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The Necessary Conditions For A (k, μ)-Paracontact Space To Be η-$EINSTEIN$ Manifold

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Abstract The object of this paper is to study the curvature tensors of (k,μ)-Paracontact manifold satisfying the conditions $R(X,Y)\cdot C=0$, $\widetilde{Z}(X,Y)\cdot C=0$ and $C(X,Y)\cdot C=0$. According these cases, (k,μ)-Paracontact manifolds have been characterized such as Einstein and η-Einstein.

1 Introduction

The study of paracontact geometry was iniated by Kaneyuki and Williams [7]. Recently, there seems to be an increasing interest in paracontact geometry. A systematic study of paracontact metric manifolds and their subclasses introduced by Zamkovoy [15]. Subsequently, many geometers have studied paracontact metric manifolds and obtained various important properties of these manifolds (see, [3, 13, 14]).

Paracontact metric manifolds have been studied from different points of view. The geometry of paracontact metric manifolds can be related to the theory of Legendre foliations. In [5], the author introduced the class of paracontact metric manifolds for which the characteristic vektor field ξ belongs to the $(k,\mu)-$nullity condition for some real constants k and μ. Such manifolds are known as $(k,\mu)-$paracontact metric manifolds. The class of $(k,\mu)-$paracontact metric manifolds contains para-Sasakian manifolds.

The notion of semi-symmetric manifolds is defined by $R(X,Y)\cdot R=0$ and such works have been studied by many authors [8, 9, 10]. The conditions $R(X,Y)\cdot P=0$ and $R(X,Y)\cdot\widetilde{C}=0$ are said to be projective semi-symmetric and quasi-conformal semi-symmetric, respectively, where $R(X,Y)$ is considered as derivation of tensor algebra at each point of the manifold.

Yano K. and Sawaki S. introduced the notion of quasi-conformal curvature tensor which is generalization of conformal curvature tensor[12]. It plays an important role in differential geometry as well as in theory of relativity. Atçeken M. studied generalized Sasakian space form satisfying certain conditions on the concircular curvature tensor [2]. De U.C., Jun J.B. and Gazi A.K. searched Sasakian manifolds with quasi-conformal curvature tensor [6]. Arslan K., Murathan C. and Özgür C. produced the works on contact manifold curvature tensor [1].

Motivated by the studies of the above authors, in this paper we classify (k,μ)-paracontact manifolds, which satify the curvature conditions $R(X,Y)\cdot C=0$, $\widetilde{Z}(X,Y)\cdot C=0$ and $C(X,Y)\cdot C=0$ where $\widetilde{Z}$ is the concircular curvature tensor, R is the Riemannian curvature tensor and C is conformal curvature tensor.

2 Preliminaries

A contact manifold is a $C^\infty-(2n+1)$ dimensional manifold M^{2n+1} equipped with a global 1-form η such that $\eta\wedge(d\eta)^n\neq 0$ everywhere on M^{2n+1}. Given such a form η, it is well known that there exists a unique vector field ξ, called the characteristic vector field, such that $\eta(\xi) = 1$ and $d\eta(X, \xi) = 0$ for every vector field X on M^{2n+1}. A Riemannian metric g is said to be associated metric if there exists a tensor field ϕ of type $(1, 1)$ such that

$$\phi^2 X = X - \eta(X)\xi, \quad \eta(\xi) = 1, \quad \eta \circ \phi = 0, \quad \phi\xi = 0, \tag{2.1}$$

$$g(\phi X, \phi Y) = -g(X, Y) + \eta(X)\eta(Y), \quad g(X, \xi) = \eta(X) \tag{2.2}$$

for all vector fields X,Y on M. Then the structure (ϕ, ξ, η, g) on M is called a paracontact metric structure and the manifold equipped with such a structure is called a almost paracontact metric manifold[15].

We now define a $(1, 1)$ tensor field h by $h = \frac{1}{2} L_\xi \phi$, where L denotes the Lie derivative. Then h is symmetric and satisfies the conditions

$$h\phi = -\phi h, \quad h\xi = 0, \quad Tr.h = Tr.\phi h = 0. \tag{2.3}$$

If ∇ denotes the Levi-Civita connection of g, then we have the following relation

$$\nabla_X \xi = -\phi X + \phi h X \tag{2.4}$$

for all $X \in \chi(M)$[15]. For a para-contact metric manifold $M^{2n+1}(\phi, \xi, \eta, g)$, if ξ is a killing vector field or equivalently, $h = 0$, then it is called a K-paracontact manifold.

A para-contact metric structure (ϕ, ξ, η, g) is normal, that is, satisfies $[\phi, \phi] + 2d\eta \otimes \xi = 0$, which is equivalent to

$$(\nabla_X \phi)Y = -g(X, Y)\xi + \eta(Y)X$$

for any $X, Y \in \chi(M)$[15]. If an almost paracontact metric manifold is normal, then it called paracontact metric manifold. Any para -Sasakian manifold is K-paracontact, and the converse holds when $n = 1$, that is, for 3-dimensional spaces. Any para-Sasakian manifold satisfies

$$R(X, Y)\xi = -(\eta(Y)X - \eta(X)Y) \tag{2.5}$$

for any $X, Y \in \chi(M)$, but this is not a sufficient condition for a paracontact manifold to be para-Sasakian. It is clear that every para-Sasakian manifold is K-paracontact. But the converse is not always true[4].

Definition 2.1. A paracontact manifold M is said to be η-Einstein if its Ricci tensor S of type $(0, 2)$ is of the from $S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$,where a, b are smooth functions on M. If $b = 0$, then the manifold is also called Einstein[17].

Definition 2.2. A paracontact metric manifold is said to be a $(k, \mu)-$paracontact manifold if the curvature tensor R satisfies

$$\widetilde{R}(X, Y)\xi = k\,[\eta(Y)X - \eta(X)Y] + \mu\,[\eta(Y)hX - \eta(X)hY] \tag{2.6}$$

for all $X, Y \in \chi(M)$, where k and μ are real constants.

This class is very wide containing the para-Sasakian manifolds as well as the paracontact metric manifolds satisfying $R(X, Y)\xi = 0$ [16].

In particular, if $\mu = 0$, then the paracontact metric $(k, \mu)-$manifold is called paracontact metric $N(k)$-manifold . Thus for a paracontact metric $N(k)$-manifold the curvature tensor satisfies the following relation

$$R(X, Y)\xi = k(\eta(Y)X - \eta(X)Y) \tag{2.7}$$

for all $X, Y \in \chi(M)$. Though the geometric behavior of paracontact metric $(k, \mu)-$spaces is different according as $k < -1$, or $k > -1$, but there are also some common results for $k < -1$ and $k > -1$.

Lemma 2.3. *There does not exist any paracontact $(k, \mu)-$manifold of dimension greater than 3 with $k > -1$ which is Einstein whereas there exits such manifolds for $k < -1$ [5].*

In a paracontact metric $(k,\mu)-$manifold $(M^{2n+1}\phi,\xi,\eta,g)$, $n>1$, the following relation hold:

$$h^2=(k+1)\phi^2,\ \text{ for } k\neq -1, \tag{2.8}$$

$$(\widetilde{\nabla}_X\phi)Y=-g(X-hX,Y)\xi+\eta(Y)(X-hX), \tag{2.9}$$

$$\begin{aligned} S(X,Y) &= [2(1-n)+n\mu]g(X,Y)+[2(n-1)+\mu]g(hX,Y) \\ &\quad +[2(n-1)+n(2k-\mu)]\eta(X)\eta(Y), \end{aligned} \tag{2.10}$$

$$S(X,\xi)=2nk\eta(X), \tag{2.11}$$

$$\begin{aligned} QY &= [2(1-n)+n\mu]Y+[2(n-1)+\mu]hY \\ &\quad +[2(n-1)+n(2k-\mu)]\eta(Y)\xi, \end{aligned} \tag{2.12}$$

$$Q\xi=2nk\xi,\ \ g(QX,Y)=S(X,Y), \tag{2.13}$$

$$Q\phi-\phi Q=2[2(n-1)+\mu]h\phi \tag{2.14}$$

for any vector fields X,Y on M^{2n+1} , where Q and S denotes the Ricci operator and Ricci tensor of (M^{2n+1},g), respectively[5].

The concept of quasi-conformal curvature tensor was defined by K. Yano and S. Sawaki [12]. Quasi-conformal curvature tensor of a $(2n+1)$-dimensional Riemanian manifold is defined as

$$\begin{aligned} \widetilde{C}(X,Y)Z &= aR(X,Y)Z+b\{S(Y,Z)X-S(X,Z)Y \\ &\quad +g(Y,Z)QX-g(X,Z)QY\} \\ &\quad -\frac{\tau}{2n+1}\{\frac{a}{2n}+2b\}\{g(Y,Z)X-g(X,Z)Y\} \end{aligned} \tag{2.15}$$

where a and b are arbitrary scalars, and r is the scalar curvature of the manifold. If $a=1$ and $b=\frac{-1}{2n-1}$, then quasi conformal curvature tensor reduces to conformal curvature tensor defined as

$$\begin{aligned} C(X,Y)Z &= R(X,Y)Z-\frac{1}{2n-1}\{S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX \\ &\quad - g(X,Z)QY\}+\frac{\tau}{2n(2n-1)}\{g(Y,Z)X-g(X,Z)Y\}. \end{aligned} \tag{2.16}$$

Let (M,g) be an $(2n+1)$-dimensional Riemanian manifold. Then the concircular curvature tensor $\widetilde{Z}$ is defined by

$$\widetilde{Z}(X,Y)Z=R(X,Y)Z-\frac{\tau}{2n(2n+1)}\{g(Y,Z)X-g(X,Z)Y\}, \tag{2.17}$$

for all $X,Y,Z\in\chi(M)$[11].

3 η-Einstein $(k,\mu)-$Paracontact Spaces

In this section, we will give the main results for this paper.

Let M be $(2n+1)-$dimensional $(k,\mu)-$paracontact metric manifold and we denote the Riemannian curvature tensor of R, from (2.6), we have for later

$$R(\xi,Y)Z = k(g(Y,Z)\xi - \eta(Z)Y) + \mu(g(hY,Z)\xi - \eta(Z)hY). \tag{3.1}$$

In (3.1), choosing $Z=\xi$, we obtain

$$R(\xi,Y)\xi = k(\eta(Y)\xi - Y) + \mu hY. \tag{3.2}$$

Also from (3.1), we have

$$\eta(R(\xi,Y)Z) = k(g(Y,Z) - \eta(Y)\eta(Z)) + \mu g(hY,Z). \tag{3.3}$$

In the same way, choosing $X=\xi$ in (2.16) and (3.1), we have

$$\begin{aligned} C(\xi,Y)Z &= (k - \frac{2nk}{2n+1} + \frac{r}{2n(2n-1)})(g(Y,Z)\xi - \eta(Z)Y) \\ &\quad +\mu(g(hY,Z)\xi - \eta(Z)hY) - \frac{1}{2n-1}(S(Y,Z)\xi - \eta(Z)QY). \end{aligned} \tag{3.4}$$

In (3.4), choosing $Z=\xi$ and using (2.11), we obtain

$$\begin{aligned} C(\xi,Y)\xi &= (k - \frac{2nk}{2n+1} + \frac{r}{2n(2n-1)})(\eta(Y)\xi - Y) \\ &\quad -\mu hY - \frac{1}{2n-1}(2nk\eta(Y))\xi - QY). \end{aligned} \tag{3.5}$$

In same way from (3.1) and (2.17), we get

$$\widetilde{Z}(\xi,Y)Z = (k - \frac{r}{2n(2n+1)})(g(Y,Z)\xi - \eta(Z)Y) + \mu(g(hY,Z)\xi - \eta(Z)hY) \tag{3.6}$$

from which

$$\widetilde{Z}(\xi,Y)\xi = (k - \frac{r}{2n(2n+1)})(\eta(Y)\xi - Y) - \mu hY. \tag{3.7}$$

Theorem 3.1. *Let M be a $(2n+1)-$dimensional (k,μ)-paracontact manifold. Then $C(X,Y)\cdot C=0$ if and only if M is an $\eta-$Einstein manifold.*

Proof. Suppose that $C(X,Y)\cdot C=0$. This implies that

$$\begin{aligned} (C(X,Y)C)(U,W)Z &= C(X,Y)C(U,W)Z - C(C(X,Y)U,W)Z \\ &\quad -C(U,C(X,Y)W)Z - C(U,W)C(X,Y)Z = 0, \end{aligned} \tag{3.8}$$

for any $X,Y,U,W,Z \in \chi(M)$. Taking $X=Z=\xi$ in (3.8) and making use of (3.4), (3.5), for $A = k - \frac{2nk}{2n-1} + \frac{r}{2n(2n-1)}$, and $b = -\frac{1}{2n-1}$, we have

$$\begin{aligned} (C(\xi,Y)C)(U,W)\xi &= C(\xi,Y)(A(\eta(W)U - \eta(U)W) + \mu(\eta(W)hU - \eta(U)hW) \\ &\quad +b(\eta(W)QU - \eta(U)QW)) - C(A(g(Y,U)\xi - \eta(U)Y) \\ &\quad +\mu(g(hY,U)\xi - \eta(U)hY) + b(S(Y,U)\xi - \eta(U)QY),W))\xi \\ &\quad -C(U,A(g(Y,W)\xi - \eta(W)Y) + \mu(g(hY,W)\xi - \eta(W)hY) \\ &\quad +b(S(Y,W)\xi - \eta(W)QY))\xi - C(U,W)(A(\eta(Y)\xi - Y) \\ &\quad -\mu hY + b(2nk\eta(Y)\xi - QY)) = 0 \end{aligned} \tag{3.9}$$

Taking into account (2.12), (3.4) and inner product both sides of (3.9) by $Z \in \chi(M)$, we obtain

$$
\begin{aligned}
&Ag(C(U,W)Y,Z) + \mu g(C(U,W)hY,Z) + bg(C(U,W)QY,Z) \\
&+Ab(\eta(W)\eta(Z)S(Y,U) - \eta(U)\eta(Z)S(Y,W)) + A^2(g(U,Y)g(W,Z) \\
&-\eta(U)\eta(Z)g(Y,W)) + A\mu(\eta(W)\eta(Z)g(Y,hU) - \eta(U)\eta(Z)g(Y,hW)) \\
&+\mu^2(k+1)(\eta(W)\eta(Z)g(Y,U) - \eta(U)\eta(Z)g(Y,W)) + b\mu(\eta(W)\eta(Z)S(Y,hU) \\
&-\eta(U)\eta(Z)S(Y,hW)) + b^2(\eta(W)\eta(Z)S(Y,QU) - \eta(U)\eta(Z)S(Y,QW)) \\
&+A\mu(g(Y,U)g(hW,Z) - g(Y,W)g(hU,Z)) + 2nkAb(\eta(U)\eta(Z)g(Y,W) \\
&-\eta(W)\eta(Z)g(Y,U)) + Ab(g(Y,U)S(W,Z) - g(Y,W)S(U,Z)) \\
&+A\mu(g(hY,U)g(W,Z) - g(hY,W)g(U,Z)) + \mu^2(g(hY,U)g(hW,Z) \\
&-g(hY,W)g(hU,Z)) + 2nkb\mu(\eta(U)\eta(Z)g(Y,hW) - \eta(W)\eta(Z)g(Y,hU)) \\
&+b\mu(g(hY,U)S(W,Z) - g(hY,W)S(U,Z)) + Ab(S(Y,U)g(W,Z) \\
&-S(Y,W)g(U,Z)) + b\mu(S(Y,U)g(hW,Z) - S(Y,W)g(hU,Z)) \\
&+2nkb^2(\eta(U)\eta(Z)S(Y,W) - \eta(W)\eta(Z)S(Y,U)) \\
&+b^2(S(Y,U)S(W,Z) - S(Y,W)S(U,Z)) = 0. \qquad (3.10)
\end{aligned}
$$

In (3.10), using (2.1), (2.8), (2.16) and choosing $W = Y = e_i, \xi$, $1 \leq i \leq n$, for orthonormal basis of $\chi(M)$, we arrive

$$
\begin{aligned}
&(A + b[2(1-n) + n\mu] - \frac{br}{2n(2n+1)})S(U,Z) + (\mu + b[2(n-1) + \mu] - b^2)S(U,hZ) \\
&+(\frac{Ar}{2n-1} + 2n\mu b(k+1)b[2(n-1) + \mu] + \frac{br^2}{2n(2n-1)} + A^2 + \mu^2(k+1) \\
&+b^2r[2(1-n) + n\mu] + 2n(1+k) + b^2[2(n-1) + \mu]^2 \\
&+2nkb^2[2(n-1) + n(2k-\mu)]])g(U,Z) \\
&+(b\mu[2(n-1) + n(2k-\mu)] - \frac{\mu r}{2n(2n-1)} - 2nA\mu - \mu br)g(U,hZ) \\
&+(-A^2(2n+1) - 2n\mu^2(1+k) - 4nb\mu(1+k)[2(n-1) + \mu] - Abr \\
&-(2nkb)^2 + 2nkAb(2n+1) + 2nkb^2(r + [2(1-n) + \mu]) - \mu^2(k+1) \\
&-b^2r[2(1-n) + \mu] - 2nb^2(1+k)[2(n-1) + \mu]^2)\eta(U)\eta(Z) = 0. \qquad (3.11)
\end{aligned}
$$

Replacing hZ of Z in (3.11) and using (2.8), we get

$$
\begin{aligned}
&(A + b[2(1-n) + n\mu] - \frac{br}{2n(2n+1)})S(U,hZ) \\
&+(1+k)(\mu + b[2(n-1) + \mu] - b^2)S(U,Z) \\
&-2nk(1+k)(\mu + b[2(n-1) + \mu] - b^2)\eta(U)\eta(Z) \\
&+(\frac{Ar}{2n-1} + 2n\mu b(k+1)b[2(n-1) + \mu] + \frac{br^2}{2n(2n-1)} + A^2 \\
&+\mu^2(k+1) + b^2r[2(1-n) + n\mu] + 2n(1+k) + b^2[2(n-1) + \mu]^2 \\
&+2nkb^2[2(n-1) + n(2k-\mu)]])g(U,hZ) \\
&+(1+k)(b\mu[2(n-1) + n(2k-\mu)] - \frac{\mu r}{2n(2n-1)} - 2nA\mu - \mu br)g(U,Z) \\
&-(1+k)(b\mu[2(n-1) + n(2k-\mu)] \\
&-\frac{\mu r}{2n(2n-1)} - 2nA\mu - \mu br)\eta(U)\eta(Z) = 0. \qquad (3.12)
\end{aligned}
$$

From (3.11), (3.12) and also using (2.10), for the sake of brevity, we put

$$\begin{aligned}
a &= (A + b[2(1-n) + n\mu] - \frac{br}{2n(2n+1)}),\\
f &= (\mu + b[2(n-1) + \mu] - b^2),\\
c &= (\frac{Ar}{2n-1} + 2n\mu b(k+1)b[2(n-1)+\mu] + \frac{br^2}{2n(2n-1)} + A^2 + \mu^2(k+1)\\
&\quad + b^2 r[2(1-n)+n\mu] + 2n(1+k) + b^2[2(n-1)+\mu]^2\\
&\quad + 2nkb^2[2(n-1)+n(2k-\mu)]),\\
d &= (b\mu[2(n-1)+n(2k-\mu)] - \frac{\mu r}{2n(2n-1)} - 2nA\mu - \mu br),
\end{aligned}$$

$$\begin{aligned}
e &= (-A^2(2n+1) - 2n\mu^2(1+k) - 4nb\mu(1+k)[2(n-1)+\mu] - Abr\\
&\quad -\mu^2(k+1) - (2nkb)^2 + 2nkAb(2n+1) + 2nkb^2(r + [2(1-n)+\mu])\\
&\quad -b^2 r[2(1-n)+\mu] - 2nb^2(1+k)[2(n-1)+\mu]^2)
\end{aligned}$$

and

$$\begin{aligned}
E &= fd(1+k) - ac) - [2(n-1)+\mu] - (fc - ad)[2(1-n)+n\mu],\\
D &= (a^2 - f^2)[2(n-1)+\mu] - fc + ad,\\
F &= (ad - fc)[2(n-1)+n(2k-\mu)] - (ae + fd(1+k) + 2nkf^2)[2(n-1)+\mu]
\end{aligned}$$

we conclude

$$DS(U,Z) = Eg(U,Z) + F\eta(U)\eta(Z).$$

So, M is an $\eta-$Einstein manifold. The converse is obvious. □

Theorem 3.2. *Let M be a $(2n+1)-$dimensional (k,μ)-paracontact manifold. Then $\widetilde{Z}(X,Y)\cdot C = 0$ if and only if M is an Einstein manifold.*

Proof. Suppose that $\widetilde{Z}(X,Y)\cdot C = 0$. Then we have

$$\begin{aligned}
(\widetilde{Z}(X,Y)C)(U,W)Z &= \widetilde{Z}(X,Y)C(U,W)Z - C(\widetilde{Z}(X,Y)U,W)Z\\
&\quad -C(U,\widetilde{Z}(X,Y)W)Z - C(U,W)\widetilde{Z}(X,Y)Z = 0, \qquad (3.13)
\end{aligned}$$

for any $X,Y,U,W,Z \in \chi(M)$. Taking $X = Z = \xi$ in (3.13) and using (3.4), for $A = k - \frac{2nk}{2n-1} + \frac{r}{2n(2n-1)}$, $B = k - \frac{r}{2n(2n+1)}$ and $b = -\frac{1}{2n-1}$, we obtain

$$\begin{aligned}
(\widetilde{Z}(\xi,Y)C)(U,W)\xi &= \widetilde{Z}(\xi,Y)(A(\eta(W)U - \eta(U)W) + \mu(\eta(W)hU - \eta(U)hW\\
&\quad +b(\eta(W)QU - \eta(U)QW)) - C(B(g(Y,U)\xi - \eta(U)Y)\\
&\quad +\mu(g(hY,U)\xi - \eta(U)hY),W)\xi - C(U,B(g(Y,W)\xi\\
&\quad -\eta(W)Y) + \mu(g(hY,W)\xi - \eta(W)hY))\xi\\
&\quad -C(U,W)(B(\eta(Y)\xi - Y) - \mu hY) = 0. \qquad (3.14)
\end{aligned}$$

Taking into account that (3.4), (3.6) and inner product both sides of (3.14) by $Z \in \chi(M)$, we get

$$
\begin{aligned}
&Bg(C(U,W)Y,Z)+\mu g(C(U,W)hY,Z)\\
&+\mu B(\eta(W)\eta(Z)g(Y,hU)-\eta(U)\eta(Z)g(Y,hW)\\
&+\mu^2(1+k)(\eta(W)\eta(Z)g(Y,U)-\eta(U)\eta(Z)g(Y,W))\\
&+Bb(\eta(W)\eta(Z)S(Y,U)-\eta(U)\eta(Z)S(Y,W))\\
&+\mu b(\eta(W)\eta(Z)S(hY,U)-\eta(U)\eta(Z)S(hY,U))\\
&+AB(g(Y,U)g(W,Z)-g(Y,W)g(U,Z))\\
&+\mu B(g(Y,U)g(hW,Z)-g(Y,W)g(hU,Z))\\
&+Bb(g(Y,U)S(W,Z)-g(Y,W)S(U,Z))\\
&+A\mu(g(hY,U)g(W,Z)-g(hY,W)g(U,Z))\\
&+2nkBb(\eta(U)\eta(Z)g(Y,W)-\eta(W)\eta(Z)g(Y,U))\\
&+\mu^2(g(hY,U)g(hW,Z)-g(hY,W)g(hU,Z))\\
&+b\mu(g(hY,U)S(W,Z)-g(hY,W)S(U,Z))\\
&+2nkb\mu(\eta(U)\eta(Z)g(hY,W)-\eta(W)\eta(Z)g(hY,U))=0. \qquad (3.15)
\end{aligned}
$$

In (3.15), using (2.1), (2.16) and choosing $U=Z=e_i$, ξ for orthonormal basis of $\chi(M)$, $1\leq i\leq n$, we arrive

$$BS(W,Y)+\mu S(W,hY)-2nkBg(W,Y)-2nk\mu g(W,hY)=0. \qquad (3.16)$$

Replacing hY of Y in (3.16) and making use of (2.8), we obtain

$$BS(W,hY)+\mu(1+k)S(W,Y)-2nkBg(W,hY)-2nk\mu(1+k)g(W,Y)=0. \qquad (3.17)$$

From (3.16), (3.17) and using (2.11), we have

$$S(W,Y)=2nkg(W,Y). \qquad (3.18)$$

Thus, M is an Einstein manifold. The converse is obvious. From (3.18), we conclude that

$$\mu=2(k+1-\frac{1}{n}).$$

□

Theorem 3.3. *Let M be a $(2n+1)-$dimensional (k,μ)-paracontact manifold. Then M is a conformal semi-symmetric if and only if M is an Einstein manifold.*

Proof. Suppose that $R(X,Y)\cdot C=0$. This means that

$$
\begin{aligned}
(R(X,Y)C)(U,W)Z &= R(X,Y)C(U,W)Z-C(R(X,Y)U,W)Z\\
&\quad -C(U,R(X,Y)W)Z-C(U,W)R(X,Y)Z=0, \qquad (3.19)
\end{aligned}
$$

for any $X,Y,U,W,Z\in\chi(M)$. Setting $X=Z=\xi$ in (3.19) and making use of (3.1), for $A=k-\frac{2nk}{2n-1}+\frac{r}{2n(2n-1)}$ and $b=-\frac{1}{2n-1}$, we obtain

$$
\begin{aligned}
(R(\xi,Y)C)(U,W)\xi &= R(\xi,Y)(A(\eta(W)U-\eta(U)W)+\mu(\eta(W)hU-\eta(U)hW)\\
&\quad +b(\eta(W)QU-\eta(U)QW))-C(k(g(Y,U)\xi-\eta(U)Y)\\
&\quad +\mu(g(hY,U)\xi-\eta(U)hY,W)\xi-C(U,k(g(Y,W)\xi\\
&\quad -\eta(W)Y)+\mu(g(hY,W)\xi-\eta(W)hY))\xi\\
&\quad -C(U,W)(k(\eta(Y)\xi-Y)-\mu hY)=0. \qquad (3.20)
\end{aligned}
$$

Inner product both sides of (3.20) by $Z \in \chi(M)$ and using of (3.1) and (3.4), we arrive

$$\begin{aligned}
&kg(C(U,W)Y,Z) + \mu g(C(U,W)hY,Z) \\
&+k\mu(\eta(W)\eta(Z)g(Y,hU) - \eta(U)\eta(Z)g(Y,hW)) \\
&+\mu^2(1+k)(\eta(W)\eta(Z)g(Y,U) - \eta(U)\eta(Z)g(Y,W)) \\
&+bk(\eta(W)\eta(Z)S(Y,U) - \eta(U)\eta(Z)S(Y,W)) \\
&+b\mu(\eta(W)\eta(Z)S(hY,U) - \eta(U)\eta(Z)S(hY,W)) \\
&+Ak(g(Y,U)g(W,Z) - g(Y,W)g(U,Z)) \\
&+k\mu(g(Y,U)g(hW,Z) - g(Y,W)g(hU,Z)) \\
&+bk(g(Y,U)S(W,Z) - g(Y,W)S(U,Z)) \\
&+b\mu(g(hY,U)S(W,Z) - g(hY,W)S(U,Z)) \\
&+\mu^2(g(hY,U)g(hW,Z) - g(hY,W)g(hU,Z)) \\
&+A\mu(g(hY,U)g(W,Z) - g(hY,W)g(U,Z)) \\
&+2nkb\mu(\eta(U)\eta(Z)g(hY,W) - \eta(W)\eta(Z)g(hY,U)) \\
&+2nk^2b(\eta(U)\eta(Z)g(Y,W) - \eta(W)\eta(Z)g(Y,U)) = 0. \qquad (3.21)
\end{aligned}$$

Making use of (2.8), (2.16) and choosing $U = Z = e_i,\ \xi,\ 1 \leq i \leq n$, for orthonormal basis of $\chi(M)$ in (3.21), we have

$$kS(W,Y) + \mu S(W,hY) - 2nk^2 g(Y,W) - 2nk\mu g(W,hY) = 0. \qquad (3.22)$$

Replacing hY of Y in (3.22) and taking into account (2.8), we get

$$kS(W,hY) + \mu(1+k)S(W,Y) - 2nk^2 g(hY,W) - 2nk\mu g(W,Y) = 0. \qquad (3.23)$$

From (3.22), (3.23) and by using (2.11), we have

$$S(Y,W) = 2nkg(Y,W).$$

Thus, M is an Einstein manifold. The converse is obvious. □

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References

[1] K. Arslan, C. Murathan and C. Özgür, On contact manifolds satisfying certain curvature conditions, an. Univ. Bucuresti Math. (2000), 49(2), 17-26.

[2] M. Atçeken , On generalized Sasakian space forms satisfying certain conditions on the concircular curvature tensor, Bulletin of Mathematical Analysis and Applications, 6(1)2014, 1-8.

[3] M. Atçeken and P. Uygun, Characterizations for totally geodesic submanifolds of (k, μ)-paracontact metric manifolds, Korcan J. Math. 28(2020), 555-571.

[4] G. Calvaruso, Homogeneous paracontact metric three-manifolds, Illinois J. Math., 55(2011), 697-718.

[5] B. Cappelletti-Montano, I. Küpeli Erken and C. Murathan, Nullity conditions in paracontact geometry, Differential Geom. Appl., 30(2012), 665-693.

[6] U.C. De, J.B. Jun and A.K. Gazi, Sasakian manifolds with quasi-conformal curvature tensor, Bull. Korean Math. Soc. 45(2008), No.2, pp. 313-319.

[7] S. Kaneyuki and F.L. Williams, Almost paracontact and parahodge structures on manifolds, Nagoya Math. J., 99(1985), 173-187.

[8] Z. I. Szabo, Classification and construction of complete hypersurfaces satisfying $R(X,Y)R = 0$, Acta. Sci. Math. (1984), 7, 321-348.

[9] Z.I. Szabo, Structure theorems on Riemannian spaces satisfying $R(X,Y)R = 0$, I Global version, Geom. Dedicate, (1983), 19, 65-108.

[10] Z.I. Szabo, Structure theorems on Riemannian spaces satisfying $R(X,Y)R = 0$, I The Local version, J. Diff. Geom. (1982), 17, 531-582.

[11] K. Yano and M. Kon, Structures manifolds, Singapore, World Scientific, (1984).

[12] K. Yano and S. Sawaki, Riemannian manifolds admitting a conformal transformation group, J. Diff. Geom. 2(1968), 161-184.

[13] Ü. Yıldırım, M. Atçeken and S. Dirik, A Normal Paracontact Metric Manifold Satisfying Some Conditions on the M-Projective Curvature Tensor. Konuralp Journal of Mathematics , vol: 7(1)(2019), 217-221.

[14] Ü. Yıldırım, M. Atçeken and S. Dirik, Pseudo projective curvature tensor satisfying some properties on a normal paracontact metric manifold, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. 68(2019),997-1006.

[15] S. Zamkovoy, Canonical connections on paracontact manifolds, Ann. Global Anal. Geom., 36(2009), 37-60.

[16] S.Zamkovoy and V. Tzanov, Non-existence of flat paracontact metric structures in dimension greater than or equal to five, Annuaire Univ. Sofia Fac. Math. Inform., 100(2011), 27-34.

[17] W.M. Boothby, An Introduction to Differentiable Manifolds and Riemanniann Geometry, Academic Press, Inc. London, (1986).

[18] Ü. Yıldırım, M. Atçeken and S. Dirik, On C-Bochner Curvature Tensor in (LCS)n-Manifolds, Hagia Sophia Journal of Geometry, (2019), Vol. 1, No. 2, 15-21.

[19] Ü. Yıldırım, M. Atçeken and S. Dirik, D-Conformal Curvature Tensor on (LCS)_n-Manifold, Turkish Journal of Mathematics and Computer Science, 10(2018) 215-221.

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