**Palestine Journal of Mathematics**

**Vol. 11(2)(2022)** , 457–468 

# On $\alpha$-Baskakov-Gamma operators with two shifted nodes

Taqseer Khan, Nadeem Rao, Shuzaat Ali Khan and Pradeep Malik

Communicated by Siraj Uddin

MSC 2010 Classification: 41A10, 41A25, 41A28, 41A35, 41A36

Keywords: Baskakov operators, modulus of continuity, rate of convergence, Korovkin theorem

*The authors like to thank the reviewer for the valuable comments which improved the quality of the paper.*

**Abstract** The goal of the present article is to introduce a new sequence of operators, $i.e.$, $\alpha$-Baskakov-Gamma operators with two shifted nodes, $0 \leq \mu \leq \nu$, to approximate a class of Lebesgue measurable functions on $[0, \infty)$. We give basic results and study the rate of convergence. Further, local and global approximation properties are investigated in terms of first and second-order modulus of smoothness, Peetre's K-functional and weight functions in several function spaces. Lastly, $A-$statistical approximation results are obtained.

## 1 Introduction

The theory of linear positive operators deals with questions that arise in the approximate representation of an arbitrary function by simpler functions. Operator theory is a growing field of research of approximation theory for the last two decades with the advent of the computer. Several mathematicians, $e.g.$, Acar *et al.* ([1], [2]), Mohiuddine *et al.* [14], Ana *et al.* [3], İçöz *et al.* ([11]), [12]), Kajla *et al.* ([13]) constructed new sequences of linear positive operators and studied the rapidity of convergence and order of approximation in diffrent function spaces in terms of several generating functions. In the recent past, for $g \in C[0,1], m \in \mathbb{N}$ and $\alpha \in [-1,1]$, Chen *et al.* [6] constructed a sequence of linear positive operators as follows

$$T_{m,\alpha}(g;y) = \sum_{i=0}^{m} g\left(\frac{i}{m}\right) p^{\alpha}_{m,i}(y) \qquad (y \in [0,1]), \tag{1.1}$$

where $p^{(\alpha)}_{1,0} = 1-y, \quad p^{(\alpha)}_{1,1} = y$ and

$$p^{\alpha}_{m,i}(y) = \left[(1-\alpha)y\binom{m-2}{i} + (1-\alpha)(1-y)\binom{m-2}{i-2} + \alpha y(1-y)\binom{m}{i}\right] y^{i-1}(1-y)^{m-i-1} \quad (m \geq 2).$$

The operators defined in (1.1) are named as $\alpha-$Bernstein operators of order m. One can note that for $\alpha = 1$, the relation (1.1) is reduced to classical Bernstein operators in [5].

The rate of convergence, shape-preserving characteristics and Voronovskaja type results for these constructed linear positive operators have been studied in [6]. The bivariate version of $\alpha-$ Bernstein-Durrmeyer operators was developed and investigated by Micláu s and Kajla [13]. Kantorovich variant of $\alpha-$Bernstein operators was constructed and studied by Mohiuddine $et\ al.$ [14]. Later, Aral and Erbay [4] introduced a parametric extension of Baskakov operators, for $f \in C_B[0,\infty)$, (space of bounded and continuous functions) as follows

$$L_{n,\alpha}(f;x) = \sum_{k=0}^{\infty} \mathcal{Q}^{(\alpha)}_{n,k}(x) f\left(\frac{k}{n}\right), \tag{1.2}$$

where $n \geq 1, x \in [0, \infty)$ and

$$\mathcal{Q}_{n,k}^{(\alpha)}(x) = \frac{x^{k-1}}{(1+x)^{n+k-1}}\left\{\frac{\alpha x}{1+x}\binom{n+k-1}{k} - (1-\alpha)(1+x)\binom{n+k-3}{k-2} + (1-\alpha)x\binom{n+k-1}{k}\right\}, \tag{1.3}$$

with $\binom{n-3}{-2} = \binom{n-2}{-1} = 0$.
Motivated by the above, we introduce a generalisation of operators (1.2) with two non-negative shifted nodes, $0 \leq \mu \leq \nu$, to approximate in a wider class, *i.e.*, space of Lebesgue integrable functions as follows:

$$\mathcal{P}_{n,\alpha}^{\mu,\nu}(f;x) = \sum_{k=0}^{\infty} \mathcal{Q}_{n,k}^{(\alpha)}(x) \frac{n^{k+\lambda}}{\Gamma(k+\lambda+1)} \int_0^{\infty} t^{k+\lambda} e^{-nt} f\left(\frac{nt+\mu}{n+\nu}\right) dt, \tag{1.4}$$

where $\lambda \geq 0$ and $\mathcal{Q}_{n,k}^{(\alpha)}(x)$ is given by (1.3) and the Gamma function is defined as

$$\Gamma n = \int_0^{\infty} x^{n-1} e^{-x} dx, \Gamma z = (z-1)\Gamma(z-1) = (z-1)!. \tag{1.5}$$

In the ensuing sections, we obtain basic lemmas and investigate rate of convergence, order of approximation, local and global approximation results in terms of modulus of continuity, Peetre‘s K-functional, second-order modulus of smoothness, Lipschitz class and Lipschitz maximal function, weighted modulus of continuity for the operators defined in (1.4). In the last section, $A-$statistical approximation properties are studied for these operators.

## 2 Preliminary Results

Let $e_r(t) = t^r$ and $\psi_x^r(t) = (t-x)^r$, $r \in \{0, 1, 2\}$ denote the test functions and central moments respectively. We have the following lemmas.

**Lemma 2.1.** *For the operators* (1.4)*, we have*

$$\mathcal{P}_{n,\alpha}^{\mu,\nu}(t^r;x) = \sum_{i=0}^{r} \binom{r}{i} \frac{n^i \alpha^{r-i}}{(n+\beta)^r} \mathcal{B}_{n,\alpha}^*(t^i;x),$$

*where* $\mathcal{B}_{n,\alpha}^*(f;x)$ *are defined by*

$$\mathcal{B}_{n,\alpha}^*(f;x) = \sum_{k=0}^{\infty} \mathcal{Q}_{n,k}^{(\alpha)}(x) \frac{n^{k+\lambda+1}}{\Gamma(k+\lambda+1)} \int_0^{\infty} t^{k+\lambda} e^{-nt} f(t) dt, \tag{2.1}$$

*where* $\mathcal{Q}_{n,k}^{(\alpha)}(x)$ *is given by* (1.3) *and the Gamma function is defined in* (1.5)

We have the following lemma.

**Lemma 2.2.** *For the operators given by (2.1), one obtains*

$$\begin{aligned}
\mathcal{B}_{n,\alpha}^*(e_0;x) &= 1, \\
\mathcal{B}_{n,\alpha}^*(e_1;x) &= \left(1 + \frac{2}{n}(\alpha-1)\right)x + \frac{\lambda+1}{n}, \\
\mathcal{B}_{n,\alpha}^*(e_2;x) &= x^2\left(1 + \frac{4\alpha-3}{n}\right) + x\left(\frac{2\lambda+3}{n} + \frac{4\alpha-4+(2\lambda+3)(\alpha-1)}{n^2}\right) \\
&+ \frac{\lambda^2+3\lambda+2}{n^2}.
\end{aligned}$$

We now prove Lemma 2.1.

*Proof.* From (1.4) and (2.1), we have

$$\begin{aligned}\mathcal{P}^{\mu,\nu}_{n,\alpha}(t^r;x) &= \sum_{k=0}^{\infty} \mathcal{Q}^{(\alpha)}_{n,k}(x) \frac{n^{k+\lambda+1}}{\Gamma(k+\lambda+1)} \int_0^\infty t^{k+\lambda} e^{-nt} f\left(\frac{nt+\mu}{n+\nu}\right)^r dt,\\ &= \sum_{i=0}^{r} \binom{r}{i} \frac{n^i \mu^{r-i}}{(n+\nu)^r} \sum_{k=0}^{\infty} \mathcal{Q}^{(\alpha)}_{n,k}(x) \frac{n^{k+\lambda+1}}{\Gamma(k+\lambda+1)} \int_0^\infty t^{k+\lambda} e^{-nt} t^i dt,\\ &= \sum_{i=0}^{r} \binom{r}{i} \frac{n^i \mu^{r-i}}{(n+\nu)^r} \mathcal{B}^*_{n,\alpha}(t^i;x)\end{aligned}$$

and hence the lemma. □

The following lemma computes central moments of our operators.

**Lemma 2.3.** *For $r \in \mathbb{N}$, we obtain the relation*

$$\mathcal{P}^{\mu,\nu}_{n,\alpha}((t-x)^r;x) = \sum_{i=0}^{r} \binom{r}{i} (-x)^{r-i} \mathcal{P}^{\mu,\nu}_{n,\alpha}(t^i;x).$$

*Proof.* We write from (1.4)

$$\begin{aligned}\mathcal{P}^{\mu,\nu}_{n,\alpha}((t-x)^r;x) &= \sum_{k=0}^{\infty} \mathcal{Q}^{(\alpha)}_{n,k}(x) \frac{n^{k+\lambda+1}}{\Gamma(k+\lambda+1)} \int_0^\infty t^{k+\lambda} e^{-nt} \left(\frac{nt+\mu}{n+\nu} - x\right)^r dt,\\ &= \sum_{i=0}^{r} \binom{r}{i} (-u)^{r-i} \mathcal{P}^{\mu,\nu}_{n,\alpha}(t^i;x).\end{aligned}$$

□

**Lemma 2.4.** *For the operators given by (1.4), we have*

$$\begin{aligned}\mathcal{P}^{\mu,\nu}_{n,\alpha}(e_0;x) &= 1,\\ \mathcal{P}^{\mu,\nu}_{n,\alpha}(e_1;x) &= \left(\frac{n+2(\alpha-1)}{n+\nu}\right)x + \frac{\mu+\lambda+1}{n+\nu},\\ \mathcal{P}^{\mu,\nu}_{n,\alpha}(e_2;x) &= \left(\frac{n(n+4\alpha-3)}{(n+\nu)^2}\right)x^2\\ &\quad + \left(\frac{n(2\mu+2\lambda+3)+4(\alpha-1)(\mu-1)+(2\lambda+3)(\alpha-1)}{(n+\nu)^2}\right)x\\ &\quad + \left(\frac{\mu(\mu+\lambda+1)+(\lambda+1)(\lambda+2)}{(n+\nu)^2}\right).\end{aligned}$$

*Proof.* Using Lemma 2.2, by Lemma 2.1, we will have

$$\begin{aligned}\mathcal{P}^{\mu,\nu}_{n,\alpha}(e_0;x) &= \binom{0}{0} \frac{n^0 \mu^0}{(n+\nu)^0} \mathcal{B}^*_{n,\alpha}(1;x),\\ &= 1,\\ \mathcal{P}^{\mu,\nu}_{n,\alpha}(e_1;x) &= \sum_{i=0}^{1} \binom{1}{i} \frac{n^i \mu^{1-i}}{(n+\nu)^1} \mathcal{B}^*_{n,\alpha}(t^i;x)\\ &= \binom{1}{0} \frac{n^0 \mu^{1-0}}{(n+\nu)^1} \mathcal{B}^*_{n,\alpha}(1;x) + \binom{1}{1} \frac{n^1 \mu^{1-1}}{(n+\nu)^1} \mathcal{B}^*_{n,\alpha}(t;x)\\ &= \frac{\mu}{n+\nu} + \frac{n}{n+\nu}\left(\left(1+\frac{2}{n}(\alpha-1)\right)x + \frac{\lambda+1}{n}\right)\end{aligned}$$

$$
\begin{aligned}
&= \left(\frac{n+2(\alpha-1)}{n+\nu}\right)x + \frac{\mu+\lambda+1}{n+\nu},\\
\mathcal{P}_{n,\alpha}^{\mu,\nu}(e_2;x) &= \sum_{i=0}^{2}\binom{2}{i}\frac{n^i\mu^{2-i}}{(n+\nu)^2}\mathcal{B}_{n,\alpha}^{*}(t^i;x)\\
&= \binom{2}{0}\frac{n^0\mu^{2-0}}{(n+\nu)^2}\mathcal{B}_{n,\alpha}^{*}(1;x) + \binom{2}{1}\frac{n^1\mu^{2-1}}{(n+\nu)^2}\mathcal{B}_{n,\alpha}^{*}(t;x)\\
&+ \binom{2}{2}\frac{n^2\mu^{2-2}}{(n+\nu)^2}\mathcal{B}_{n,\alpha}^{*}(t^2;x)\\
&= \frac{\mu^2}{(n+\nu)^2} + \frac{2n\mu}{(n+\nu)^2}\left(\left(1+\frac{2}{n}(\alpha-1)\right)x + \frac{\lambda+1}{n}\right)\\
&+ \frac{n^2}{(n+\nu)^2}\left(x^2\left(1+\frac{4\alpha-3}{n}\right)\right.\\
&+ \left. x\left(\frac{2\lambda+3}{n} + \frac{4\alpha-4+(2\lambda+3)(\alpha-1)}{n^2}\right)\right)\\
&+ \frac{n^2}{(n+\nu)^2}\left(\frac{\lambda^2+3\lambda+2}{n^2}\right),\\
&= \left(\frac{n(n+4\alpha-3)}{(n+\nu)^2}\right)x^2\\
&+ \left(\frac{n(2\mu+2\lambda+3)+4(\alpha-1)(\mu-1)+(2\lambda+3)(\alpha-1)}{(n+\nu)^2}\right)x\\
&+ \left(\frac{\mu(\mu+\lambda+1)+(\lambda+1)(\lambda+2)}{(n+\nu)^2}\right),
\end{aligned}
$$

which completes the proof of Lemma 2.4. □

**Lemma 2.5.** *For the operators $\mathcal{P}_{n,\alpha}^{\mu,\nu}(.;,)$ introduced in (1.4), we obtain*

$$
\begin{aligned}
&\mathcal{P}_{n,\alpha}^{\mu,\nu}(\psi_x^0;x) = 1,\\
&\mathcal{P}_{n,\alpha}^{\mu,\nu}(\psi_x^1;x) = \left(\frac{n+2(\alpha-1)}{n+\nu} - 1\right)x + \frac{\mu+\lambda+1}{n+\nu},\\
&\mathcal{P}_{n,\alpha}^{\mu,\nu}(\psi_x^2;x) = \left(\left(\frac{n(n+4\alpha-3)}{(n+\nu)^2}\right) - 2\left(\frac{n+2(\alpha-1)}{n+\nu}\right)+1\right)x^2\\
&\qquad + \left(\left(\frac{n(2\mu+2\lambda+3)+4(\alpha-1)(\mu-1)+(2\lambda+3)(\alpha-1)}{(n+\nu)^2}\right)\right.\\
&\qquad \left. - 2\left(\frac{\mu+\lambda+1}{n+\nu}\right)\right)x + \left(\frac{\mu(\mu+\lambda+1)+(\lambda+1)(\lambda+2)}{(n+\nu)^2}\right).
\end{aligned}
$$

*Proof.* Making use of Lemma 2.4, from Lemma 2.3, we get

$$
\begin{aligned}
\mathcal{P}_{n,\alpha}^{\mu,\nu}(\psi_x^0;x) &= 1,\\
\mathcal{P}_{n,\alpha}^{\mu,\nu}(\psi_x^1;x) &= \binom{1}{0}(-x)^{1-0}\mathcal{P}_{n}^{\mu,\nu}(1;x) + \binom{1}{1}(-x)^{1-1}\mathcal{P}_{n}^{\mu,\nu}(t;x)\\
&= -x + \left(\frac{n+2(\alpha-1)}{n+\nu}\right)x + \frac{\mu+\lambda+1}{n+\nu}\\
&= \left(\frac{n+2(\alpha-1)}{n+\nu} - 1\right)x + \frac{\mu+\lambda+1}{n+\nu}
\end{aligned}
$$

$$
\begin{aligned}
\mathcal{P}_{n,\alpha}^{\mu,\nu}(\psi_x^2;x) &= \binom{2}{0}(-x)^{2-0}\mathcal{P}_n^{\mu,\nu}(1;x) + \binom{2}{1}(-x)^{2-1}\mathcal{P}_n^{\mu,\nu}(t;x) \\
&+ \binom{2}{2}(-x)^{2-2}\mathcal{P}_n^{\mu,\nu}(t^2;x) \\
&= x^2 - 2x^2\left(\frac{n+2(\alpha-1)}{n+\nu}\right) - 2x\left(\frac{\mu+\lambda+1}{n+\nu}\right) \\
&+ \left(\frac{n(n+4\alpha-3)}{(n+\nu)^2}\right)x^2 \\
&+ \left(\frac{n(2\mu+2\lambda+3)+4(\alpha-1)(\mu-1)+(2\lambda+3)(\alpha-1)}{(n+\nu)^2}\right)x \\
&+ \left(\frac{\mu(\mu+\lambda+1)+(\lambda+1)(\lambda+2)}{(n+\nu)^2}\right) \\
&= \left(\left(\frac{n(n+4\alpha-3)}{(n+\nu)^2}\right) - 2\left(\frac{n+2(\alpha-1)}{n+\nu}\right)+1\right)x^2 \\
&+ \left(\left(\frac{n(2\mu+2\lambda+3)+4(\alpha-1)(\mu-1)+(2\lambda+3)(\alpha-1)}{(n+\nu)^2}\right)\right. \\
&\left. - 2\left(\frac{\mu+\lambda+1}{n+\nu}\right)\right)x + \left(\frac{\mu(\mu+\lambda+1)+(\lambda+1)(\lambda+2)}{(n+\nu)^2}\right),
\end{aligned}
$$

which proves Lemma 2.5. □

**Definition 2.6.** Let $f \in C[0,\infty)$. Then, modulus of continuity for a uniformly continuous function $f$ is defined as

$$\omega(f;\delta) = \sup_{|t_1-t_2|\leq\delta} |f(t_1)-f(t_2)|, \qquad t_1,t_2 \in [0,\infty).$$

The following relation holds.

$$|f(t_1)-f(t_2)| \leq \left(1+\frac{(t_1-t_2)^2}{\delta^2}\right)\omega(f;\delta).$$

We prove the following theorems.

**Theorem 2.7.** *Let $\mathcal{P}_{n,\alpha}^{\mu,\nu}(.;.)$ be the operators given by (1.4). Then, $\mathcal{P}_{n,\alpha}^{\mu,\nu} \rightrightarrows f$ on each compact subset of $[0,\infty)$ where $\rightrightarrows$ stands for uniform convergence and $f \in C[0,\infty)\bigcap\left\{f : x\geq 0, \frac{f(x)}{1+x^2}\right.$ is convergent as $\left.x\to\infty\right\}$.*

*Proof.* In the light of Korovkin's theorem, it is sufficient to prove that

$$\mathcal{P}_{n,\alpha}^{\mu,\nu}(.;.) \to e_i(x), \text{ for } i=0,1,2.$$

Using Lemma 2.4, it follows that $\mathcal{P}_{n,\alpha}^{\mu,\nu}(e;x) \to e_i(x)$ as $n \to \infty$, for $i=0,1,2$ and hence the proof of the theorem. □

**Example 2.8.** One can note that, for the following set of parameters $\mu=0.1,\ \nu=0.4,\ \lambda=1.5$ and $\alpha=0.5$, the operators $P_{n,k}^{\alpha}(f;x)$ converge uniformly to the function $f(x)=x^3-5x+4$ (refer to Fig. 1, 2 below).

We state the following theorem from [16].

**Theorem 2.9.** *Let $L : C([a,b]) \to B([a,b])$ be a linear and positive operator and let $\varphi_x$ be the function defined by*

$$\varphi_x(t) = |t-x|,\ (x,t)\in[a,b]\times[a,b].$$

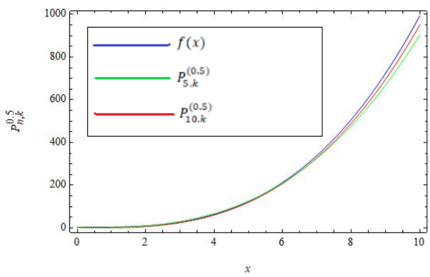


**Figure 1.** Approximation by operator $P^{\alpha}_{n,k}(;,;)$ for the function $f(x) = x^3 - 5x + 4$

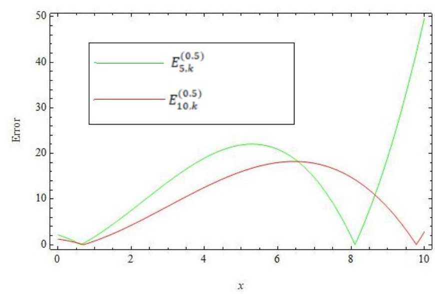


**Figure 2.** Graphical analysis of error estimation of operators $P^{\alpha}_{n,k}(;,;)$ for different values of $n$

*Then for $f \in C_B[a,b]$ and $x \in [a,b]$ and any $\delta > 0$, the operator $L$ verifies*

$$|(Lf)(x) - f(x)| \leq |f(x)||(Le_0)(x) - 1| + \{(Le_0)(x) + \delta^{-1}\sqrt{(Le_0)(x)(L\varphi_x^2)(x)}\}\omega_f(\delta).$$

Using this theorem, the following estimate is obtained.

**Theorem 2.10.** *Let the operators $\mathcal{P}^{\mu,\nu}_{n,\alpha}(.;.)$ be introduced by (1.4) and $f \in C_B[0,\infty)$. Then*

$$|\mathcal{P}^{\mu,\nu}_{n,\alpha}(f;x) - f(x)| \leq 2\omega(f;\delta),$$

*where $\delta = \sqrt{\mathcal{P}^{\mu,\nu}_{n,\alpha}(\psi_x^2;x)}$.*

*Proof.* Making use of Lemmas 2.4, 2.5 and Theorem 2.9, the following is obtained

$$|\mathcal{P}^{\mu,\nu}_{n,\alpha}(f;x) - f(x)| \leq \{1 + \delta^{-1}\sqrt{\mathcal{P}^{\mu,\nu}_{n,\alpha}(f;x)(\psi_x^2;x)}\}\omega(f;\delta).$$

On choosing $\delta = \sqrt{\mathcal{P}^{\mu,\nu}_{n,\alpha}(\psi_u^2;u)}$, we arrive at the desired result. □

## 3 Pointwise Approximation Results

Let $C_B[0,\infty)$ be the space of real valued continuous and bounded functions equipped with the norm $\|f\| = \sup_{0\leq x<\infty} |f(x)|$. For any $f \in C_B[0,\infty)$ and $\delta > 0$, Peetre's K-functional is defined as

$$K_2(g,\delta) = inf\{\|f - h\| + \delta\|h''\| : h \in C_B^2[0,\infty)\}$$

where $C_B^2[0,\infty) = \{h \in C_B[0,\infty) : h', h'' \in C_B[0,\infty)\}$. From DeVore and Lorentz [[7], p.177, Theorem 2.4], there exists an absolute constant $C > 0$ in such a way that

$$K_2(f;\delta) \le C\omega_2(f;\sqrt{\delta}).$$

**Lemma 3.1.** *Consider the auxiliary operators as*

$$\widehat{\mathcal{P}}_n^{\mu,\nu}(f;x) = \mathcal{P}_{n,\alpha}^{\mu,\nu}(f;x) + f(x) - f\left(\frac{n+2(\alpha-1)}{n+\nu}\right)x + \frac{\mu+\lambda+1}{n+\nu}.$$

*Then, for $f \in C_B^2[0,\infty)$ one has*

$$|\widehat{\mathcal{P}}_{n,\alpha}^{\mu,\nu}(f;x) - f(x)| \le \xi_n^x \|h''\|,$$

*where*

$$\xi_n^x = \mathcal{P}_{n,\alpha}^{\mu,\nu}(\psi_x^2;x) + \left(\mathcal{P}_{n,\alpha}^{\mu,\nu}(\psi_x^1;x)\right)^2.$$

*Proof.* Using definition of operators in (1.4), one obtains

$$\widehat{\mathcal{P}}_{n,\alpha}^{\mu,\nu}(1;x) = 1,\ \widehat{\mathcal{P}}_{n,\alpha}^{\mu,\nu}(\psi_x;x) = 0 \text{ and } |\widehat{\mathcal{P}}_{n,\alpha}^{\mu,\nu}(f;x)| \le 3\|f\|. \tag{3.1}$$

In the direction of Taylor's series, for $g \in C_B^2[0,\infty)$, one can write

$$g(t) = g(x) + (t-x)g'(x) + \int_x^t (x-v)g''(v)dv. \tag{3.2}$$

On applying operators (1.4) on both sides of (3.2), we get

$$\widehat{\mathcal{P}}_{n,\alpha}^{\mu,\nu}(h;x) - h(x) = h'(x)\widehat{\mathcal{P}}_{n,\alpha}^{\mu,\nu}(t-x;x) + \widehat{\mathcal{P}}_{n,\alpha}^{\mu,\nu}\left(\int_x^t (t-v)h''(v)dv;u\right),$$

which, with the help of (3.1), gives

$$\begin{aligned}\widehat{\mathcal{P}}_{n,\alpha}^{\mu,\nu}(f;x) - h(x) &= \widehat{\mathcal{P}}_{n,\alpha}^{\mu,\nu}\left(\int_x^t (t-v)h''(v)dv;u\right)\\ &= \mathcal{P}_{n,\alpha}^{\mu,\nu}\left(\int_x^t (t-v)h''(v)dv;u\right)\\ &- \int_x^{\left(\frac{n+2(\alpha-1)}{n+\nu}\right)x+\frac{\mu+\lambda+1}{n+\nu}} \left(\left(\frac{n+2(\alpha-1)}{n+\nu}\right)x + \frac{\mu+\lambda+1}{n+\nu} - v\right)\\ &g''(v)dv.\end{aligned}$$

$$\begin{aligned}|\widehat{\mathcal{P}}_{n,\alpha}^{\mu,\nu}(f;x) - f(x)|\mathcal{P}_{n,\alpha}^{\mu,\nu} \le &\left|\left(\int_x^t (t-v)h''(v)dv;x\right)\right|\\ &+\left|\int_x^{\left(\frac{n+2(\alpha-1)}{n+\nu}\right)x+\frac{\mu+\lambda+1}{n+\nu}}\right.\\ &\left.\left(\left(\frac{n+2(\alpha-1)}{n+\nu}\right)x + \frac{\mu+\lambda+1}{n+\nu} - v\right)h''(v)dv\right|.\end{aligned} \tag{3.3}$$

Since

$$\left|\int_x^t (t-v)h''(v)dv\right| \le (t-v)^2 \parallel h'' \parallel, \tag{3.4}$$

we have

$$\left| \int_x^{\mathcal{P}_{n,\alpha}^{\mu,\nu}(e_1;x)} \left(\mathcal{P}_{n,\alpha}^{\mu,\nu}(e_1;x) - v\right) h''(v)dv \right| \leq \left(\mathcal{P}_{n,\alpha}^{\mu,\nu}(t-v;x)\right)^2 \parallel h'' \parallel . \tag{3.5}$$

In the light of (3.3), (3.4) and (3.5), we obtain

$$\widehat{\mathcal{P}}_{n,\alpha}^{\mu,\nu}(h;x) - h(x)| \leq \xi_n^x \|h''\|,$$

which completes the proof of the lemma. □

**Theorem 3.2.** *Let* $f \in C_B^2[0,\infty)$ *and operators* $\mathcal{P}_{n,\alpha}^{\mu,\nu}(.;.)$ *be constructed in* (1.4). *Then, there exists a constant* $C > 0$ *such that*

$$| \mathcal{P}_{n,\alpha}^{\mu,\nu}(f;x) - f(x) | \leq C\omega_2\left(f; \frac{1}{2}\sqrt{\xi_n^x}\right) + \omega(f; \mathcal{P}_{n,\alpha}^{\mu,\nu}(\psi_x;x)),$$

*where* $\xi_n^x$ *is defined in Lemma 3.1.*

*Proof.* For $h \in C_B^2[0,\infty)$ and $f \in C_B[0,\infty)$ and by the definition of $\widehat{\mathcal{P}}_{n,\alpha}^{\mu,\nu}(.;.)$, we have

$$\begin{aligned} |\mathcal{P}_{n,\alpha}^{\mu,\nu}(f;x) - f(x)| &\leq |\widehat{\mathcal{P}}_{n,\alpha}^{\mu,\nu}(f-h;x)| + |(f-h)(x)| + |\widehat{\mathcal{P}}_{n,\alpha}^{\mu,\nu}(h;x) - h(x))| \\ &\quad + \left| f\Big(\frac{n+2(\alpha-1)}{n+\nu}\Big)x + \frac{\mu+\lambda+1}{n+\nu}\Big) - g(x) \right|. \end{aligned}$$

Lemma 3.1 and relations in (3.1) yield

$$\begin{aligned} |\mathcal{P}_{n,\alpha}^{\mu,\nu}(f;x) - f(x)| &\leq 4\|f-h\| + |\widehat{\mathcal{P}}_{n,\alpha}^{\mu,\nu}(h;x) - h(x)| \\ &\quad + \left| f\Big(\frac{n+2(\alpha-1)}{n+\nu}\Big)x + \frac{\mu+\lambda+1}{n+\nu}\Big) - g(x) \right| \\ &\leq 4\|f-h\| + \xi_n^x\|h''\| + \omega\Big(f; \mathcal{P}_{n,\alpha}^{\mu,\nu}(\psi_x;x)\Big). \end{aligned}$$

With the aid of definition of Peetre's K-functional, we get

$$|\mathcal{P}_{n,\alpha}^{\mu,\nu}(f;x) - f(x)| \leq C\omega_2\left(f; \frac{1}{2}\sqrt{\xi_n^x}\right) + \omega(f; \mathcal{P}_{n,\alpha}^{\mu,\nu}(\psi_x;x),$$

which is the desired result. □

We consider the Lipschitz type space [15] as

$$Lip_M^{k_1,k_2}(\rho) := \left\{ f \in C_B[0,\infty) : |f(t) - f(x)| \leq M \frac{|t-x|^\rho}{(t+k_1x+k_2x^2)^{\frac{\rho}{2}}} : x,t \in (0,\infty) \right\},$$

where $M \geq 0$ is a real constant; $k_1, k_2 > 0$, $\rho > 0$ and $\rho \in (0,1]$.
The following estimate is obtained.

**Theorem 3.3.** *For* $f \in Lip_M^{k_1,k_2}(\rho)$, *the following estimate is obtained*

$$|\mathcal{P}_{n,\alpha}^{\mu,\nu}(f;x) - f(x)| \leq M\left(\frac{\eta_n^*(x)}{k_1x + k_2x^2}\right)^{\frac{\rho}{2}},$$

*where* $x > 0$ *and* $\eta_n^*(x) = \mathcal{P}_{n,\alpha}^{\mu,\nu}(\psi_x^2;x)$.

*Proof.* For $\rho = 1$, we have

$$|\mathcal{P}^{\mu,\nu}_{n,\alpha}(f;x) - f(x)| \leq \mathcal{P}^{\mu,\nu}_{n,\alpha}(|f(t) - f(x)|)(x)$$
$$\leq M\mathcal{P}^{\mu,\nu}_{n,\alpha}\left(\frac{|t-x|}{(t+k_1x+k_2x^2)^{\frac{1}{2}}};x\right).$$

Since $\frac{1}{t+k_1x+k_2x^2} < \frac{1}{k_1x+k_2x^2}$ for all $t, x \in (0,\infty)$, we get

$$|\mathcal{P}^{\mu,\nu}_{n,\alpha}(f;x) - f(x)| \leq \frac{M}{(k_1x+k_2x^2)^{\frac{1}{2}}}(\mathcal{P}^{\mu,\nu}_{n,\alpha}((t-x)^2;x))^{\frac{1}{2}}$$
$$\leq M\left(\frac{\eta^*_n(x)}{k_1x+k_2x^2}\right)^{\frac{1}{2}}.$$

This implies that for $\rho = 1$, this result stands good. Now, for $\rho \in (0,1)$ and using Hölder's inequality, on choosing $p = \frac{2}{\rho}$ and $q = \frac{2}{2-\rho}$, one gets

$$|\mathcal{P}^{\mu,\nu}_{n,\alpha}(f;x) - f(x)| \leq \left(\mathcal{P}^{\mu,\nu}_{n,\alpha}((|f(t) - f(x)|)^{\frac{2}{\rho}};x)\right)^{\frac{\rho}{2}}$$
$$\leq M\left(\mathcal{P}^{\mu,\nu}_{n,\alpha}\left(\frac{|t-x|^2}{(t+k_1x+k_2x^2)};x\right)\right)^{\frac{\rho}{2}}.$$

Since $\frac{1}{t+k_1x+k_2x^2} < \frac{1}{k_1x+k_2x^2}$ for all $t, x \in (0,\infty)$, we obtain

$$|\mathcal{P}^{\mu,\nu}_{n,\alpha}(f;x) - f(x)| \leq M\left(\frac{\mathcal{P}^{\mu,\nu}_{n}\left(|t-x|^2;x\right)}{k_1x+k_2x^2}\right)^{\frac{\rho}{2}}$$
$$\leq M\left(\frac{\eta^*_n(x)}{k_1x+k_2x^2}\right)^{\frac{\rho}{2}}.$$

Hence, we arrive at the desired result. □

## 4 Global Approximation

From [9], we recall some notation to prove the global approximation results.
For the weight function $1 + x^2$ and $0 \leq x < \infty$, we define $B_{1+x^2}[0,\infty) = \{f(x) : |f(x)| \leq M_f(1+x^2), M_f$ is constant depending on $f\}$, $C_{1+x^2}[0,\infty) \subset B_{1+x^2}[0,\infty)$, space of continuous functions endowed with the norm $\|f\|_{1+x^2} = \sup\limits_{x\in[0,\infty)} \frac{|f(x)|}{1+x^2}$ and

$$C^k_{1+x^2}[0,\infty) = \left\{f \in C_{1+x^2}[0,\infty) : \lim_{x\to\infty}\frac{f(x)}{1+x^2} = k, \text{ where } k \text{ is a constant}\right\}.$$

**Theorem 4.1.** *Let $\mathcal{P}^{\mu,\nu}_{n,\alpha}(.;.)$ be the operators given by (1.4) and $\mathcal{P}^{\mu,\nu}_{n,\alpha}(.;.) : C^k_{1+x^2}[0,\infty) \to B_{1+x^2}[0,\infty)$. Then, we have*

$$\lim_{n\to\infty}\|\mathcal{P}^{\mu,\nu}_{n,\alpha}(f;x) - f\|_{1+x^2} = 0,$$

*where $f \in C^k_{1+x^2}[0,\infty)$.*

*Proof.* To prove this result, it is sufficient to show that

$$\lim_{n\to\infty}\|\mathcal{P}^{\mu,\nu}_{n,\alpha}(e_i;x) - x^i\|_{1+x^2} = 0, \; i = 0,1,2.$$

From Lemma 2.4, we get

$$\|\mathcal{P}^{\mu,\nu}_{n,\alpha}(e_0;x) - x^0\|_{1+x^2} = \sup_{x\in[0,\infty)}\frac{|\mathcal{P}^{\mu,\nu}_{n,\alpha}(1;x) - 1|}{1+x^2} = 0 \text{ for } i = 0.$$

For $i = 1$, we have

$$\begin{aligned}\|\mathcal{P}_{n,\alpha}^{\mu,\nu}(e_1;x)-x^1\|_{1+x^2} &= \sup_{x\in[0,\infty)} \frac{\left(\frac{n+2(\alpha-1)}{n+\nu}-1\right)x+\frac{\mu+\lambda+1}{n+\nu}}{1+x^2}\\ &= \left(\frac{n+2(\alpha-1)}{n+\nu}-1\right)\sup_{x\in[0,\infty)}\frac{x}{1+x^2}\\ &+\frac{\mu+\lambda+1}{n+\nu}\sup_{x\in[0,\infty)}\frac{1}{1+x^2},\end{aligned}$$

which implies that $\|\mathcal{P}_{n,\alpha}^{\mu,\nu}(e_1;x)-x^1\|_{1+x^2}\to 0$ an $n\to\infty$.
Finally, for $i = 2$, one obtains the following

$$\begin{aligned}\|\mathcal{P}_{n,\alpha}^{\mu,\nu}(e_2;x)-x^2\|_{1+x^2} &= \sup_{x\in[0,\infty)}\frac{\left|\mathcal{P}_{n,\alpha}^{\mu,\nu}(e_2;x)-x^2\right|}{1+x^2}\\ &= \Bigg|\left(\left(\frac{n(n+4\alpha-3)}{(n+\nu)^2}\right)-2\left(\frac{n+2(\alpha-1)}{n+\nu}\right)\right)x^2\\ &+\Bigg(\left(\frac{n(2\mu+2\lambda+3)+4(\alpha-1)(\mu-1)+(2\lambda+3)(\alpha-1)}{(n+\nu)^2}\right)\\ &-2\left(\frac{\mu+\lambda+1}{n+\nu}\right)\Bigg)x+\left(\frac{\mu(\mu+\lambda+1)+(\lambda+1)(\lambda+2)}{(n+\nu)^2}\right)\Bigg|\\ &\le \left(\left(\frac{n(n+4\alpha-3)}{(n+\nu)^2}\right)-2\left(\frac{n+2(\alpha-1)}{n+\nu}\right)\right)\sup_{x\in[0,\infty)}\frac{x^2}{1+x^2}\\ &+\Bigg(\left(\frac{n(2\mu+2\lambda+3)+4(\alpha-1)(\mu-1)+(2\lambda+3)(\alpha-1)}{(n+\nu)^2}\right)\\ &-2\left(\frac{\mu+\lambda+1}{n+\nu}\right)\Bigg)\sup_{x\in[0,\infty)}\frac{x}{1+x^2}\\ &+\left(\frac{\mu(\mu+\lambda+1)+(\lambda+1)(\lambda+2)}{(n+\nu)^2}\right)\sup_{x\in[0,\infty)}\frac{1}{1+x^2},\end{aligned}$$

which implies that $\|\mathcal{P}_{n,\alpha}^{\mu,\nu}(e_2;x)-x^2\|_{1+x^2}\to 0$ as $n\to\infty$ and this completes the proof of the theorem. □

## 5 A-statistical Approximation

Gadjiev *et al.* [10] were the first to introduce statistical approximation theorems in operators theory. We recall some notation from [10]. Let $A = (a_{nk})$ be a non-negative infinite summability matrix. For a given sequence $x := (x_k)$, the $A$-transform of $x$ denoted by $Ax : ((Ax)_n)$ is defined as

$$(Ax)_n = \sum_{k=1}^{\infty} a_{nk}x_k,$$

provided the series converges for each $n$. A is said to be regular if $\lim(Ax)_n = L$ whenever $\lim x = L$. Then $x = (x_n)$ is said to be a $A$-statistically convergent to $L$ i.e. $st_A - \lim x = L$ if for every $\epsilon > 0$, $\lim_n \sum_{k:|x_k-L|\ge\epsilon} a_{nk} = 0$.
We prove the following theorem.

**Theorem 5.1.** *Let $A = (a_{nk})$ be a non-negative regular summability matrix and $x \ge 0$. Then, we have*

$$st_A - \lim_n \|\mathcal{P}_{n,\alpha}^{\mu,\nu}(f;x)-f\|_{1+x^2} = 0, \textit{ for all } f\in C_{1+x^2}^k[0,\infty).$$

*Proof.* From ([8], p. 191, Th. 3), it is sufficient to show that for $\lambda_1 = 0$,

$$st_A - \lim_n \|\mathcal{P}_{n,\alpha}^{\mu,\nu}(e_i;x) - e_i\|_{1+x^2} = 0, \text{ for } i \in \{0,1,2\}.$$

From Lemma 2.4, we have

$$\begin{aligned}\|\mathcal{P}_{n,\alpha}^{\mu,\nu}(e_1;x) - x\|_{1+x^2} &= \sup_{x\in[0,\infty)} \frac{1}{1+x^2}\left|\left(\frac{n+2(\alpha-1)}{n+\nu} - 1\right)x + \frac{\mu+\lambda+1}{n+\nu}\right| \\ &\leq \left|\frac{n+2(\alpha-1)}{n+\nu} - 1\right| \sup_{x\in[0,\infty)} \frac{x}{1+x^2} \\ &+ \left|\frac{\mu+\lambda+1}{n+\nu}\right| \sup_{x\in[0,\infty)} \frac{1}{1+x^2} \\ &\leq \frac{1}{2}\left|\frac{n+2(\alpha-1)}{n+\nu} - 1\right| + \left|\frac{\mu+\lambda+1}{n+\nu}\right|.\end{aligned}$$

Then, we get

$$st_A - \lim_n \frac{1}{2}\left|\frac{n+2(\alpha-1)}{n+\nu} - 1\right| = st_A - \lim_n \left|\frac{\mu+\lambda+1}{n+\nu}\right| = 0. \tag{5.1}$$

Now, for a given $\epsilon > 0$, we define the following sets

$$\begin{aligned} J_1 &:= \left\{n : \|\mathcal{P}_{n,\alpha}^{\mu,\nu}(e_1;x) - x\| \geq \epsilon\right\}, \\ J_2 &:= \left\{n : \frac{1}{2}\left|\frac{n+2(\alpha-1)}{n+\nu} - 1\right| \geq \frac{\epsilon}{2}\right\}, \\ J_3 &:= \left\{n : \left|\frac{\mu+\lambda+1}{n+\nu}\right| \geq \frac{\epsilon}{2}\right\}. \end{aligned}$$

This implies that $J_1 \subseteq J_2 \cup J_3$, which shows that $\sum_{k_1\in J_1} a_{nk_1} \leq \sum_{k_1\in J_2} a_{nk} + \sum_{k_1\in J_3} a_{nk}$. Hence, from (5.1) we get

$$st_A - \lim_n \|\mathcal{P}_{n,\alpha}^{\mu,\nu}(e_1;x) - x\|_{1+x^2} = 0.$$

Similarly, one can show that

$$st_A - \lim_n \|\mathcal{P}_{n,\alpha}^{\mu,\nu}(e_2;x) - x^2\|_{1+x^2} = 0.$$

This completes the proof of Theorem 5.1. □

## 6 Conclusion

The motive of the present paper was to give a better error estimation of the generalised $\alpha-$Baskakov-Gamma operators with two shifted nodes $0 \leq \mu \leq \nu$. This type of generalisation yields better error estimation for certain functions in comparison to the $\alpha-$Baskakov-Gamma operators. We investigated some approximation results by means of the well-known Korovkin-type theorem and given graphical presentation. We have also calculated the rate of convergence by means of Peetre's K-functional and second-order modulus of continuity. Lastly, we studied the global approximation results.

## References

[1] T. Acar, A. Aral, and S. A. Mohiuddine, Approximation by bivariate $(p,q)$-Bernstein-Kantorovich operators, *Iran. J. Sci. Technol. Trans. Sci.,* **42**, 655–662 (2018).

[2] T. Acar and A. Kajla, Degree of Approximation for bivariate generalized Bernstein type operators, *Results Math.,* **73,** 79 (2018) (in press).

[3] A. M. Acu, T. Acar, and V. A. Radu, Approximation by modified $U_n^\rho$ operators, *Rev. R. Acad. Ciene. Exactas Fis. Nat. Ser. A Math. (Racsam)*, **113,** 2715–2729, (2019).

[4] A. Aral and H. Erbay, Parametric generalization of Baskakov operators, *Math. Commun.*, **24**, 119–131, (2019).

[5] S. N. Berstein, Démonstration du théorém de Weierstrass fondée sur le calcul des probabilitiés, *Comm. Kharkov Math. Soc.* **13**, 1–2, (1912/13).

[6] X. Chen, J. Tan, Z. Liu and J. Xie, Approximation of functions by a new family of generalized Bernstein operators, *J. Math. Anal. Appl.*, **450**, 244–261, (2017).

[7] R. A. DeVore and G. G. Lorentz, Constructive Approximation, *Grudlehren Math. Wiss.*, Springer-Verlag, Berlin, 1993.

[8] O. Duman and C. Orhan, Statistical approximation by positive linear operators, *Studio Math.,* **16**(2), 187–197, (2004).

[9] A. D. Gadjiev, Theorems of the type of P.P. Korovkin's theorems, *Mat. Zametki*, **20** (5), 781–786, (1976) (in Russian), (*Math. Notes*, **20** (5-6), 995–998, (1976) (English Translation).

[10] A. D. Gadjiev and C. Orhan, Some approximation theorems via statistical convergence, *Rocky Mountain J. Math.,* **32** (1), 129–138, (2007).

[11] G. İçöz and B. Çekim, Dunkl generalization of Szász operators via q-calculus, *J. Inequal. Appl.*, 2015:284, 11p, (2015).

[12] G. İçöz and B. Çekim, Stancu-type generalization of Dunkl analogue of Szász-Kantorovich operators, *Math. Met. Appl. Sci.*, DOI: 10.1002/mma.3602, (2015).

[13] A. Kajla, and D. Micláus, Blending type approximation by GBS operators of generalized Bernstein-Durrmeyer type, *Results Math.*, 73:1, (2018).

[14] S. A. Mohiuddine, T. Acar and A. Alotaibi, Construction of a new family of Bernsteion Kantorovich operators, *Math. Meth. Appl. Sci.*, **40,** 7749–7759, (2017).

[15] M. A. Özarslan and H. Aktuğlu, Local approximation for certain King type operators, *Filomat*, **27:1**, 173–181, (2013).

[16] O. Shisha and B. Bond, The degree of convergence of linear positive operators, *Proc. Nat. Acad. Sci. USA*, **60**(4), 1196–1200, (1968).

## Author information

Taqseer Khan, Department of Mathematics, Faculty of Natural Sciences, Jamia Millia Islamia (a central university), New Delhi-110025, India.
E-mail: `tkhan4@jmi.ac.in`

Nadeem Rao, Department of Mathematics, Shree Guru Gobind Singh Tricentenary University, Gurugram-122505, Haryana, India.
E-mail: `nadeemrao1990@gmail.com`

Shuzaat Ali Khan, Department of Mathematics, Faculty of Natural Sciences, Jamia Millia Islamia (a central university), New Delhi-110025, India.
E-mail: `shuzaatkhan786@gmail.com`

Pradeep Malik, Department of Mathematics, Shree Guru Gobind Singh Tricentenary University, Gurugram-122505, Haryana, India.
E-mail: `pradeepmaths@gmail.com`

Received: January 29, 2021
Accepted: July 12, 2021