan generalized derivation, derivation and centralizer on R, respectively.

1 Introduction

Let R be an associative ring with identity e and center $Z(R)$. For $x, y \in R$, the Lie commutator $[x, y] = xy - yx$. Recall that, if $aRb = 0$ implies $a = 0$ or $b = 0$, then R is prime and if $aRa = 0$ implies $a = 0$, then R is semiprime. For $n > 1$, R is n-torsion free if $nx = 0$, for all $x \in R$ implies $x = 0$.

An additive mapping D from R into itself is said to be a derivation if $D(xy) = D(x)y + xD(y)$, for all $x, y \in R$ and is said to be a Jordan derivation if $D(x^2) = D(x)x + xD(x)$, for all $x \in R$. Clearly, every derivation is a Jordan derivation but converse, in general, is not true. Herstein [6], asserts that on a 2-torsion free prime ring every Jordan derivation is a derivation. Further, Cusack [3] generalized Herstein's result to 2-torsion free semiprime rings. An additive mapping $D : R \to R$ is a Jordan triple derivation if $D(xyx) = D(x)yx + xD(y)x + xyD(x)$, for all $x, y \in R$. One can easily see that any derivation is a Jordan triple derivation. Bresar [2] has proved that, on a 2-torsion free semiprime ring any Jordan triple derivation is a derivation.

An additive mapping T from R into itself is said to be a left (right) centralizer if $T(xy) = T(x)y$ ($T(xy) = xT(y)$), for all $x, y \in R$ and is said to be a left (right) Jordan centralizer if $T(x^2) = T(x)x$ ($T(x^2) = xT(x)$), for all $x \in R$. An additive mapping $T : R \to R$ is called a centralizer if it is both left and right centralizer. In [13], it is proved that, on a 2-torsion free semiprime ring any left (right) Jordan centralizer is a centralizer. Further, T is a left (right) centralizer if and only if T is of the form $T(x) = \alpha x$ ($T(x) = x\alpha$), for some fixed $\alpha \in R$. Vukman and Kosi-Ulbl [10] proved that if T is an additive mapping and R is a 2-torsion free semiprime ring such that $3T(xyx) = T(x)yx + xT(y)x + xyT(x)$, for all $x, y \in R$, then there exists an element $\lambda \in C$ such that $T(x) = \lambda x$, for all $x \in R$. In [11], they proved that if $n \geq 2$ and R is a 2, n-torsion free semiprime ring such that $2T(x^{n+1}) = T(x)x^n + x^nT(x)$, for all $x \in R$, then T is a centralizer. Further, if R is a 2-torsion free semiprime ring such that $2T(xyx) = T(x)yx + xyT(x)$, for all $x, y \in R$, then T is a centralizer. In [7], it is proved that if $m, n > 1$ and R is an $(m+n+2)!$-torsion free semiprime ring such that $T(x^{m+n+1}) = x^mT(x)x^n$, for all $x \in R$, then T is a centralizer.

An additive mapping F from R into itself is called a generalized derivation if $F(xy) = F(x)y + xD(y)$, for all $x, y \in R$, where D is a derivation. We can easily see that F is a generalized derivation if and only if F can be written in the form $F = D + T$, where D is a derivation and T is a left centralizer. An additive mapping is called a generalized Jordan derivation if $F(x^2) = F(x)x + xD(x)$, for all $x \in R$, where D is a Jordan derivation. An additive mapping F is called a generalized Jordan triple derivation if $F(xyx) = F(x)yx + xD(y)x + xyD(x)$, for all $x, y \in R$, where D is a Jordan triple derivation. Vukman [9], proved that if R is a 2-torsion free semiprime ring and F is either a generalized Jordan derivation or a generalized Jordan triple derivation, then F is a generalized derivation.

In [4, 5], Dhara and Sharma show that, an additive map satisfying an identity having $(n+2)$

terms is a derivation, centralizer and Jordan generalized derivation whereas in [12], Yadav and Sharma proved that, an additive map satisfying an identity having just five terms is a derivation, centralizer and Jordan generalized derivation. In this article, we show that, an additive map satisfying an identity having at most four terms is a derivation, centralizer and Jordan generalized derivation.

2 Centralizers

First we prove some results on additive mappings which are centralizers.

Theorem 2.1. *Let R be an $(n+1)!$-torsion free semiprime ring, where $n \geq 1$ is a fixed integer. If $T : R \to R$ is an additive mapping such that $2T(x^{n+1}) = T(x^n)x + xT(x^n)$ for all $x \in R$, then T is a centralizer.*

Proof. For $x \in R$, we have

$$2T(x^{n+1}) = T(x^n)x + xT(x^n). \tag{2.1}$$

Replacing x by $x + ke$ in (2.1), where k is any positive integer, we get

$$2T(x+ke)^{n+1} = T(x+ke)^n(x+ke) + (x+ke)T(x+ke)^n.$$

Expanding the powers of $x + ke$,

$$\begin{aligned} &2T\left(x^{n+1} + \cdots + \binom{n+1}{n-1}k^{n-1}x^2 + \binom{n+1}{n}k^n x + k^{n+1}e\right) \\ =&T\left(x^n + \cdots + \binom{n}{n-2}k^{n-2}x^2 + \binom{n}{n-1}k^{n-1}x + k^n e\right)(x+ke) \\ &+ (x+ke)T\left(x^n + \cdots + \binom{n}{n-2}k^{n-2}x^2 + \binom{n}{n-1}k^{n-1}x + k^n e\right). \end{aligned}$$

Using (2.1) and rearranging the above terms by collecting the terms involving equal powers of k, we have $\sum_{i=1}^{n} k^i f_i(x,e) = 0$, for all $x \in R$. Now by putting $k = 1, 2, \ldots, n$, we get a system of n homogeneous equations, with coefficient matrix

$$V = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 2 & 2^2 & 2^3 & \cdots & 2^n \\ . & . & . & & . \\ . & . & . & & . \\ n & n^2 & n^3 & \cdots & n^n \end{bmatrix}.$$

Since $|V|$ is a product of positive integers, each of which is $\leq n$ and since R is $(n+1)!$-torsion free, this system has only a trivial solution. In particular, $f_n(x,e) = 0$ implies that

$$2\binom{n+1}{n}T(x) = T(e)x + 2\binom{n}{n-1}T(x) + xT(e).$$

So,

$$2T(x) = T(e)x + xT(e) \tag{2.2}$$

for all $x \in R$. Now $f_{n-1}(x,e) = 0$, gives

$$2\binom{n+1}{n-1}T(x^2) = \binom{n}{n-1}\big(T(x)x + xT(x)\big) + 2\binom{n}{n-2}T(x^2).$$

Multiplying both sides by 2 in the above equation, we get

$$4nT(x^2) = 2n(T(x)x + xT(x)).$$

Since R is $(n+1)!$-torsion free ring, the above equations can be reduced to

$$4T(x^2) = 2(T(x)x + xT(x)). \tag{2.3}$$

Using (2.2) in (2.3) gives $[[T(e), x], x] = 0$, for all $x \in R$. Now by [8, Theorem 2], $T(e) \in Z(R)$. Thus $T(x) = T(e)x = xT(e)$ and T is a centralizer. □

Theorem 2.2. *Let R be an $(n+2)!$-torsion free semiprime ring, where $n \geq 1$ is a fixed integer. If $T : R \to R$ is an additive mapping such that $3T(x^{n+2}) = T(x)x^{n+1} + xT(x^n)x + x^{n+1}T(x)$ for all $x \in R$, then T is a centralizer.*

Proof. For $x \in R$, we have

$$3T(x^{n+2}) = T(x)x^{n+1} + xT(x^n)x + x^{n+1}T(x). \tag{2.4}$$

Replacing x by $x + ke$ in (2.4), where k is any positive integer, we get

$$\begin{aligned} 3T(x+ke)^{n+2} =& T(x+ke)(x+ke)^{n+1} + (x+ke)T(x+ke)^n(x+ke) \\ &+ (x+ke)^{n+1}T(x+ke). \end{aligned}$$

Expanding the powers of $x + ke$, we get

$$\begin{aligned} &3T\left(x^{n+2} + \cdots + \binom{n+2}{n}k^n x^2 + \binom{n+2}{n+1}k^{n+1}x + k^{n+2}e\right) \\ =&T(x+ke)\left(x^{n+1} + \cdots + \binom{n+1}{n-1}k^{n-1}x^2 + \binom{n+1}{n}k^n x + k^{n+1}e\right) \\ &+ (x+ke)T\left(x^n + \cdots + \binom{n}{n-2}k^{n-2}x^2 + \binom{n}{n-1}k^{n-1}x + k^n e\right)(x+ke) \\ &+ \left(x^{n+1} + \cdots + \binom{n+1}{n-1}k^{n-1}x^2 + \binom{n+1}{n}k^n x + k^{n+1}e\right)T(x+ke). \end{aligned}$$

Using (2.4) and rearranging the above terms, we get $\sum_{i=1}^{n+1} k^i f_i(x,e) = 0$, for all $x \in R$. Proceeding in the same way as in the proof of the Theorem 2.1, $f_i(x,e) = 0$, for all $x \in R$ and $i \in \{1, 2, \ldots, n+1\}$. In particular, $f_{n+1}(x,e) = 0$ implies that

$$\begin{aligned} 3\binom{n+2}{n+1}T(x) =& 2T(x) + \binom{n}{n-1}T(x) + \binom{n+1}{n}T(e)x + xT(e) + T(e)x \\ &+ \binom{n+1}{n}xT(e). \end{aligned}$$

We have,

$$2(n+2)T(x) = (n+2)(T(e)x + xT(e))$$

for all $x \in R$. Since R is $(n+2)!$-torsion free,

$$2T(x) = T(e)x + xT(e) \tag{2.5}$$

for all $x \in R$. Now $f_n(x,e) = 0$, gives

$$\begin{aligned} 3\binom{n+2}{n}T(x^2) =& \binom{n+1}{n}T(x)x + \binom{n+1}{n-1}T(e)x^2 + xT(e)x + \binom{n}{n-1}xT(x) \\ &+ \binom{n}{n-1}T(x)x + \binom{n}{n-2}T(x^2) + \binom{n+1}{n}xT(x) \\ &+ \binom{n+1}{n-1}x^2T(e). \end{aligned}$$

We get,

$$(n^2+5n+3)T(x^2) = (2n+1)(T(x)x+xT(x)) + \frac{n(n+1)}{2}(T(e)x^2+x^2T(e)) + xT(e)x.$$

Multiplying both sides by 2 in the above equation and using (2.5), we get $(n+1)[[T(e),x],x] = 0$, which yields $[[T(e),x],x] = 0$ for all $x \in R$. Now by [8, Theorem 2], $T(e) \in Z(R)$. Thus $T(x) = T(e)x = xT(e)$ and T is a centralizer. □

Theorem 2.3. *Let R be an $(n+2)!$ and $(3n+1)$-torsion free semiprime ring, where $n \geq 1$ is a fixed integer. If $T : R \to R$ is an additive mapping such that $2T(x^{n+2}) = xT(x)x^n + x^nT(x)x$ for all $x \in R$, then T is a centralizer.*

Proof. For $x \in R$, we have

$$2T(x^{n+2}) = xT(x)x^n + x^nT(x)x. \tag{2.6}$$

Replacing x by $x+ke$ in (2.6) and expanding the powers of $x+ke$, we get

$$\begin{aligned} &2T\left(x^{n+2}+\cdots+\binom{n+2}{n}k^nx^2+\binom{n+2}{n+1}k^{n+1}x+k^{n+2}e\right)\\ =&(x+ke)T(x+ke)\left(x^n+\cdots+\binom{n}{n-2}k^{n-2}x^2+\binom{n}{n-1}k^{n-1}x+k^ne\right)\\ &+\left(x^n+\cdots+\binom{n}{n-2}k^{n-2}x^2+\binom{n}{n-1}k^{n-1}x+k^ne\right)T(x+ke)(x+ke). \end{aligned}$$

Using (2.6) and rearranging the above terms, we get $\sum_{i=1}^{n+1} k^i f_i(x,e) = 0$, for all $x \in R$. Hence $f_i(x,e) = 0$ for all $x \in R$ and $i \in \{1,2,\ldots,n+1\}$. In particular, $f_{n+1}(x,e) = 0$ implies that

$$2(n+1)T(x) = (n+1)(T(e)x + xT(e))$$

for all $x \in R$. Since R is $(n+2)!$-torsion free, we get

$$2T(x) = T(e)x + xT(e) \tag{2.7}$$

for all $x \in R$. Now $f_n(x,e) = 0$, gives

$$\begin{aligned} 2\binom{n+2}{n}T(x^2) =& xT(x)+\binom{n}{n-1}xT(e)x+\binom{n}{n-1}T(x)x+\binom{n}{n-2}T(e)x^2+\\ &T(x)x+\binom{n}{n-1}xT(e)x+\binom{n}{n-1}xT(x)+\binom{n}{n-2}x^2T(e). \end{aligned}$$

That is,

$$\begin{aligned} (n+2)(n+1)T(x^2) =& (n+1)(xT(x)+T(x)x)+2nxT(e)x+\\ &\frac{n(n-1)}{2}(T(e)x^2+x^2T(e)). \end{aligned}$$

Multiplying both sides by 2 in the above equation and using (2.7), we get

$$(3n+1)[[T(e),x],x] = 0. \tag{2.8}$$

Since R is $(n+2)!$ and $(3n+1)$-torsion free, $[[T(e),x],x] = 0$, for all $x \in R$. Now by [8, Theorem 2], $T(e) \in Z(R)$ and (2.7) implies that T is a centralizer. □

3 Derivations

In this section, we discuss derivations and Jordan derivations.

Theorem 3.1. *Let R be an $(n+1)!$-torsion free semiprime ring, where $n \geq 1$ is a fixed integer. If $D : R \to R$ is an additive mapping such that $D(x^{n+1}) = D(x^n)x + x^nD(x)$ for all $x \in R$, then D is a derivation.*

Proof. For $x \in R$, we have

$$D(x^{n+1}) = D(x^n)x + x^nD(x). \tag{3.1}$$

Replacing x by e in (3.1), we get $D(e) = 2D(e)$ which implies $D(e) = 0$. Further, replacing x by $x + ke$ and expanding the powers of $x + ke$, we get

$$\begin{aligned} &D\left(x^{n+1} + \cdots + \binom{n+1}{n-1}k^{n-1}x^2 + \binom{n+1}{n}k^nx + k^{n+1}e\right) \\ =&D\left(x^n + \cdots + \binom{n}{n-2}k^{n-2}x^2 + \binom{n}{n-1}k^{n-1}x + k^ne\right)(x+ke) \\ &+ \left(x^n + \cdots + \binom{n}{n-2}k^{n-2}x^2 + \binom{n}{n-1}k^{n-1}x + k^ne\right)D(x+ke). \end{aligned}$$

Using (3.1), rearranging the above terms and using the fact that $D(e) = 0$, we get $\sum_{i=1}^{n} k^i f_i(x) = 0$, for all $x \in R$. So $f_i(x) = 0$, for all $x \in R$ and $i \in \{1, 2, \ldots, n\}$. In particular, $f_{n-1}(x) = 0$ implies that

$$\binom{n+1}{n-1}D(x^2) = \binom{n}{n-1}D(x)x + \binom{n}{n-2}D(x^2) + \binom{n}{n-1}xD(x).$$

We have

$$D(x^2) = D(x)x + xD(x)$$

for all $x \in R$. Thus D is a Jordan Derivation. Therefore by [3, Theorem 6], D is a derivation. □

Theorem 3.2. *Let R be an $(n+2)!$ and $(2n+1)$-torsion free semiprime ring, where $n \geq 1$ is a fixed integer. If $D : R \to R$ is an additive mapping such that $D(x^{n+2}) = D(x)x^{n+1} + xD(x^n)x + x^{n+1}D(x)$ for all $x \in R$, then D is a derivation.*

Proof. For $x \in R$, we have

$$D(x^{n+2}) = D(x)x^{n+1} + xD(x^n)x + x^{n+1}D(x). \tag{3.2}$$

Replacing x by e in (3.2) gives, $2D(e) = 0$. But R is 2-torsion free, so $D(e) = 0$. Now, replacing x by $x + ke$ in (3.2) and expanding the powers of $x + ke$, we get

$$\begin{aligned} &D\left(x^{n+2} + \cdots + \binom{n+2}{n}k^nx^2 + \binom{n+2}{n+1}k^{n+1}x + k^{n+2}e\right) \\ =&D(x+ke)\left(x^{n+1} + \cdots + \binom{n+1}{n-1}k^{n-1}x^2 + \binom{n+1}{n}k^nx + k^{n+1}e\right) \\ &+ (x+ke)D\left(x^n + \cdots + \binom{n}{n-2}k^{n-2}x^2 + \binom{n}{n-1}k^{n-1}x + k^ne\right)(x+ke) \\ &+ \left(x^{n+1} + \cdots + \binom{n+1}{n-1}k^{n-1}x^2 + \binom{n+1}{n}k^nx + k^{n+1}e\right)D(x+ke). \end{aligned}$$

Using (3.2), rearranging the above terms and using the fact that $D(e) = 0$, we get $\sum_{i=1}^{n} k^i f_i(x) = 0$, for all $x \in R$. So $f_i(x) = 0$ for all $x \in R$ and $i \in \{1, 2, \ldots, n\}$. In particular, $f_n(x) = 0$ implies that

$$\begin{aligned} \binom{n+2}{n}D(x^2) =&\binom{n+1}{n}D(x)x + \binom{n}{n-1}D(x)x + \binom{n}{n-1}xD(x)+ \\ &\binom{n}{n-2}D(x^2) + \binom{n+1}{n}xD(x). \end{aligned}$$

After simplification, we get $(2n+1)\{D(x^2) - D(x)x - xD(x)\} = 0$ for all $x \in R$ which yields $D(x^2) = D(x)x + xD(x)$. Thus, D is a Jordan Derivation and by [3, Theorem 6], a derivation. □

Theorem 3.3. *Let R be an $(n+2)!$ and $(2n+1)$-torsion free semiprime ring, where $n \geq 1$ is a fixed integer. If $F, D : R \to R$ are additive mappings such that $F(x^{n+2}) = F(x)x^{n+1} + xD(x^n)x + x^{n+1}D(x)$ for all $x \in R$, then D is a Jordan derivation and F is a Jordan generalised derivation.*

Proof. For $x \in R$, we have

$$F(x^{n+2}) = F(x)x^{n+1} + xD(x^n)x + x^{n+1}D(x). \tag{3.3}$$

Replacing x by e, we get $2D(e) = 0$ and so $D(e) = 0$. Now, replacing x by $x + ke$ in (3.3) and expanding the powers of $x + ke$, we get

$$\begin{aligned} &F\left(x^{n+2} + \cdots + \binom{n+2}{n}k^n x^2 + \binom{n+2}{n+1}k^{n+1}x + k^{n+2}e\right) \\ =&F(x+ke)\left(x^{n+1} + \cdots + \binom{n+1}{n-1}k^{n-1}x^2 + \binom{n+1}{n}k^n x + k^{n+1}e\right) \\ &+ (x+ke)D\left(x^n + \cdots + \binom{n}{n-2}k^{n-2}x^2 + \binom{n}{n-1}k^{n-1}x + k^n e\right)(x+ke) \\ &+ \left(x^{n+1} + \cdots + \binom{n+1}{n-1}k^{n-1}x^2 + \binom{n+1}{n}k^n x + k^{n+1}e\right)D(x+ke). \end{aligned}$$

Using (3.3), rearranging the above terms and using the fact that $D(e) = 0$, we get $\sum_{i=1}^{n+1} k^i f_i(x,e) = 0$, for all $x \in R$. Hence, $f_i(x,e) = 0$ for all $x \in R$ and $i \in \{1, 2, \ldots, n+1\}$. In particular, $f_{n+1}(x) = 0$ implies that

$$(n+1)F(x) = (n+1)\{F(e)x + D(x)\} = 0$$

for all $x \in R$. Since R is $(n+2)!$-torsion free,

$$F(x) = F(e)x + D(x) \tag{3.4}$$

for all $x \in R$. Now $f_n(x,e) = 0$, gives

$$\begin{aligned} \binom{n+2}{n}F(x^2) =&\binom{n+1}{n}F(x)x + \binom{n+1}{n-1}F(e)x^2 + \binom{n}{n-1}xD(x) + \\ &\binom{n}{n-1}D(x)x + \binom{n}{n-2}D(x^2) + \binom{n+1}{n}xD(x). \end{aligned}$$

Using (3.4), we get $(2n+1)\{D(x^2) - D(x)x - xD(x)\} = 0$ which yields $D(x^2) = D(x)x + xD(x)$ for all $x \in R$. Hence D is a Jordan derivation. Again by (3.4), we have $F(x^2) = F(e)x^2 + D(x^2) = F(e)x^2 + D(x)x + xD(x) = F(x)x + xD(x)$ for all $x \in R$ and F is a Jordan generalized derivation. □

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