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A Study On Bi-f-Harmonic Curves

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Abstract In this manuscript, we obtain necessary and sufficient conditions for a spacelike curve with timelike first binormal vector field to be a bi-f-harmonic curve in a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold.

1 Introduction

Harmonic maps between Riemannian manifolds, which can be viewed as a generalization of geodesics when the domain is 1-dimensional, or of harmonic functions when the ranges are Euclidean spaces, have an extensive study area and there exist many applications of such mappings in mathematics and physics.

As a generalization of harmonic maps, biharmonic maps between Riemannian manifolds were introduced by J. Eells and J. H. Sampson in [5]. B. Y. Chen [3] defined biharmonic submanifolds of the Euclidean space and stated a well-known conjecture: Any biharmonic submanifold of the Euclidean space is harmonic, thus minimal. If one use the definition of biharmonic maps to Riemannian immersions into Euclidean space, it is easy to see that Chen's definition of biharmonic submanifold coincides with the definition given by using bienergy functional. In recent years, there has been an important literature survey on biharmonic submanifold theory including many results on the non-existence of biharmonic submanifolds in manifolds with non-positive sectional curvature. These non-existence consequences (see [7], [10]) as well as *Generalized Chen's conjecture*: Any biharmonic submanifold in a Riemannian manifold with non-positive sectional curvature is minimal, which was proposed by R. Caddeo, S. Montaldo and C. Oniciuc [2], led the studies to spheres and other non-negatively curved spaces. For some studies of general biharmonic maps see ([2], [8], [11], [13], [14], [15], [17], [18]) and the references therein.

In 1970, f-harmonic maps between Riemannian manifolds were first studied by A. Lichnerowicz (see also [4]). They have also some physical meanings by considering them as solutions of continuous spin systems and inhomogenous Heisenberg spin systems [1].

There are two ways to formalize such a link between biharmonic maps and f-harmonic maps. The first formalization is that by mimicking the theory for biharmonic maps, the authors of [19] extended bienergy functional to bi-f-energy functional and obtained a new type of harmonic maps called bi-f-harmonic maps. This idea was already considered by Ouakkas, Nasri and Djaa [16]. They used the terminology "f-biharmonic maps" for the critical points of bi-f-energy functional. In [19], as parallel to "biharmonic maps", they think that it is more reasonable to call them "bi-f-harmonic maps". The second formalization is that by following the definition of f-harmonic map, to extend the f-energy functional to the f-bienergy functional and obtain another type of harmonic maps called f-biharmonic maps as critical points of f-bienergy functional.

As a generalization of biharmonic maps, the term of f-biharmonic maps has been introduced by W.-J. Lu [9] . A differentiable map between Riemannian manifolds is said to be f-biharmonic if it is a critical point of the f-bienergy functional, where f is a smooth positive function on the domain. If $f = 1$, then f-biharmonic maps are biharmonic. To avoid the confusion with the types of maps called by the same name in [16] and defined as critical points of the square-norm of the f-tension field, some authors (see [9], [12]) called the map defined in [16] as *bi-f-harmonic map*, which we shall examine in this manuscript.

In [25], the study of Lorentzian almost paracontact manifold was introduced by K. Matsumoto. He also studied the notion of Lorentzian para-Sasakian manifold. In [21], the same notion was defined by authors independently and thereafter many authors ([22, 27, 31]) initiated Lorentzian para-Sasakian manifolds.

In this article, we study bi-f-harmonic spacelike curve parametrized by arclength with timelike first binormal B_1 on 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold M.

2 Preliminaries

2.1 Harmonic maps

Harmonic maps $\phi : (\tilde{M}, g) \to (\tilde{N}, h)$ between two Riemannian manifolds are critical points of the energy functional:

$$E(\phi) = \frac{1}{2}\int_{\Omega} |d\phi|^2 \vartheta_g, \tag{2.1}$$

where $\Omega \subset \phi$ is a compact domain. The corresponding Euler-Lagrange equation is [5]:

$$\tau(\phi) \equiv trace \nabla d\phi = 0, \tag{2.2}$$

where ∇ is the connection induced from the Levi-Civita connection $\nabla^{\tilde{M}}$ of $\tilde{M}$ and the pull-back connection ∇^{ϕ}. $\tau(\phi)$ is called the *tension field* of the map ϕ.

2.2 Biharmonic maps

Biharmonic maps $\phi : (\tilde{M}, g) \to (\tilde{N}, h)$ between two Riemannian manifolds are critical points of the bienergy functional:

$$E_2(\phi) = \frac{1}{2}\int_{\Omega} |\tau(\phi)|^2 \vartheta_g, \tag{2.3}$$

where $\Omega \subset \tilde{M}$ is a compact domain. The corresponding Euler-Lagrange equation is [6]:

$$\tau_2(\phi) \equiv trace\left(\nabla^{\phi}\nabla^{\phi}\tau(\phi) - \nabla^{\phi}_{\nabla^M}\tau(\phi) + R^{\tilde{N}}(\tau(\phi), d\phi)d\phi\right) = 0, \tag{2.4}$$

where $\tau(\phi)$ is the tension field of ϕ and $R^{\tilde{N}}(X,Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ is the curvature operator on $\tilde{N}$. $\tau_2(\phi)$ is called the *bitension field* of the map φ.

From the expression of the bitension field τ_2, it is clear that a harmonic map is automatically a biharmonic map. So non-harmonic biharmonic maps, which are called proper biharmonic maps, are more interesting to be studied.

2.3 f-harmonic maps

f-harmonic maps $\phi : (\tilde{M}, g) \to (\tilde{N}, h)$ between two Riemannian manifolds are critical points of the f-energy functional:

$$E_f(\phi) = \frac{1}{2}\int_{\Omega} f\,|d\phi|^2 \vartheta_g, \tag{2.5}$$

where $\Omega \subset \tilde{M}$ is a compact domain. The corresponding Euler-Lagrange equation is [16]:

$$\tau_f(\phi) \equiv f\,\tau(\phi) + \phi(gradf) = 0, \tag{2.6}$$

where $\tau(\phi)$ is the tension field of ϕ. $\tau_f(\phi)$ is called the *f-tension field* of the map ϕ.

If f is a constant function, then it is obvious that f-harmonic maps are harmonic. So f-harmonic maps, where f is a non-constant function, which are called proper f-harmonic maps, are more interesting to be studied.

There are two ways to formalize a link between biharmonic maps and f-harmonic maps. Both of them motivate the following definitions.

2.4 *f*-biharmonic maps

f-biharmonic maps $\phi : (\tilde{M}, g) \to (\tilde{N}, h)$ between two Riemannian manifolds are critical points of the f-bienergy functional:

$$E_{2,f}\,(\phi) = \frac{1}{2} \int_{\Omega} f \; |\tau(\phi)|^2 \vartheta_g, \tag{2.7}$$

where $\Omega \subset \tilde{M}$ is a compact domain. The corresponding Euler-Lagrange equation is [9]:

$$\tau_{2,f}\,(\phi) \equiv f\,\tau_2(\phi) + (\Delta f)\tau(\phi) + 2\nabla^{\phi}_{grad f}\;\tau(\phi) = 0, \tag{2.8}$$

where $\tau(\phi)$ and $\tau_2(\phi)$ are the tension and bitension fields of ϕ, respectively. $\tau_{2,f}\,(\phi)$ is called the *f-bitension field* of the map ϕ.

2.5 Bi-*f*-harmonic maps

Bi-f-harmonic maps $\phi : (\tilde{M}, g) \to (\tilde{N}, h)$ between two Riemannian manifolds are critical points of the bi-f-energy functional:

$$E_{f,2}\,(\phi) = \frac{1}{2} \int_{\Omega} |\tau_f\,(\phi)|^2 \vartheta_g, \tag{2.9}$$

where $\Omega \subset \tilde{M}$ is a compact domain. The corresponding Euler-Lagrange equation is [16]:

$$\tau_{f,2}\,(\phi) \equiv -trace\left(\nabla^{\phi} f\;\left(\nabla^{\phi}\tau_f\;(\phi)\right)\right) - f\,\nabla^{\phi}_{\nabla^M}\tau_f\;(\phi) + f\,R^N\left(\tau_f\;(\phi)\,, d\phi\right) d\phi\right) = 0, \tag{2.10}$$

where $\tau_f(\phi)$ is the f-tension field of ϕ. $\tau_{f,2}\,(\phi)$ is called the *bi-f-tension field* of the map ϕ.

The following inclusions illustrate the relations among these different types of harmonic maps:

Harmonic maps $\subset$ biharmonic maps $\subset$ f-biharmonic maps,

Harmonic maps $\subset$ f-harmonic maps $\subset$ bi-f-harmonic maps.

2.6 Lorentzian almost paracontact manifolds

Let $\tilde{M}$ be an n-dimensional differentiable manifold with a Lorentzian metric g, i.e., g is a smooth symmetric tensor field of type $(0, 2)$ such that at every point $p \in \tilde{M}$, the tensor

$$g_p : T_p\tilde{M} \times T_p\tilde{M} \to R,$$

is a non-degenerate inner product of signature $(-, +, +, ..., +)$, where $T_p\tilde{M}$ is the tangent space of $\tilde{M}$ at the point p. Then $(\tilde{M}, g)$ is called a *Lorentzian manifold.*

A non-zero vector $X_p \in T_p\tilde{M}$ can be spacelike, null or timelike, if it satisfies

$$g_p(X_p, X_p) > 0, \;\; g_p(X_p, X_p) = 0 \;\text{ or } g_p(X_p, X_p) < 0,$$

respectively.

Let $\tilde{M}$ be an n-dimensional differentiable manifold equipped with a structure (φ, ξ, η), where φ is a $(1, 1)$-tensor field, ξ is a vector field, η is a 1-form on $\tilde{M}$ such that [25];

$$\varphi^2 X = X + \eta(X)\xi, \tag{2.11}$$

$$\eta(\xi) = -1. \tag{2.12}$$

From above equations, we get

$$\eta \circ \varphi = 0, \quad \varphi\xi = 0, \quad rank(\varphi) = n - 1.$$

Then $\tilde{M}$ admits a Lorentzian metric g, such that

$$g(\varphi X, \varphi Y) = g(X, Y) + \eta(X)\eta(Y),$$

and $\tilde{M}$ is said to admit a Lorentzian almost paracontact structure (φ, ξ, η, g). Then we get

$$g(X, \xi) = \eta(X).$$

The manifold $\tilde{M}$ endowed with a Lorentzian almost paracontact structure (φ, ξ, η, g) is called a Lorentzian almost paracontact manifold [25, 26].

Moreover, in (2.11) and (2.12) if we replace ξ by $-\xi$, we obtain an almost paracontact structure on $\tilde{M}$ defined by I. Sato [23].

A Lorentzian almost paracontact manifold $\tilde{M}$ equipped with the structure (φ, ξ, η, g) is called a Lorentzian para-Sasakian manifold (for short, *LP*-Sasakian manifold) [25] if

$$(\nabla_X \varphi)Y = g(X, Y)\xi + \eta(Y)X + 2\eta(X)\eta(Y)\xi. \tag{2.13}$$

The conformal curvature tensor C is given by

$$\begin{aligned} C(X,Y)U &= R(X,Y)U \\ &\quad -\frac{1}{n-2}\left\{ \begin{array}{c} S(Y,U)X - S(X,U)Y \\ +g(Y,U)QX - g(X,U)QY \end{array} \right\} \\ &\quad +\frac{r}{(n-1)(n-2)}\left\{g(Y,U)X - g(X,U)Y\right\}, \end{aligned}$$

where $S(X, Y) = g(QX, Y)$. The Lorentzian para-Sasakian manifold is called conformally flat if conformal curvature tensor vanishes i.e., $C = 0$.

The quasi-conformal curvature tensor $\hat{C}$ is defined by

$$\begin{aligned} \hat{C}(X,Y)U &= aR(X,Y)U \\ &\quad -b\left\{ \begin{array}{c} S(Y,U)X - S(X,U)Y \\ +g(Y,U)QX - g(X,U)QY \end{array} \right\} \\ &\quad -\frac{r}{n}\left(\frac{a}{(n-1)} + 2b\right)\left\{g(Y,U)X - g(X,U)Y\right\}, \end{aligned}$$

where a, b constants such that $ab \neq 0$. Similarly the Lorentzian para-Sasakian manifold is called quasi-conformally flat if $\hat{C} = 0$.

We know that a conformally flat and quasi-conformally flat Lorentzian para-Sasakian manifold $\tilde{M}^n$ $(n > 3)$ is of constant curvature (equal $+1$) and also a Lorentzian para-Sasakian manifold is locally isometric to a Lorentzian unit sphere if the relation $R(X, Y) \cdot C = 0$ satisfies on $\tilde{M}$ [28]. For a conformally symmetric Riemannian manifold [29], we get $\nabla C = 0$. Thus for a conformally symmetric space the relation $R(X, Y) \cdot C = 0$ satisfies. Hence a conformally symmetric Lorentzian para-Sasakian manifold is locally isometric to a Lorentzian unit sphere [28].

After these explanations, for a conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold, we get [28]

$$R(X, Y)U = g(Y, U)X - g(X, U)Y, \tag{2.14}$$

for any $X, Y, U \in T\tilde{M}$.

Let M be a 4-dimensional *LP*-Sasakian manifold. Denote by $\{T, N, B_1, B_2\}$ the moving Frenet frame along the curve γ in M. Then T, N, B_1, B_2 are respectively, the tangent, the principal normal, the first binormal and the second binormal vector fields.

Let $\gamma(s)$ be a curve in a 4-dimensional *LP*-Sasakian manifold parametrized by arclength function s. Then for the curve γ the following Frenet equations are given in [24]:

γ **is a spacelike curve:**

Then T is a spacelike vector, so depending on the casual character of the principal normal vector N and the first binormal vector B_1, we have different Frenet formulas. In this paper we only assume spacelike curves with N and B_2 are spacelike. In this case we have

$$\begin{bmatrix} \nabla_T T \\ \nabla_T N \\ \nabla_T B_1 \\ \nabla_T B_2 \end{bmatrix} = \begin{bmatrix} 0 & \kappa_1 & 0 & 0 \\ -\kappa_1 & 0 & \kappa_2 & 0 \\ 0 & \kappa_2 & 0 & \kappa_3 \\ 0 & 0 & \kappa_3 & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B_1 \\ B_2 \end{bmatrix}, \tag{2.15}$$

where T, N, B_1, B_2 are mutually orthogonal vectors satisfying the equations

$$g(T,T) = g(N,N) = g(B_2,B_2) = 1, \qquad g(B_1,B_1) = -1$$

and κ_1, κ_2 and κ_3 are the first, the second and the third curvature of the γ.

2.7 Bi-f-harmonic Curve

In this section we derive the bi-f-harmonic equation for curves in Riemannian manifolds. The following proposition for Euler-Lagrange equation of bi-f-harmonic maps originates from [16].

Proposition 2.1. *Let $\phi : (\tilde{M}, g) \to (\tilde{N}, h)$ be a smooth map between Riemannian manifolds. Then, in terms of Euler-Lagrange equation, ϕ is a bi-f-harmonic map if and only if its bi-f-tension field $\tau_{f,2}(\phi)$ vanishes, i.e.*

$$trace\left(\nabla^{\phi} f \ (\nabla^{\phi} \tau_f \ (\phi)) - f \, \nabla^{\phi}_{\nabla^{\tilde{M}}} \tau_f \ (\phi) + f \, R^N \left(\tau_f \ (\phi), d\phi\right) d\phi\right) = 0, \tag{2.16}$$

where $f : I \to (0, \infty)$ is a smooth map defined on a real interval I and $\tau_f \ (\phi)$ is the f-tension field given by (2.6) [20].

Clearly, it is observed from (2.16) that bi-f-harmonic map is a much wider generalization of harmonic map, because it is not only a generalization of f-harmonic map (as $f \neq 1$ and $\tau_f(\phi) = 0$), but also a generalization of biharmonic map (as $f = 1$). Therefore, it would be interesting to know whether there is any non-trivial or proper bi-f-harmonic map which is neither harmonic map nor f-harmonic map with $f \neq$ constant.

Let $\gamma : I \to (\tilde{N}, h)$ be a curve in a Riemannian manifold $(\tilde{N}, h)$, defined on an open real interval I and parametrized by its arclength, and $\gamma' =: T$. We have

$$\tau \ (\gamma) = \nabla^N_T T$$

$$\tau_f \ (\gamma) = f \, \nabla^N_T T + f' T$$

and in order to obtain the bi-f-tension field of γ, we compute:

$$\begin{aligned} trace\left(\nabla^{\gamma} f \ (\nabla^{\gamma} \tau_f \ (\gamma)) - f \, \nabla^{\gamma}_{\nabla^{\tilde{M}}} \tau_f \ (\gamma)\right) &= \nabla^{\gamma}_{\frac{d}{dt}} f \left(\nabla^{\gamma}_{\frac{d}{dt}} \tau_f \,(\gamma)\right) - f \, \nabla^{\gamma}_{\nabla^I_{\frac{d}{dt}} \frac{d}{dt}} \tau_f \,(\gamma) \\ &= \nabla^N_T f \left(\nabla^N_T (f \, \nabla^N_T T + f' T)\right) \\ &= (f f''' + f' f'') T + (3 f f'' + 2 \, (f')^2) \nabla^N_T T \\ &\quad + 4 f f' \, \nabla^N_T \nabla^N_T T + f^2 \nabla^N_T \nabla^N_T \nabla^N_T T \end{aligned} \tag{2.17}$$

and

$$\begin{aligned} trace\left(R^N \left(\tau_f \,(\gamma), d\gamma\right) d\gamma\right) &= R^N \left(\tau_f \,(\gamma), d\gamma \left(\frac{d}{dt}\right)\right) d\gamma \left(\frac{d}{dt}\right) \\ &= f R^N \left(\nabla^N_T T, T\right) T. \end{aligned} \tag{2.18}$$

3 Bi-f-harmonic Curves

In this section, some charecterizations for a spacelike proper bi-f-harmonic curves in a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold are given. Firstly, we give the following:

Definition 3.1. An arbitrary curve $\gamma : I \to \tilde{M}$, $\gamma = \gamma(s)$, on a Lorentzian para-Sasakian manifold is called spacelike, timelike or lightlike (null), if all velocity vectors $\gamma'(s)$ are spacelike, timelike or lightlike (null), respectively.

Let γ be a spacelike curve parametrized by arclength in a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold $\tilde{M}$. Using the Frenet formulas given in (2.15) and equation (2.14), we get

$$\begin{aligned} \nabla_T T &= \kappa_1 N, \\ \nabla_T \nabla_T T &= -\kappa_1^2 T + \kappa_1' N + \kappa_1 \kappa_2 B_1, \\ \nabla_T \nabla_T \nabla_T T &= -3\kappa_1 \kappa_1' T \\ &\quad +(\kappa_1'' - \kappa_1^3 + \kappa_1 \kappa_2^2) N \\ &\quad +(2\kappa_1' \kappa_2 + \kappa_1 \kappa_2') B_1 \\ &\quad +(\kappa_1 \kappa_2 \kappa_3) B_2, \\ R(T, \nabla_T T) T &= -\kappa_1 N, \end{aligned} \tag{3.1}$$

where κ_1, κ_2 and κ_3 are the first, the second and the third curvature of the γ, respectively.

Proposition 3.2. *Let $\gamma : I \to (\tilde{M}, h)$ be a curve in a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold $\tilde{M}$, parametrized by its arclength, and $\gamma' = T$. Then γ is a bi-f-harmonic curve if and only if*

$$\begin{aligned} 0 &= (ff''' + f'f'')T \\ &\quad +(3ff'' + 2(f')^2)\nabla_T^N T \\ &\quad +4ff'\nabla_T^2 T + f^2 \nabla_T^3 T \\ &\quad +f^2 R^N\left(\nabla_T^N T, T\right) T, \end{aligned} \tag{3.2}$$

where $f : I \to (0, \infty)$ is a smooth map, $\nabla_T^2 T =: \nabla_T^N \nabla_T^N T$ and $\nabla_T^3 T =: \nabla_T^N \nabla_T^N \nabla_T^N T$.

In this case, using (3.1) in (3.2), we have following theorem.

Theorem 3.3. *Let $\gamma : I \to \tilde{M}$ be a spacelike curve parametrized by arclength with timelike first binormal B_1 and M be a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold. Suppose that $\{T, N, B_1, B_2\}$ be an orthonormal Frenet frame field along γ. Then γ is a bi-f-harmonic curve if and only if*

$$\begin{aligned} 0 &= \begin{pmatrix} -3k_1k_1'f^2 - 4k_1^2 f f' \\ +f f''' + f'f'' \end{pmatrix} T \\ &\quad + \begin{pmatrix} 3k_1 ff'' + 2k_1(f')^2 + 4k_1' ff' \\ +k_1'' f^2 - k_1^3 f^2 + k_1 k_2^2 f^2 + f^2 k_1 \end{pmatrix} N \\ &\quad + \left(\begin{pmatrix} 2k_1' k_2 f + k_1 k_2' f \\ +4k_1 k_2 f' \end{pmatrix} f \right) B_1 \\ &\quad + \left(k_1 k_2 k_3 f^2\right) B_2, \end{aligned} \tag{3.3}$$

which yields

$$\left\{ \begin{array}{c} -3k_1k_1'f^2 - 4k_1^2 f f' + f f''' + f'f'' = 0, \\ -k_1^3 f^2 + k_1 k_2^2 f^2 + k_1'' f^2 + 4k_1' ff' \\ +3k_1 ff'' + 2k_1(f')^2 + k_1 f^2 = 0, \\ 2k_1' k_2 f + k_1 k_2' f + 4k_1 k_2 f' = 0, \\ k_1 k_2 k_3 = 0. \end{array} \right. \tag{3.4}$$

CASE I: If $k_1 = 0$, namely γ is a geodesic curve, then from (3.4) we obtain that it is bi-f-harmonic if and only if $ff'' = constant$.

Theorem 3.4. *A geodesic curve is bi-f-harmonic if and only if $ff'' = constant$.*

CASE II: If $k_1 = constant \neq 0$ and $k_2 = 0$, then (3.4) reduces to

$$\begin{cases} -4k_1^2 ff' + ff''' + f'f'' = 0, \\ -k_1^2 f^2 + 3ff'' + 2(f')^2 + f^2 = 0. \end{cases} \tag{3.5}$$

From the second equation above we obtain

$$ff'' = \frac{(k_1^2 - 1)f^2 - 2(f')^2}{3}, \tag{3.6}$$

which implies

$$f'\left((5k_1^2 + 1)f + 2f''\right) = 0, \tag{3.7}$$

via the first equation of (3.5) and we get

Theorem 3.5. *Let $\gamma : I \to \tilde{M}$ be a spacelike curve parametrized by arclength with timelike first binormal B_1 and $\tilde{M}$ be a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold, with $k_1 = constant \neq 0$ and $k_2 = 0$. Then γ is a bi-f-harmonic curve if and only if either f is a constant function or f is given by*

$$f(s) = c_1 \cos\left(\sqrt{\frac{5k_1^2 + 1}{2}} s\right) + c_2 \sin\left(\sqrt{\frac{5k_1^2 + 1}{2}} s\right),$$

for $s \in I$ and $c_1, c_2 \in \mathbb{R}$.

CASE III: If $k_1 = constant \neq 0$ and $k_2 = constant \neq 0$, then (3.4) reduces to

$$\begin{cases} -4k_1^2 ff' + ff''' + f'f'' = 0, \\ -k_1^2 f^2 + k_2^2 f^2 + 3ff'' + 2(f')^2 + f^2 = 0, \\ f' = 0, \\ k_3 = 0, \end{cases} \tag{3.8}$$

which implies

$$\begin{cases} k_1^2 - k_2^2 = 1, \\ f' = 0, \\ k_3 = 0, \end{cases} \tag{3.9}$$

and we deduce

Theorem 3.6. *Let $\gamma : I \to \tilde{M}$ be a spacelike curve parametrized by arclength with timelike first binormal B_1 and $\tilde{M}$ be a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold, with $k_1 = constant \neq 0$ and $k_2 = constant \neq 0$. Then γ is a bi-f-harmonic curve if and only if γ is a helix with $k_1^2 - k_2^2 = 1$.*

CASE IV: If $k_1 = constant \neq 0$ and $k_2 \neq constant$, then (3.4) reduces to

$$\begin{cases} -4k_1^2 ff' + ff''' + f'f'' = 0, \\ -k_1^2 f^2 + k_2^2 f^2 + 3ff'' \\ +2(f')^2 + f^2 = 0, \\ k_2' f + 4k_2 f' = 0, \\ k_2 k_3 = 0, \end{cases} \tag{3.10}$$

and we have

Theorem 3.7. *Let $\gamma : I \to \tilde{M}$ be a spacelike curve parametrized by arclength and M be a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold, with $k_1 = constant \neq 0$ and $k_2 \neq constant$ and nowhere zero. Then γ is a bi-f-harmonic curve if and only if $f = ck_2^{-\frac{1}{4}}$ (with c a positive constant), $k_3 = 0$ and the curvatures k_1 and k_2 satisfy:*

$$\begin{cases} 32k_1^2k_2^2k_2' - 25(k_2')^3 + 32k_2k_2'k_2'' - 8k_2^2k_2''' = 0, \\ -16k_1^2k_2^2 + 16k_2^4 + 16k_2^2 + 17(k_2')^2 - 12k_2k_2'' = 0. \end{cases} \tag{3.11}$$

CASE V: If $k_1 \neq constant$ and $k_2 = 0$, we can state

Theorem 3.8. *Let $\gamma : I \to \tilde{M}$ be a spacelike curve parametrized by arclength with timelike first binormal B_1 and $\tilde{M}$ be a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold, with $k_1 \neq constant$ and $k_2 = 0$. Then γ is a bi-f-harmonic curve if and only if the curvatures k_1 and k_2 satisfy:*

$$\begin{cases} -3k_1k_1'f^2 - 4k_1^2 f f' + f f''' + f'f'' = 0, \\ -k_1^3 f^2 + k_1'' f^2 + 4k_1' f f' + 3k_1 f f'' \\ +2k_1(f')^2 + k_1 f^2 = 0. \end{cases} \tag{3.12}$$

CASE VI: If $k_1 \neq constant$ and $k_2 = constant \neq 0$, then (3.4) reduces to

$$\begin{cases} -3k_1k_1'f^2 - 4k_1^2ff' + ff''' + f'f'' = 0, \\ -k_1^3f^2 + k_1k_2^2f^2 + k_1''f^2 + 4k_1'ff' \\ +3k_1ff'' + 2k_1(f')^2 = 0, \\ k_1'f + 2k_1f' = 0, \\ k_1k_3 = 0, \end{cases} \tag{3.13}$$

Theorem 3.9. *Let $\gamma : I \to \tilde{M}$ be a spacelike curve parametrized by arclength with timelike first binormal B_1 and $\tilde{M}$ be a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold, with $k_1 \neq constant$ and nowhere zero and $k_2 = constant \neq 0$. Then γ is a bi-f-harmonic curve if and only if $f = ck_1^{-\frac{1}{2}}$ (with c a positive constant), $k_3 = 0$ and the curvatures k_1 and k_2 satisfy:*

$$\begin{cases} -9(k_1')^3 - 4k_1^4k_1' + 10k_1k_1'k_1'' - 2k_1^2k_1''' = 0, \\ 3(k_1')^2 - 4k_1^4 + 4k_1^2 + 4k_1^2k_2^2 - 2k_1k_1'' = 0. \end{cases} \tag{3.14}$$

CASE VII: Concerning the case $k_1 \neq constant$ and $k_2 \neq constant$, we can state

Theorem 3.10. *Let $\gamma : I \to \tilde{M}$ be a spacelike curve parametrized by arclength with timelike first binormal B_1 and $\tilde{M}$ be a 4-dimensional conformally flat, quasi-conformally flat and conformally symmetric Lorentzian para-Sasakian manifold, with $k_1 \neq constant$ and $k_2 \neq constant$ and k_1, k_2 are nowhere zero. Then γ is a bi-f-harmonic curve if and only if $f = ck_1^{-\frac{1}{2}}k_2^{-\frac{1}{4}}$ (with c a positive constant), $k_3 = 0$ and the curvatures k_1 and k_2 satisfy:*

$$\begin{cases} -3k_1k_1'f^2 - 4k_1^2ff' + ff''' + f'f'' = 0, \\ -k_1^3f^2 + k_1k_2^2f^2 + k_1''f^2 + k_1f^2 \\ +4k_1'ff' + 3k_1ff'' + 2k_1(f')^2 = 0. \end{cases} \tag{3.15}$$

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