Palestine Journal of Mathematics

Vol. 11(2)(2022) , 439–448 © Palestine Polytechnic University-PPU 2022

ROMAN DOMINATION POLYNOMIAL OF PATHS

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Communicated by Ayman Badawi

MSC 2010 Classifications: 05C31, 05C69, 05C76.

Keywords and phrases: Domination polynomial of paths, Roman domination polynomial of graphs, Roman domination polynomial of paths.

Abstract A Roman dominating function on a graph $G = (V, E)$ is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value $W(f(V)) = \sum_{u \in V(G)} f(u)$. The minimum weight of a Roman dominating function on a graph G is called the Roman domination number of G and is denoted by $\gamma_R(G)$. In [8], we have introduced and established the study of the Roman domination polynomial of graphs and obtained some important properties about the polynomial and we have computed the polynomial for some specific graphs and graph operations. In this paper, as a continuing of this study, we study the Roman domination polynomial of a path P_n on n vertices. Exact formula for the polynomial, important properties of its coefficients and interesting results have obtained.

1 Introduction

Let $G = (V, E)$ be a simple graph, where V and E are the set of vertices and edges of G, respectively. The open neighborhood and the closed neighborhood of a vertex $v \in V(G)$ are defined by $N(v) = \{u \in V(G) : uv \in E\}$ and $N[v] = N(v) \cup \{v\}$, respectively. The cardinality of $N(v)$ is called the degree of the vertex v and denoted by $deg(v)$ in G. For more terminology and notations about graph, the reader is referred to [6, 9].

A subset D of $V(G)$ is a dominating set of G, if for every vertex $v \in V - D$, there exists a vertex $u \in D$ such that v is adjacent to u. A dominating set of G of cardinality $\gamma(G)$ is called the domination number of G. For more details about domination of graphs, we refer to [10].

A Roman dominating function of a graph $G = (V, E)$ (or in brief RDF of G) is a function $f : V(G) \to \{0, 1, 2\}$ satisfying the condition that every vertex u for which $f(u) = 0$ is adjacent to at least one vertex v for which $f(v) = 2$. The weight of a Roman dominating function is the value $W(f(V)) = \sum_{u \in V(G)} f(u)$. A Roman dominating function of a graph G with weight $\gamma_R(G)$ is called the Roman domination number of G. For more details about Roman domination and its properties, the reader is referred to [7]. The next proposition obtained the exact value of γ_R of a path P_n and a cycle C_n on n vertices.

Proposition 1.1 ([7]). *For the classes of paths P_n and cycles C_n,*

$$\gamma_R(P_n) = \gamma_R(C_n) = \left\lceil \frac{2n}{3} \right\rceil.$$

The domination polynomial $D(G, x)$ of a graph G is defined by $D(G, x) = \sum_{i=\gamma(G)}^{n} d(G, i)x^i$, where $d(G, i)$ is the number of all the dominating sets of G of size i [5]. The dominating sets and the domination polynomial of graphs have been studied extensively in [5, 3, 4, 2]. Recently, the injective domination polynomial of graphs has been studied in [1].

In [8], we have introduced the Roman domination polynomial of graph as $R(G,x) = \sum_{j=\gamma_R(G)}^{2n} r(G,j)x^j$, where $r(G,j)$ is the number of Roman dominating functions of G of weight j. We have established this study by obtaining some important properties of the polynomial and its coefficients, and determining the exact formula of the polynomial for some families of graphs and graph operations.

In the next proposition, we obtain some important properties of $R(G,x)$ of a graph G which we need to use in this paper.

Proposition 1.2 ([8]). *Let G be a non trivial graph on n vertices. Then*

(i) $R(G,x)$ *has no constant term.*

(ii) $R(G,x)$ *has no term of degree one.*

(iii) Zero is a root of $R(G,x)$*, with multiplicity* $\gamma_R(G)$.

(iv) $R(G,x)$ *never equal* x^p *for any* $2 \le p \le 2n$.

(v) For any graph G, $r(G,2n) = 1$ *and* $r(G,2n-1) = n$.

(vi) $r(G,j) = 0$ *if and only if* $j < \gamma_R(G)$ *or* $j > 2n$.

(vii) $R(G,x)$ *is a strictly increasing function in* $[0,\infty)$.

(viii) The only polynomial of degree two can $R(G,x)$ *be equal is* $x^2 + x$ *if and only if* $G \cong K_1$.

(ix) Let H be any induced subgraph of G. Then

$$deg\big(R(G,x)\big) \ge deg\big(R(H,x)\big).$$

In this paper, we study the Roman domination polynomial of a path P_n on n vertices. Exact formula for $R(P_n,x)$, important properties and relations between the coefficient of $R(P_n,x)$ are obtained.

2 Roman domination polynomial of a path

In [4], Alikhani and Peng have showed that the number of all dominating sets with cardinality i of a path P_n equal to the sum of the number of all dominating sets of the path P_{n-1} with cardinality $i-1$, the path P_{n-2} with cardinality $i-1$ and the path P_{n-3} with cardinality $i-1$, and then they have found the exact formula of the domination polynomial of paths, as following.

Theorem 2.1 ([4]).

(i) If $\mathcal{P}_n^i$ is the family of all dominating sets with cardinality i of a path P_n, then

$$|\mathcal{P}_n^i| = |\mathcal{P}_{n-1}^{i-1}| + |\mathcal{P}_{n-2}^{i-1}| + |\mathcal{P}_{n-3}^{i-1}|.$$

(ii) For every $n \ge 4$,

$$D(P_n,x) = x\big[D(P_{n-1},x) + D(P_{n-2},x) + D(P_{n-3},x)\big],$$

with initial values $D(P_1,x) = x$, $D(P_2,x) = x^2 + 2x$ *and* $D(P_3,x) = x^3 + 3x^2 + x$.

In this section, we find the Roman domination polynomial of a path P_n on n vertices, and then we study some of its properties, and finally, we illustrate in a table the coefficients of all Roman domination polynomials of paths P_n with $n \le 10$.

Let $\mathbb{P}_n^j$ be the set of all RDFs of P_n with weight j. Actually, to find a RDF of P_n, we do not need to consider RDFs of P_{n-4} with weight $j-2$ (weight $j-1$ is impossible here), that's what we will show in the next lemma. Note that, when we talk about a RDF f with weight $j-1$ or $j-2$ in P_{n-r}, where $r = 1,2,3$ such that $f \in \mathbb{P}_n^j$, we mean a RDF f of P_n minus only one vertex $v \in P_n \setminus P_{n-r}$ taking a value $f(v) = 1$ or $f(v) = 2$, respectively.

Lemma 2.2. *Let $f \in \mathbb{P}_n^j$. Then, if $f \in \mathbb{P}_{n-4}^{j-2}$, this implies that $f \in \mathbb{P}_{n-3}^{j-2}$.*

Proof. Suppose $f \in \mathbb{P}_{n-4}^{j-2}$. Then for the vertex $v \in V(P_{n-4})$ labeled $n-4$, either $f(v) = 0$ or 1 or 2. Now, if $f(v) = 0$ or 1, then $f \notin \mathbb{P}_n^j$, a contradiction. Therefore, $f(v) = 2$ and hence $f \in \mathbb{P}_{n-3}^{j-2}$. □

In the next theorem, according to Theorem 2.1 part (i) (since every RDF of a graph G it just a labeling on some dominating set of the graph G itself) and Lemma 2.2, we state the Roman domination polynomial of P_n in terms of the Roman domination polynomial of P_{n-1}, P_{n-2} and P_{n-3}.

Theorem 2.3. *Let P_n be a path on $n \geq 4$ vertices. Then*

$$R(P_n, x) = (x^2 + x)R(P_{n-1}, x) + x^2 R(P_{n-2}, x) + (x^3 + x^2)R(P_{n-3}, x),$$

with initial values $R(P_3, x) = x^6 + 3x^5 + 6x^4 + 5x^3 + x^2$, $R(P_2, x) = x^4 + 2x^3 + 3x^2$ and $R(P_1, x) = x^2 + x$.

Proof. Consider $V(P_n) = \{v_1, v_2, \ldots, v_n\}$. Let $f \in \mathbb{P}_n^j$. Then we have the following cases:
Case 1. Suppose that $f \in \mathbb{P}_{n-1}^{j-1}$ or $f \in \mathbb{P}_{n-1}^{j-2}$ (this means that, for the last vertex v_n either $f(v_n) = 1$ or $f(v_n) = 2$, respectively). Then we get the term $(x^2 + x)R(P_{n-1}, x)$.
Case 2. Suppose that $f \in \mathbb{P}_{n-2}^{j-1}$ or $f \in \mathbb{P}_{n-2}^{j-2}$. Actually, the case $f \in \mathbb{P}_{n-2}^{j-1}$ is included in Case 1, so we will take only $f \in \mathbb{P}_{n-2}^{j-2}$ in this case. Thus, we have two subcases:
Subcase i. Suppose $f(v_{n-1}) = 2$ and $f(v_n) = 0$. Then we get the term $x^2 R(P_{n-2}, x)$.
Subcase ii. Suppose $f(v_{n-1}) = 0$ and $f(v_n) = 2$. This situation has some connection with Case 1, so to avoid the repetition, we will take only the situations when $\mathbb{P}_{n-1}^{j-2} = \phi$. Therefore, we will choose $f(v_{n-2}) = 1$ or $f(v_{n-2}) = 0$.

- If $f(v_{n-2}) = 1$, then we have the term $x^3 R(P_{n-3}, x)$.
- If $f(v_{n-2}) = 0$, then $f(v_{n-3})$ must equal 2, and thus we have the term $x^4 R(P_{n-4}, x)$.

Case 3. In this case, we have remaining the situation when $f(v_{n-2}) = 0$, $f(v_{n-1}) = 2$ and $f(v_n) = 0$ such that $\mathbb{P}_{n-2}^{j-2} = \phi$ (here already $\mathbb{P}_{n-1}^{j-1} = \mathbb{P}_{n-1}^{j-2} = \phi$). Therefore, $f(v_{n-3})$ must equal 1 or 0. Hence, we get the term

$$x^2 R(P_{n-3}, x) - x^4 R(P_{n-4}, x),$$

and this completes the proof. □

Using Theorem 2.3, we obtain $r(P_n, j)$ for $1 \leq n \leq 10$ as shown in Table 1.

j	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
n																				
1	1	1																		
2	0	3	2	1																
3	0	1	5	6	3	1														
4	0	0	2	11	14	10	4	1												
5	0	0	0	6	23	34	28	15	5	1										
6	0	0	0	1	14	51	80	76	48	21	6	1								
7	0	0	0	0	3	34	113	189	198	144	75	28	7	1						
8	0	0	0	0	0	10	80	255	444	506	410	246	110	36	8	1				
9	0	0	0	0	0	1	28	189	579	1044	1272	1129	758	391	154	45	9	1		
10	0	0	0	0	0	0	4	76	444	1325	2454	3164	3030	2236	1294	589	208	55	10	1

Table 1. $r(P_n, j)$, the number of Roman dominating functions of P_n with cardinality j.

In the following theorem, we obtain some important properties about the coefficients of the Roman domination polynomial of a path P_n.

Theorem 2.4. *The following properties are satisfied for the Roman domination polynomial $R(P_n, x)$ of a path P_n:*

(i) $r(P_n, j) = r(P_{n-1}, j-1) + r(P_{n-1}, j-2) + r(P_{n-2}, j-2) + r(P_{n-3}, j-2) + r(P_{n-3}, j-3)$.

(ii) $r(P_{3k}, 2k) = 1$, *where* $n = 3k$ *for some* $k \in \mathbb{N}$.

(iii) If $n = 3k+1$ *for some* $k \in \mathbb{N}$, *then* $r(P_{3k+1}, 2k+1) = k+1$.

(iv) If $n = 3k+2$ *for some* $k \in \mathbb{N}$, *then* $r(P_{3k+2}, 2k+2) = \frac{(k+2)(k+3)}{2}$.

(v) If $n = 3k$ *for some* $k \in \mathbb{N}$, *then* $r(P_{3k}, 2k+1) = \frac{k(k+4)(k+5)}{6}$.

(vi) If $n = 3k+1$ *for some* $k \in \mathbb{N}$, *then* $r(P_{3k+1}, 2k+2) = 1 + \frac{k(k+2)(k+7)(k+9)}{24}$.

(vii) If $n = 3k+2$ *for some* $k \in \mathbb{N}$, *then*

$$r(P_{3k+2}, 2k+3) = 2 + \frac{k(k+2)(k+4)(k+11)(k+13)}{120}.$$

(viii) If $n = 3k$ *for some* $k \in \mathbb{N}$, *then*

$$r(P_{3k}, 2k+2) = \frac{k(k+7)(k^4 + 32k^3 + 281k^2 + 418k - 192)}{720}.$$

(ix) $r(P_n, 2n) = 1$.

(x) $r(P_n, 2n-1) = n$.

(xi) $r(P_n, 2n-2) = \frac{n(n+1)}{2}$.

(xii) $r(P_n, 2n-3) = \frac{(n^2-4)(n+3)}{6}$.

(xiii) $r(P_n, 2n-4) = \frac{n(n-1)(n^2+7n-6)}{24} - (3n-4)$.

(xiv) $r(P_n, 2n-5) = \frac{n(n^2-1)(n-2)(n+12)}{120} + 2 - 2n(n-2)$.

(xv) For every $k \in \mathbb{N}$,

$$1 = r(P_k, 2k) < r(P_{k+1}, 2k) < r(P_{k+2}, 2k) < \cdots < r(P_{2k}, 2k) > \cdots > r(P_{3k-1}, 2k) > r(P_{3k}, 2k) = 1.$$

(xvi) For every $k \in \mathbb{N}$,

$$k+1 = r(P_{k+1}, 2k+1) < r(P_{k+2}, 2k+1) < r(P_{k+3}, 2k+1) < \cdots < r(P_{2k+1}, 2k+1) > \cdots > r(P_{3k}, 2k+1) > r(P_{3k+1}, 2k+1) = k+1.$$

(xvii) If $\alpha_n = \sum_{j=\lceil \frac{2n}{3} \rceil}^{2n} r(P_n, j)$, *then for every* $n \geq 4$, $\alpha_n = 2\alpha_{n-1} + \alpha_{n-2} + 2\alpha_{n-3}$, *with initial values* $\alpha_1 = 2$, $\alpha_2 = 6$ *and* $\alpha_3 = 16$.

(xviii) For $j \geq 2$,

$$\sum_{i=j}^{3j} r(P_i, 2j) = \sum_{i=j}^{3j-2} r(P_i, 2j-1) + 3 \sum_{i=j-1}^{3j-3} r(P_i, 2j-2) + \sum_{i=j-1}^{3j-5} r(P_i, 2j-3).$$

(xix) For $j \geq 3$,

$$\sum_{i=j}^{3j-2} r(P_i, 2j-1) = \sum_{i=j-1}^{3j-3} r(P_i, 2j-2) + 3\sum_{i=j-1}^{3j-5} r(P_i, 2j-3) + \sum_{i=j-2}^{3j-6} r(P_i, 2j-4).$$

(xx) For every $k \in \mathbb{N}$ *and* $m = 0, 1, 2, \ldots, 2k-1$, $r(P_{2k-m}, 2k) = r(P_{2k+m}, 2k)$.

(xxi) For every $k \in \mathbb{N}$ *and* $m = 0, 1, 2, \ldots, 2k-1$,

$$r(P_{2k-m+1}, 2k+1) = r(P_{2k+m+1}, 2k+1).$$

Proof. Let P_n be a path on n vertices with $V(P_n) = \{v_1, v_2, \ldots, v_n\}$.

(i) The proof of this result is straightforward from Theorem 2.3.

(ii) Let $n = 3k$ for some $k \in \mathbb{N}$. Since $\mathcal{P}_{3k}^{k} = \big\{\{v_2, v_5, \ldots, v_{3k-4}, v_{3k-1}\}\big\}$, then we have only one RDF of P_n, in this case, such that each vertex taking the value 2. Hence, $r(P_{3k}, 2k) = 1$.

(iii) Proof by induction on k. If $k = 1$, then $r(P_4, 3) = 2$ (see Table 1). Therefore, the result is true for $k = 1$. Now, suppose the result is true for all natural numbers less than or equal $k-1$. We will prove that the result still true for k. By parts (i) and (ii), the induction hypothesis and Proposition 1.2 part (vi), we get.

$$\begin{aligned} r(P_{3k+1}, 2k+1) =& r(P_{3k}, 2k) + r(P_{3k}, 2k-1) + r(P_{3k-1}, 2k-1) \\ &+ r(P_{3k-2}, 2k-1) + r(P_{3k-2}, 2k-2) \\ =& 1 + 0 + 0 + r\big(P_{3(k-1)+1}, 2(k-1)+1\big) + 0 = k+1. \end{aligned}$$

(iv) By induction on k. If $k = 1$, then $r(P_5, 4) = 6 = \dfrac{(1+2)(1+3)}{2}$ (see Table 1). Suppose now the result is true for all natural numbers less than or equal $k-1$. Then by parts (i), (ii) and (iii) and Proposition 1.2 part (vi), we have.

$$\begin{aligned} r(P_{3k+2}, 2k+2) =& r(P_{3k+1}, 2k+1) + r(P_{3k+1}, 2k) + r(P_{3k}, 2k) \\ &+ r(P_{3k-1}, 2k) + r(P_{3k-1}, 2k-1) \\ =& k+1+0+1+r\big(P_{3(k-1)+2}, 2(k-1)+2\big) + 0 \\ =& k+2+\frac{(k+1)(k+2)}{2} = \frac{(k+2)(k+3)}{2}. \end{aligned}$$

(v) Proof by induction on k. If $k = 1$, then $r(P_3, 3) = 5 = \dfrac{1(1+4)(1+5)}{6}$ (see Table 1). Suppose the result is true for all natural numbers less than k. Then by using parts (i), (ii), (iii) and (iv) and Proposition 1.2 part (vi), we obtain.

$$\begin{aligned} r(P_{3k}, 2k+1) =& r(P_{3(k-1)+2}, 2(k-1)+2) + r(P_{3(k-1)+2}, 2(k-1)+1) \\ &+ r(P_{3(k-1)+1}, 2(k-1)+1) + r(P_{3(k-1)}, 2(k-1)+1) \\ &+ r(P_{3(k-1)}, 2(k-1)) \\ =& \frac{(k+1)(k+2)}{2} + 0 + k + \frac{(k-1)(k+3)(k+4)}{6} + 1 \\ =& \frac{k(k+4)(k+5)}{6}. \end{aligned}$$

(vi) By induction on k. When $k = 1$, $r(P_4, 4) = 11 = 1 + \dfrac{1(1+2)(1+7)(1+9)}{24}$ (see Table 1). Suppose the result is true for all natural numbers less than k. Then by using parts (i),

(ii), (iii), (iv) and (v), we get.

$$\begin{aligned} r(P_{3k+1}, 2k+2) =& r(P_{3k}, 2k+1) + r(P_{3k}, 2k) + r(P_{3(k-1)+2}, 2(k-1)+2) \\ & + r(P_{3(k-1)+1}, 2(k-1)+2) + r(P_{3(k-1)+1}, 2(k-1)+1) \\ =& \frac{k(k+4)(k+5)}{6} + 1 + \frac{(k+1)(k+2)}{2} + 1 + \frac{(k-1)(k+1)(k+6)(k+8)}{24} + k \\ =& 1 + \frac{k(k+2)(k+7)(k+9)}{24}. \end{aligned}$$

(vii) By induction on k. If $k = 1$, then $r(P_5, 5) = 23 = 2 + \frac{1(1+2)(1+4)(1+11)(1+13)}{120}$ (see Table 1). Now, suppose the result is true for all natural numbers less than k. Then by using parts (i), (iii), (iv), (v) and (vi), we get.

$$\begin{aligned} r(P_{3k+2}, 2k+3) =& r(P_{3k+1}, 2k+2) + r(P_{3k+1}, 2k+1) + r(P_{3k}, 2k+1) \\ & + r(P_{3(k-1)+2}, 2(k-1)+3) + r(P_{3(k-1)+2}, 2(k-1)+2) \\ =& 1 + \frac{k(k+2)(k+7)(k+9)}{24} + k + 1 + \frac{k(k+4)(k+5)}{6} \\ & + 2 + \frac{(k-1)(k+1)(k+3)(k+10)(k+12)}{120} + \frac{(k+1)(k+2)}{2} \\ =& 2 + \frac{k(k+2)(k+4)(k+11)(k+13)}{120}. \end{aligned}$$

(viii) By induction on k. If $k = 1$, then $r(P_3, 4) = 6 = \frac{1(1+7)(1+32+281+418-192)}{720}$ (see Table 1). Now, suppose the result is true for all natural numbers less than k. Then by using parts (i), (iv), (v), (vi) and (vii), we get.

$$\begin{aligned} r(P_{3k}, 2k+2) =& r(P_{3(k-1)+2}, 2(k-1)+3) + r(P_{3(k-1)+2}, 2(k-1)+2) \\ & + r(P_{3(k-1)+1}, 2(k-1)+2) + r(P_{3(k-1)}, 2(k-1)+2) \\ & + r(P_{3(k-1)}, 2(k-1)+1) \\ =& 2 + \frac{(k-1)(k+1)(k+3)(k+10)(k+12)}{120} + \frac{(k+1)(k+2)}{2} \\ & + 1 + \frac{(k-1)(k+1)(k+6)(k+8)}{24} + \frac{(k-1)(k+3)(k+4)}{6} \\ & + \frac{(k-1)(k+6)\left[(k-1)^4 + 32(k-1)^3 + 281(k-1)^2 + 418(k-1) - 192\right]}{720} \\ =& \frac{k(k+7)(k^4 + 32k^3 + 281k^2 + 418k - 192)}{720}. \end{aligned}$$

(ix) We need RDFs from $V(P_n)$ to $\{0, 1, 2\}$ with weight $2n$. Clearly, there is only one function satisfies that in which all the vertices of P_n taking the value 2. Hence, $r(P_n, 2n) = 1$.

(x) Clearly, for every vertex $v \in V(P_n$ the function $f : V(P_n) \to \{0, 1, 2\}$ with $f(v) = 1$ and weight $W(f(V)) = 2n - 1$ is a Roman dominating function of G. Hence, $r(P_n, 2n-1) = \binom{n}{1} = n$.

(xi) By induction on n. The result is true for $n = 2$, since $r(P_2, 2) = 3$ (see Table 1). Suppose the result is true for every natural number less than n. Then by parts (i), (ix) and (x) and

Proposition 1.2 part (vi), we have.

$$\begin{aligned} r(P_n, 2n-2) =& r\big(P_{n-1}, 2(n-1)-1\big) + r\big(P_{n-1}, 2(n-1)-2\big) + r\big(P_{n-2}, 2(n-2)\big) \\ &+ r\big(P_{n-3}, 2(n-3)+2\big) + r\big(P_{n-3}, 2(n-3)+1\big) \\ =& n-1 + \frac{n(n-1)}{2} + 1 + 0 + 0 = \frac{n(n+1)}{2}. \end{aligned}$$

(xii) By induction on n. The result is true for $n = 3$, since $r(P_3, 3) = 5$ (see Table 1). Suppose the result is true for every natural number less than n. Then by parts (i), (ix), (x) and (xi) and Proposition 1.2 part (vi), we have.

$$\begin{aligned} r(P_n, 2n-3) =& r\big(P_{n-1}, 2(n-1)-2\big) + r\big(P_{n-1}, 2(n-1)-3\big) + r\big(P_{n-2}, 2(n-2)-1\big) \\ &+ r\big(P_{n-3}, 2(n-3)+1\big) + r\big(P_{n-3}, 2(n-3)\big) \\ =& \frac{n(n-1)}{2} + \frac{(n-1)(n-2)(n+3)}{6} - 2 + n - 2 + 0 + 1 \\ =& \frac{(n^2-4)(n+3)}{6}. \end{aligned}$$

(xiii) By induction on n. If $n = 5$, then $r(P_5, 6) = 34$. Therefore, the result is true for $n = 4$(see Table 1). Suppose now the result is true for every natural number less than n. Then by parts (i), (ix), (x), (xi) and (xii), we have.

$$\begin{aligned} r(P_n, 2n-4) =& r\big(P_{n-1}, 2(n-1)-3\big) + r\big(P_{n-1}, 2(n-1)-4\big) + r\big(P_{n-2}, 2(n-2)-2\big) \\ &+ r\big(P_{n-3}, 2(n-3)\big) + r\big(P_{n-3}, 2(n-3)-1\big) \\ =& \frac{(n-1)(n-2)(n+3)}{6} - 2 + \frac{(n-1)(n-2)\big[(n-1)^2 + 7(n-1) - 6\big]}{24} \\ &- (3n-7) + \frac{(n-1)(n-2)}{2} + 1 + n - 3 \\ =& \frac{n(n-1)(n^2+7n-6)}{24} - (3n-4). \end{aligned}$$

(xiv) By induction on n. The result is true for $n = 5$, since $r(P_5, 5) = 23$ (see Table 1). Suppose now the result is true for every natural number less than n. Then by parts (i), (x), (xi), (xii) and $(xiii)$, we have.

$$\begin{aligned} r(P_n, 2n-5) =& r\big(P_{n-1}, 2(n-1)-4\big) + r\big(P_{n-1}, 2(n-1)-5\big) + r\big(P_{n-2}, 2(n-2)-3\big) \\ &+ r\big(P_{n-3}, 2(n-3)-1\big) + r\big(P_{n-3}, 2(n-3)-2\big) \\ =& \frac{(n-1)(n-2)\big[(n-1)^2 + 7(n-1) - 6\big]}{24} - (3n-7) \\ &+ \frac{(n-1)\big[(n-1)^2 - 1\big](n-3)(n+11)}{120} + 2 - 2(n-1)(n-3) \\ &+ \frac{(n-2)(n-3)(n+2)}{6} - 2 + n - 3 + \frac{(n-3)(n-2)}{2} \\ =& \frac{n(n^2-1)(n-2)(n+12)}{120} + 2 - 2n(n-2). \end{aligned}$$

(xv) We need to prove that for every $k \in \mathbb{N}$, $r(P_i, 2k) < r(P_i, 2k)$ for $k \leq i \leq 2k-1$ and $r(P_i, 2k) > r(P_i, 2k)$ for $2k \leq i \leq 3k$. By induction on k. The result is true for $k = 1$. Now, suppose that the result is true for every i less than or equal k. We will prove it for $i = k+1$ which means $r(P_i, 2k+2) < r(P_{i+1}, 2k+2)$ for $k+1 \leq i \leq 2k+1$. By part

(i) and the induction hypothesis, we have

$$\begin{aligned} r(P_i, 2k+2) =& r(P_{i-1}, 2k+1) + r(P_{i-1}, 2k) + r(P_{i-2}, 2k) \\ &+ r(P_{i-3}, 2k) + r(P_{i-3}, 2k-1) \\ &< r(P_i, 2k+1) + r(P_i, 2k) + r(P_{i-1}, 2k) \\ &+ r(P_{i-2}, 2k) + r(P_{i-2}, 2k-1) = r(P_{i+1}, 2k+2). \end{aligned}$$

Similarly for the other inequality.

(xvi) Similar to the prove of part (xv), we will prove that for every $k \in \mathbb{N}$, $r(P_i, 2k+1) < r(P_i, 2k+1)$ for $k+1 \leq i \leq 2k$ and $r(P_i, 2k+1) > r(P_i, 2k+1)$ for $2k+1 \leq i \leq 3k+1$. By induction on k. The result is true for $k = 1$. Now, suppose that the result is true for every i less than or equal $k+1$. We will prove it for $i = k+2$ which means $r(P_i, 2k+3) < r(P_{i+1}, 2k+3)$ for $k+2 \leq i \leq 2k+2$. By part (i) and the induction hypothesis, we have

$$\begin{aligned} r(P_i, 2k+3) =& r(P_{i-1}, 2k+2) + r(P_{i-1}, 2k+1) + r(P_{i-2}, 2k+1) \\ &+ r(P_{i-3}, 2k+1) + r(P_{i-3}, 2k) \\ &< r(P_i, 2k+2) + r(P_i, 2k+1) + r(P_{i-1}, 2k+1) \\ &+ r(P_{i-2}, 2k+1) + r(P_{i-2}, 2k) = r(P_{i+1}, 2k+3). \end{aligned}$$

Similarly for the other inequality.

(xvii) By Theorem 2.3, we have

$$\begin{aligned} R(P_n, x) =& \sum_{j=\gamma_R(P_n)}^{2n} r(P_n, j)\, x^j = (x^2+x)R(P_{n-1}, x) + x^2 R(P_{n-2}, x) + (x^3+x^2)R(P_{n-3}, x) \\ =& \sum_{j=\lceil \frac{2n-2}{3} \rceil}^{2n-2} r(P_{n-1}, j)\big[x^{j+2} + x^{j+1}\big] + \sum_{j=\lceil \frac{2n-4}{3} \rceil}^{2n-4} r(P_{n-2}, j)x^{j+2} \\ &+ \sum_{j=\lceil \frac{2n-6}{3} \rceil}^{2n-6} r(P_{n-3}, j)\big[x^{j+3} + x^{j+2}\big]. \end{aligned}$$

Now, if $\alpha_n = \sum_{j=\lceil \frac{2n}{3} \rceil}^{2n} r(P_n, j)$, we can see that all the coefficients of $R(P_{n-1}, x)$ and $R(P_{n-3}, x)$ counted twice and all the coefficients of $R(P_{n-2}, x)$ counted once in α_n. Hence, $\alpha_n = 2\alpha_{n-1} + \alpha_{n-2} + 2\alpha_{n-3}$.

(xviii) If $j = 2$, then

$$\begin{aligned} \sum_{i=2}^{6} r(P_i, 4) =& \sum_{i=2}^{4} r(P_i, 3) + 3\sum_{i=1}^{3} r(P_i, 2) + \sum_{i=1}^{1} r(P_i, 1) \\ 25 =& 9 + 3(5) + 1 = 25. \end{aligned}$$

By part (i), we have

$$\begin{aligned} \sum_{i=j}^{3j} r(P_i, 2j) =& \sum_{i=j}^{3j} r(P_{i-1}, 2j-1) + \sum_{i=j}^{3j} r(P_{i-1}, 2j-2) + \sum_{i=j}^{3j} r(P_{i-2}, 2j-2) \\ &+ \sum_{i=j}^{3j} r(P_{i-3}, 2j-2) + \sum_{i=j}^{3j} r(P_{i-3}, 2j-3). \end{aligned}$$

Now, by Proposition 1.2 part (vi), we have

$$\sum_{i=j}^{3j} r(P_{i-1}, 2j-1) = \sum_{i=j-1}^{3j} r(P_i, 2j-1) = \sum_{i=j}^{3j-2} r(P_i, 2j-1),$$

$$\sum_{i=j}^{3j} r(P_{i-1}, 2j-2) = \sum_{i=j-1}^{3j} r(P_i, 2j-2) = \sum_{i=j-1}^{3j-3} r(P_i, 2j-2),$$

$$\sum_{i=j}^{3j} r(P_{i-2}, 2j-2) = \sum_{i=j-2}^{3j} r(P_i, 2j-2) = \sum_{i=j-1}^{3j-3} r(P_i, 2j-2),$$

$$\sum_{i=j}^{3j} r(P_{i-3}, 2j-2) = \sum_{i=j-3}^{3j} r(P_i, 2j-2) = \sum_{i=j-1}^{3j-3} r(P_i, 2j-2),$$

and

$$\sum_{i=j}^{3j} r(P_{i-3}, 2j-3) = \sum_{i=j-3}^{3j} r(P_i, 2j-3) = \sum_{i=j-1}^{3j-5} r(P_i, 2j-3).$$

(xix) The proof is similar to the proof of part $(xviii)$.

(xx) By induction on k. If $k=1$, then $r(P_1, 2) = r(P_3, 2) = 1$, therefore, the result is true for $k=1$. Suppose now the result holds for all natural numbers less than k. We will prove it for k, as follows

$$\begin{aligned} r(P_{2k-m}, 2k) = {} & r(P_{2k-m-1}, 2k-1) + r(P_{2k-m-1}, 2k-2) + r(P_{2k-m-2}, 2k-2) \\ & + r(P_{2k-m-3}, 2k-2) + r(P_{2k-m-3}, 2k-3) \\ = {} & r\big(P_{2(k-1)+1-m}, 2(k-1)+1\big) + r\big(P_{2(k-1)+1-m}, 2(k-1)\big) + r\big(P_{2(k-1)-m}, 2(k-1)\big) \\ & + r\big(P_{2(k-1)-1-m}, 2(k-1)\big) + r\big(P_{2(k-1)-1-m}, 2(k-1)-1\big) \\ = {} & r\big(P_{2(k-1)+1+m}, 2(k-1)+1\big) + r\big(P_{2(k-1)+m-1}, 2(k-1)\big) + r\big(P_{2(k-1)+m}, 2(k-1)\big) \\ & + r\big(P_{2(k-1)+m+1}, 2(k-1)\big) + r\big(P_{2(k-1)-1+m}, 2(k-1)-1\big) \\ = {} & r(P_{2k+m-1}, 2k-1) + r(P_{2k+m-3}, 2k-2) + r(P_{2k+m-2}, 2k-2) \\ & + r(P_{2k+m-1}, 2k-2) + r(P_{2k+m-3}, 2k-3) = r(P_{2k+m}, 2k) \end{aligned}$$

(xxi) The proof is similar to the proof of part (xx).

□

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Received: January 28, 2021
Accepted: May 20, 2021