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FILTERS OF DISTRIBUTIVE LATTICES GENERATED BY DENSE ELEMENTS

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Abstract: The notions of D-filters and $\bar{D}$-ideals are introduced in a distributive lattice. A set of equivalent conditions is given for a D-filter to become a maximal filter. It is proved that the intersection of all maximal ideals is containing $\bar{D}$. A one-to-one correspondence is obtained between the set of all D-filters of a distributive lattice L and the set of all ideals of the lattice of all principal D-filters of L. A set of equivalent conditions is given, in terms of D-filters, for a quasicomplemented lattice to become a Boolean algebra. Finally, the class of D-filters is characterized in terms of co-kernels of a congruence.

1 Introduction

In 1968, the theory of relative annihilators was introduced in lattices by Mark Mandelker [8] who characterized distributive lattices in terms of their relative annihilators. Later many authors introduced the concept of annihilators in the structures of rings as well as lattices and characterized several algebraic structures in terms of annihilators. T.P. Speed [11] and W.H. Cornish [3, 4, 5] made an extensive study of annihilators in distributive lattices and then characterized some algebraic structures like normal lattices and quasicomplemented lattices. In 2013, Rao [9] studied the properties of D-filters in MS-algebras. Later in 2016, Rao and Badawy [10] studied the properties of co-annihilator filters of distributive lattices.

In this note, the concepts of D-filters and $\bar{D}$-ideals are introduced in distributive lattices. A set of equivalent conditions is given for every D-filter of a distributive lattice to become a maximal filter. A set of equivalent conditions is given for every filter of a distributive lattice to become a D-filter. It is proved that the smallest $\bar{D}$-ideal is contained in the set intersection of all maximal ideals of the distributive lattice L. The notion of principal D-filters is introduced in a distributive lattice L and observed that the set of all principal D-filters is a sublattice to the set of all D-filters of L. Later, the class of all D-filters is characterized in terms of principal D-filters. A one-to-one correspondence is obtained between the set of all D-filters of a distributive lattice L and the set of all ideals of the lattice of all principal D-filters of L.

In the final section, we introduce two different congruences on a distributive lattice L: one in terms of $\bar{D}$ and the other in terms of principal D-filters of L. A necessary and sufficient condition is given for every filter of a distributive lattice to become a D-filter.

2 Preliminaries

The reader is referred to [1] and [2] for the elementary notions and notations on distributive lattices. However, some of the preliminary definitions and results of [10] and [11] are presented for the ready reference of the reader.

Definition 2.1. [2] An algebra $(L, \wedge, \vee)$ of type $(2, 2)$ is called a distributive lattice if for all $x, y, z \in L$, it satisfies the following properties $(1), (2), (3)$ and (4) along with (5) or $(5')$

(1) $x \wedge x = x$, $x \vee x = x$,

(2) $x \wedge y = y \wedge x$, $x \vee y = y \vee x$,

(3) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $(x \vee y) \vee z = x \vee (y \vee z)$,
(4) $(x \wedge y) \vee x = x$, $(x \vee y) \wedge x = x$,
(5) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,
(5′) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

A non-empty subset A of a distributive lattice L is called an ideal(filter) of L if $a \vee b \in A(a \wedge b \in A)$ and $a \wedge x \in A(a \vee x \in A)$ whenever $a, b \in A$ and $x \in L$. The set $\mathcal{I}(L)$ of all ideals of $(L, \vee, \wedge, 0)$ forms a complete distributive lattice as well as the set $\mathcal{F}(L)$ of all filters of $(L, \vee, \wedge, 1)$ forms a complete distributive lattice. A proper ideal(filter) M of a lattice is called maximal if there exists no proper ideal(filter) N such that $M \subset N$. The set $(a] = \{x \in L \mid x \leq a\}$ is called the principal ideal generated by a and the set of all principal ideals is a sublattice of $\mathcal{I}(L)$. For any subset S of a lattice L, the set $[S) = \{x \vee (\bigwedge_{i=1}^{n} s_i) \mid x \in L, s_i \in S, n \in \mathbb{N}\}$ is called the principal filter generated by the set S. For $a \in L$, the set $[a) = \{x \in L \mid a \leq x\}$ is called the principal filter generated by the element a and the set of all principal filters is a sublattice of $\mathcal{F}(L)$. For any element a of a distributive lattice $(L, \vee, \wedge, 0)$, the *annihilator* of a is defined as the set $(a)^* = \{ x \in L \mid x \wedge a = 0 \}$.

Lemma 2.2. [11] *For any two elements a, b of a distributive lattice L with 0, we have*

(1) $a \leq b$ *implies* $(b)^* \subseteq (a)^*$,
(2) $(a \vee b)^* = (a)^* \cap (b)^*$,
(3) $(a \wedge b)^{**} = (a)^{**} \cap (b)^{**}$,
(4) $(a)^* = L$ *if and only if* $a = 0$.

An element a of a lattice L is called a *dense element* if $(a)^* = \{0\}$. The set D of all dense elements of a distribute lattice L forms a filter of L. A lattice L with 0 is called *quasicomplemented* [5] if for each $x \in L$, there exists $y \in L$ such that $x \wedge y = 0$ and $x \vee y$ is dense.

Definition 2.3. [10] Let L be a lattice and $S \subseteq L$. Define $S^+ = \{x \in L \mid s \vee x = 1 \text{ for all } s \in S\}$.

Here S^+ is called the *co-annihilator* of S. For $S = \{x\}$, then we denote simply $(x)^+$ for $(\{x\})^+$. Then clearly $L^+ = \{1\}$ and $(1)^+ = L$. For any subset S of a distributive lattice L, it is clear that S^+ is a filter of L.

Lemma 2.4. [10] *For any two elements a, b of a distributive lattice L with 1, we have*

(1) $a \leq b$ *implies* $(a)^+ \subseteq (b)^+$,
(2) $(a \wedge b)^+ = (a)^+ \cap (b)^+$,
(3) $(a \vee b)^{++} = (a)^{++} \cap (b)^{++}$,
(4) $(a)^+ = L$ *if and only if* $a = 1$.

An equivalence relation θ on a lattice L is called congruence if $(x, y) \in \theta, (z, w) \in \theta$ implies $(x \wedge z, y \wedge w) \in \theta$ and $(x \vee z, y \vee w) \in \theta$. For any filter F of a lattice L, define the Co-kernel of the congruence θ_F as $Coker\ \theta_F = \{x \in F \mid (x, 1) \in \theta_F\}$. Throughout this article, all lattices are bounded distributive lattices unless otherwise mentioned.

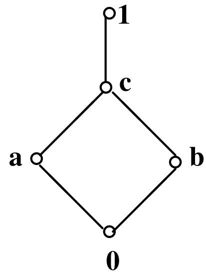
3 *D*-filters of a lattice

In this section, the notion of D-filters is introduced in a lattice L and then some of the properties of D-filters are investigated. A set of equivalent conditions is given for every D-filter of L to become a maximal filter. Some equivalent conditions are also given for the set D of all dense elements to become a maximal filter.

Definition 3.1. A filter F of a lattice L is called a *D-filter* if $D \subseteq F$.

Clearly D is a D-filter and in fact it is the smallest D-filter of L. If L has a unique dense element, precisely 1, then $\{1\}$ is the smallest D-filter of L.

Example 3.2. Consider the lattice $L = \{0, a, b, c, 1\}$ whose Hasse diagram is given in the following figure.



Consider the filters $F_1 = \{a, c, 1\}$ and $F_2 = \{b, c, 1\}$. Clearly $D \subseteq F_1$ and $D \subseteq F_2$. Hence F_1 and F_2 are D-filters of L. But the filter $F_3 = \{1\}$ is not a D-filter of L.

Denote that $\mathcal{F}(L)$ is the set of all filters of a lattice L. It is well-known that $\mathcal{F}(L)$ is a distributive lattice.

Lemma 3.3. *The set of all D-filters of a lattice L is a sublattice to $\mathcal{F}(L)$.*

Proof. Let F and G be two D-filters of a lattice L. Then $D \subseteq F$ and $D \subseteq G$. Hence $D \subseteq F \cap G$. Therefore $F \cap G$ is a D-filter of L. Since $D \subseteq F \subseteq F \vee G$, we get $F \vee G$ is a D-filter. Therefore the set of all D-filters of L is a sublattice to $\mathcal{F}(L)$. □

Definition 3.4. For any non-empty subset S of lattice L, define $\langle S \rangle_D = [S) \vee D$. For $S = \{a\}$, we have $\langle \{a\} \rangle_D = \langle a \rangle_D = [a) \vee D$. Here $\langle a \rangle_D$ is called a principal D-filter generated by a.

Lemma 3.5. *Let S be a non empty subset of a lattice L. Then $\langle S \rangle_D$ is the smallest D-filter of L which is containing S.*

Proof. Let S be a nonempty subset of L. Since $[S)$ and D are filters of L, we get $[S) \vee D$ is a filter of L. Thus $\langle S \rangle_D$ = $[S) \vee D$ is filter of L. Since $D \subseteq \langle S \rangle_D$, we get $\langle S \rangle_D$ is a D-filter of L. Let F be a D-filter of L that is containing S. Then $D \subseteq F$. Since $S \subseteq F$, we get $[S) \subseteq [F) = F$. Now $\langle S \rangle_D$ = $[S) \vee D \subseteq F \vee D$ = F. Therefore $\langle S \rangle_D \subseteq F$. □

The following corollary is a direct consequence of the above lemma:

Corollary 3.6. *For any $a \in L$, $\langle a \rangle_D$ = $[a) \vee D$ is the smallest D-filter containing a.*

Proposition 3.7. *For any lattice L, the set of all D-filters of L satisfies the following properties:*

(1) *If G is a D-filter and F is a filter of L such that $G \subseteq F$, then F is also a D-filter of L;*
(2) *If F_1 and F_2 are two D-filters of L, then both $F_1 \cap F_2$ and $F_1 \vee F_2$ are also D-filters of L*
(3) *If $\{F_\alpha\}_{\alpha \in \Delta}$ be an indexed family of D-filters of L, then $\prod_{\alpha \in \Delta} F_\alpha$ is also a D-filter of L.*

Proof. (1) and (2) are clear from the definition of D-filters of L.
(3) Assume that F_α is a D-filter for each $\alpha \in \Delta$. Hence $D \subseteq F_\alpha$ for each $\alpha \in \Delta$. It is enough to prove that $\prod_{\alpha \in \Delta} D_\alpha \subseteq \prod_{\alpha \in \Delta} F_\alpha$, where $D_\alpha = D$ for each α. Let $(d_1, d_2, d_3, d_4..., d_n, ...) \in \prod_{\alpha \in \Delta} F_\alpha$, where $d_1, d_2, d_3, d_4, \ldots, d_n, \ldots \in D_\alpha = D$. Since each F_α is a D-filter, we get $D = D_\alpha \subseteq F_\alpha$ for each α. Hence $d_1, d_2, \ldots, d_n, \ldots \in F_\alpha$ for each α. Thus $(d_1, d_2, \ldots, d_n, \ldots) \in \prod_{\alpha \in \Delta} F_\alpha$. Therefore $\prod_{\alpha \in \Delta} F_\alpha$ is a D-filter of L. □

Proposition 3.8. *Every maximal filter of a lattice L is a D-filter.*

Proof. Let M be a maximal filter of L. Let $x \in D$. Then $(x)^*$ = $\{0\}$. Let us suppose that $x \notin M$. Then $M \vee [x)$ = L. Since $0 \in L$ = $M \vee [x)$, we get $0 = a \wedge x$ for some $0 \neq a \in L$. By the definition of $(x)^*$, we get $a \in (x)^*$ = $\{0\}$. Hence $a = 0$, which is a contradiction to $0 \neq a \in L$. Thus $x \in M$. Therefore M is a D-filter of L. □

Corollary 3.9. *For any lattice L, $D \subseteq \bigcap\{M \mid M$ is a maximal filter of $L\}$.*

Theorem 3.10. *In any lattice L, the following assertions are equivalent:*

(1) *Every D-filter is maximal;*
(2) *D is a maximal filter of L;*
(3) *Every element of L is dense;*
(4) *L contains a unique D-filter.*

Proof. $(1) \Rightarrow (2)$: Assume that every D-filter is maximal. Since D is a D-filter, we get D is a maximal filter.
$(2) \Rightarrow (3)$: Assume that D is a maximal filter of L. Suppose $(x)^* \neq \{0\}$ for some $0 \neq x \in L$. Then $x \notin D$. Hence $D \subset D \vee [x)$. Since D is maximal, we get $D \vee [x) = L$. Thus $0 \in D \vee [x)$. Hence $0 = d \wedge x$ for some $d \in D$. Thus $x \in (d)^* = \{0\}$. Hence $x = 0$, which is a contradiction. Therefore $(x)^* = \{0\}$ for all $0 \neq x \in L$.
$(3) \Rightarrow (4)$: Assume that every element of L is dense. Suppose L has a proper D-filter M such that $D \neq M$. Since M is a D-filter, we get $D \subset M$. Choose $x \in M - D$. Since M is proper, we get $x \neq 0$. By condition (3), x should be a dense element. Hence $x \in D$, which is a contradiction to $x \in M - D$. Therefore $D = M$. Thus L has a unique D-filter, precisely D.
$(4) \Rightarrow (1)$: Let F be a proper D-filter of L. Suppose G is a proper filter such that $F \subset G$. Since $D \subseteq F \subset G$, we get that G is also a D-filter. Since L has the unique D-filter D, we must have $D = F = G$. Therefore F is a maximal filter of L. □

Theorem 3.11. *The following assertions are equivalent in a lattice L:*

(1) *Every filter is a D-filter;*
(2) *Every principal filter is a D-filter;*
(3) *L contains a unique dense element.*

Proof. $(1) \Rightarrow (2)$: It is clear.
$(2) \Rightarrow (3)$: Assume that every principal filter is a D-filter of L. Since $1 \in F$, we get $[1)$ is a D-filter of L. Hence $D \subseteq [1)$. Thus $D = \{1\}$. Thus L has a unique dense element, precisely 1.
$(3) \Rightarrow (1)$: Let F be a filter of L. By (3), we get $D = \{1\}$. Hence $D = \{1\} \subseteq F$. Therefore F is a D-filter of L. □

4 $\bar{D}$-ideals of a lattice

In this section, the notion of $\bar{D}$-ideals is introduced in a lattice L. A set of equivalent conditions is given for a $\bar{D}$-ideal of a lattice L to become a maximal ideal. It is proved that the intersection of all maximal ideals of a lattice is containing the set $\bar{D}$.

Definition 4.1. Let $\bar{D}$ be the set of all dual dense elements of L. i.e $\bar{D} = \{x \in L \mid (x)^+ = \{1\}\}$.

We know that an element $a \in L$ is called *dual dense* if $(a)^+ = \{1\}$. Clearly 0 is the dual dense element. Since 1 is the only element such that $0 \vee 1 = 1$, we get $(0)^+ = \{1\}$.

Lemma 4.2. *In any lattice L, $\bar{D}$ is an ideal of L.*

Proof. Clearly $0 \in L$ and $(0)^+ = \{1\}$. Hence $\bar{D} \neq \emptyset$. Let $x, y \in X$. Suppose $x, y \in \bar{D}$. Then we get $(x \vee y)^{++} = (x)^{++} \cap (y)^{++} = L \cap L = L$. Thus $(x \vee y)^{+++} = L^+ = \{1\}$. Hence $(x \vee y)^+ = \{1\}$. Therefore $x \vee y \in \bar{D}$. Let $x \in \bar{D}$ and $y \leq x$. Then $(y)^+ \subseteq (x)^+$. Hence $y \in \bar{D}$. Therefore $\bar{D}$ is the ideal of L. □

Definition 4.3. An ideal I of a lattice L is called a *$\bar{D}$-ideal* of L if $\bar{D} \subseteq I$.

Lemma 4.4. *Every maximal ideal of lattice L is a $\bar{D}$-ideal.*

Proof. Let M be a maximal ideal of L. Let $x \in \bar{D}$. Then $(x)^+ = \{1\}$. Let us suppose that $x \notin M$. Then $M \vee (x] = L$. Since $1 \in L = M \vee (x]$, we get $1 = a \vee x$ for some $1 \neq a \in L$. From the definition of $(x)^+$, we get $a \in (x)^+ = \{1\}$. Hence $a = 1$, which is a contradiction to that $1 \neq a \in L$. Hence $x \in M$. Thus $\bar{D} \subseteq M$. Therefore M is a $\bar{D}$-ideal. □

Corollary 4.5. *For any lattice L, $\bar{D} \subseteq \bigcap\{M \mid M \text{ is maximal ideal of } L\}$.*

Proposition 4.6. *For any lattice L, the set of all $\bar{D}$-ideals of L satisfies the following properties:*

(1) *If J is a $\bar{D}$-ideal and I is an ideal of L such that $J \subseteq I$, then I is also a $\bar{D}$-ideal of L;*
(2) *If I, J are two $\bar{D}$-ideals of L, then both $I \cap J$ and $I \vee J$ are also $\bar{D}$-ideals of L;*
(3) *If $\{I_\alpha\}_{\alpha\in\Delta}$ be an indexed family of $\bar{D}$-ideals of L, then $\prod\limits_{\alpha\in\Delta} I_\alpha$ is also a $\bar{D}$-ideal of L.*

Proof. (1) and (2) are clear from the definition of $\bar{D}$-ideals of L.
(3) Assume that I_α is $\bar{D}$-ideal for each $\alpha \in \Delta$. Hence $\bar{D} \subseteq I_\alpha$ for each $\alpha \in \Delta$. It is enough to prove that $\prod\limits_{\alpha\in\Delta} \bar{D} \subseteq \prod\limits_{\alpha\in\Delta} I_\alpha$. Let $(d_1, d_2, d_3, d_4, \ldots, d_n, \ldots) \in \prod\limits_{\alpha\in\Delta} I_\alpha$, where $(d_1, d_2, d_3, d_4 \ldots, d_n, \ldots) \in \bar{D}_\alpha$. Since each I_α is a $\bar{D}$-ideal, we get $D = \bar{D}_\alpha \subseteq I_\alpha$, where $d_1, d_2, \ldots, d_n, \ldots \in I_\alpha$. Thus $(d_1, d_2, \ldots, d_n, \ldots) \in \prod\limits_{\alpha\in\Delta} I_\alpha$. Therefore $\prod\limits_{\alpha\in\Delta} I_\alpha$ is a $\bar{D}$-ideal of L. □

Theorem 4.7. *In any lattice L, the following assertions are equivalent:*

(1) *Every $\bar{D}$-ideal is a maximal ideal of L;*
(2) *$\bar{D}$ is a maximal ideal of L;*
(3) *Every element of L is dual dense;*
(4) *L contains a unique $\bar{D}$-ideal.*

Proof. $(1) \Rightarrow (2)$: Assume that every $\bar{D}$-ideal of the lattice L is a maximal ideal. Since $\bar{D}$ is a $\bar{D}$-ideal of L, we get $\bar{D}$ is the maximal ideal of L.
$(2) \Rightarrow (3)$: Assume that $\bar{D}$ is a maximal ideal of L. Suppose $(x)^+ \neq \{1\}$ for some $1 \neq x \in L$. Hence $x \notin \bar{D}$. Thus $\bar{D} \subset \bar{D} \vee (x] = L$. Since $1 \in L$, we get $1 \in \bar{D} \vee (x]$. Hence $1 = d \vee x$ for some $d \in \bar{D}$. Thus $x \in (d)^+ = \{1\}$. Hence $x = 1$, which is a contradiction. Therefore $(x)^+ = \{1\}$ for all $1 \neq x \in L$.
$(3) \Rightarrow (4)$: Assume that every element of L is dual dense. Suppose L has a proper $\bar{D}$-ideal M such that $\bar{D} \neq M$. Since M is a $\bar{D}$-ideal, we get $\bar{D} \subset M$. Choose $x \in M - \bar{D}$. Since M is proper, we get $x \neq 1$. By condition (3), x should be a dual dense element. Hence $x \in \bar{D}$, which is a contradiction to $x \in M - \bar{D}$. Therefore $\bar{D} = M$. Thus L has a unique $\bar{D}$-ideal, precisely $\bar{D}$.
$(4) \Rightarrow (1)$: Let I be a proper $\bar{D}$-ideal of L. Suppose J is proper ideal of L such that $I \subset J$. Since $\bar{D} \subseteq I \subset J$, we get J is also a $\bar{D}$-ideal. Since L has the unique $\bar{D}$-ideal $\bar{D}$, we must have $\bar{D} = I = J$. Therefore I is a maximal ideal of L. □

Theorem 4.8. *The following assertions are equivalent in a lattice L:*

(1) *Every ideal is a $\bar{D}$-ideal;*
(2) *Every principal ideal is a $\bar{D}$-ideal;*
(3) *L contains a unique dual dense element.*

Proof. $(1) \Rightarrow (2)$: It is clear.
$(2) \Rightarrow (3)$: Assume that every principal ideal of L is a $\bar{D}$-ideal. Since $0 \in I$, we get that $(0]$ is a $\bar{D}$-ideal of L. Hence $\bar{D} \subseteq (0]$. Thus $\bar{D} = \{0\}$. Therefore L has a unique dual dense element, precisely 0.
$(3) \Rightarrow (1)$: Let I be an ideal of L. By (3), we get $\bar{D} = \{0\}$. Hence $\bar{D} = \{0\} \subseteq I$. Therefore I is a $\bar{D}$-ideal of L. □

Theorem 4.9. *The following statements are equivalent in a lattice L:*

(1) *Every element of L is dual dense;*
(2) *$\bar{D}$ is maximal;*
(3) *L contains a unique $\bar{D}$-ideal.*

Proof. $(1) \Rightarrow (2)$: Assume that every element of L is dual dense. Suppose there exists a proper ideal M of L such that $\bar{D} \subset M$. Choose $x \in M - \bar{D}$. By (1), x is dual dense. Hence $x \in \bar{D}$, which is a contradiction. Therefore $\bar{D}$ is maximal.
$(2) \Rightarrow (3)$: Assume that $\bar{D}$ is a maximal ideal of L. Let us suppose that L has a $\bar{D}$-ideal M such

that $\bar{D} \neq M$. Since M is a $\bar{D}$-ideal, we get $\bar{D} \subset M$. Choose $x \in M - \bar{D}$. Then $\bar{D} \vee (x] = L$. Hence $d \vee x = 1$ for some $d \in \bar{D}$. Thus $x \in (d)^+$ and $x \neq 1$. Hence $d \notin \bar{D}$, which is a contradiction. Therefore L has a unique $\bar{D}$-ideal, precisely $\bar{D}$.
(3) $\Rightarrow$ (1): Assume that L has a unique $\bar{D}$-ideal, precisely $\bar{D}$. Let $1 \neq x \in L$. Suppose $(x)^+ \neq \{1\}$. Then $x \notin \bar{D}$. Hence $\bar{D} \subset \bar{D} \vee (x]$. Thus $\bar{D} \vee (x]$ is a $\bar{D}$-ideal, which is different from $\bar{D}$. Hence $x \in \bar{D}$. Therefore every element of L is dual dense. □

5 Principal D-filters

In this section, the concept of principal D-filters is introduced in a lattice L and its properties are studied. A one-to-one correspondence is obtained between the set of all D-filters of L and the set of all ideals of $\mathcal{PDF}(L)$, where $\mathcal{PDF}(L)$ is the lattice of all principal D-filters of L.

Lemma 5.1. *Let L be a lattice and $a, b \in L$. Then we have*

(1) *$a \in D$ if and only if $\langle a \rangle_D = D$,*
(2) *If $a \leq b$, then $\langle b \rangle_D \subseteq \langle a \rangle_D$,*
(3) *If $b \in \langle a \rangle_D$, then $\langle b \rangle_D \subseteq \langle a \rangle_D$,*
(4) *$\langle a \vee b \rangle_D = \langle a \rangle_D \cap \langle b \rangle_D$,*
(5) *$\langle a \wedge b \rangle_D = \langle a \rangle_D \vee \langle b \rangle_D$.*

Proof. (1) Let $a, b \in L$. Suppose $a \in D$. Then $(a)^* = \{0\}$. Now $D \subseteq [a) \vee D = \langle a \rangle_D$. Hence $D \subseteq \langle a \rangle_D$. Conversely, let $x \in \langle a \rangle_D = [a) \vee D$. Then $x = y \wedge d$ for some $y \in [a)$ and $d \in D$. Since $y \in [a)$, we get $y = t \vee a$ for some $t \in L$. Now $(x)^{**} = (y \wedge d)^{**} = [(t \vee a) \wedge d]^{**} = (t \vee a)^{**} \cap (d)^{**} = (t \vee a)^{**} \cap L = (t \vee a)^{**} = [(t)^* \cap (a)^*]^* = [(t)^* \cap \{0\}] = \{0\}^* = L$. Thus $(x)^{***} = L^* = \{0\}$. Hence $(x)^* = \{0\}$, which means $x \in D$. Thus $\langle a \rangle_D \subseteq D$. There fore $\langle a \rangle_D = D$. Conversely, assume that $\langle a \rangle_D = D$. Then $[a) \vee D = D$, which gives $a \in [a) \subseteq D$.
(2) Let $a \leq b$. Then $[b) \subseteq [a)$. Now $\langle a \rangle_D \cap \langle b \rangle_D = ([a) \vee D) \cap ([b) \vee D) = ([a) \cap [b)) \vee D = [b) \vee D = \langle b \rangle_D$. Hence $\langle a \rangle_D \cap \langle b \rangle_D = \langle b \rangle_D$. Therefore $\langle b \rangle_D \subseteq \langle a \rangle_D$.
(3) Suppose $b \in \langle a \rangle_D$. Let $t \in \langle b \rangle_D$. Then $t \geq b \wedge d_1$ for some $d_1 \in D$. Since $b \in \langle a \rangle_D$, we get $b \geq a \wedge d_2$ for some $d_2 \in D$. Since $d_1, d_2 \in D$, we get $d_1 \wedge d_2 \in D$. Then $t \geq b \wedge d_1 \geq (a \wedge d_2) \wedge d_1 = a \wedge (d_1 \wedge d_2) \in \langle a \rangle_D$. Hence $t \in \langle a \rangle_D$. Therefore $\langle b \rangle_D \subseteq \langle a \rangle_D$.
(4) Let $a, b \in L$. Then $\langle a \vee b \rangle_D = [a \vee b) \vee D = ([a) \cap [b)) \vee D = ([a) \vee D) \cap ([b) \vee D) = \langle a \rangle_D \cap \langle b \rangle_D$.
(5) Let $a, b \in L$. Then $\langle a \wedge b \rangle_D = [a \wedge b) \vee D = ([a) \vee [b)) \vee D = ([a) \vee D) \vee ([b) \vee D) = \langle a \rangle_D \vee \langle b \rangle_D$. □

The filter $\langle a \rangle_D$ is called the *principal D-filter* of L. Then the following proposition is clear from the above lemma.

Proposition 5.2. *The set $\mathcal{PDF}(L)$ of all principal D-filters of a lattice L forms a bounded distributive lattice in which the smallest element is D and the greatest element is L.*

Definition 5.3. For any D-filter F of a lattice L, define $F^e = \{\langle a \rangle_D \mid a \in F\}$.

Lemma 5.4. *For any D-filter F of a lattice L, F^e is an ideal of $\mathcal{PDF}(L)$ such that $D^e \subseteq F^e$.*

Proof. Since $1 \in F$, we get $\langle 1 \rangle_D \in F^e$. Hence F^e is non-empty. Let $\langle a \rangle_D, \langle b \rangle_D \in F^e$. Then $a, b \in F$. Since F is a filter, we get $a \wedge b \in F$. Hence $\langle a \rangle_D \vee \langle b \rangle_D = \langle a \wedge b \rangle_D \in F^e$. Let $\langle a \rangle_D \in F^e$ and $\langle x \rangle_D \in \mathcal{PDF}(L)$. Then $a \in F$ and $x \in L$. Since $a \vee x \in F$, we get $\langle a \rangle_D \cap \langle x \rangle_D = \langle a \vee x \rangle_D \in F^e$. Therefore F^e is an ideal of $\mathcal{PDF}(L)$. It is enough to prove that $D^e \subseteq F^e$. Let $\langle x \rangle_D \in D^e$. Then $x \in D \subseteq F$. Hence $\langle x \rangle_D \in F^e$. Therefore $D^e \subseteq F^e$. □

Lemma 5.5. *Let L be a lattice. For any ideal K of the lattice $\mathcal{PDF}(L)$, define $K^c = \{a \in L \mid \langle a \rangle_D \in K\}$. Then K^c is a D-filter of L.*

Proof. Let K be an ideal of $\mathcal{PDF}(L)$. Then $\langle 1 \rangle_D \in K$. Hence $1 \in K^c$. Thus K is non-empty. Let $a, b \in K^c$. Then $\langle a \rangle_D, \langle b \rangle_D \in K$. Since K is an ideal, we get $\langle a \wedge b \rangle_D = \langle a \rangle_D \vee \langle b \rangle_D \in K$. Thus $a \wedge b \in K^c$. Let $a \in K^c$ and $x \in L$. Then $\langle a \rangle_D \in K$ and $\langle x \rangle_D \in \mathcal{PDF}(L)$. Now

$\langle a \vee x \rangle_D = \langle a \rangle_D \cap \langle x \rangle_D \in K$. Hence $a \vee x \in K^c$. Therefore K^c is a filter of L. It is enough to prove that K^c is a D-filter. Let $x \in D$. Then $\langle x \rangle_D = D = \langle 1 \rangle_D \in K$. Since $\langle 1 \rangle_D$ is the least element of $\mathcal{PDF}(L)$, we get $x \in K^c$. Therefore $D \subseteq K^c$. □

Lemma 5.6. *In any lattice L, we have the following properties:*

(1) *For any two D-filters F, G of L, $F \subseteq G$ implies $F^e \subseteq G^e$,*
(2) *For any two ideals I, J of $\mathcal{PDF}(L)$, $I \subseteq J$ implies $I^c \subseteq J^c$,*
(3) *For any D-filter F of L, $(F^e)^c = F$,*
(4) *For any ideal I of $\mathcal{PDF}(L)$, $(I^c)^e = I$.*

Proof. (1) Let F and G be two D-filters of L. Suppose $F \subseteq G$. Let $\langle a \rangle_D \in F^e$. Then $a \in F \subseteq G$. Thus $\langle a \rangle_D \in G^e$. Therefore $F^e \subseteq G^e$.
(2) Let I and J be two ideals of $\mathcal{PDF}(L)$. Suppose $I \subseteq J$. Let $a \in I^c$. Then $\langle a \rangle_D \in I \subseteq J$. Hence $\langle a \rangle_D \in J$, which means $a \in J^c$. Therefore $I^c \subseteq J^c$.
(3) Let F be a D-filter of L. Now $x \in (F^e)^c \Leftrightarrow \langle x \rangle_D \in F^e \Leftrightarrow x \in F$. Therefore $(F^e)^c = F$.
(4) Let I be an ideal of $\mathcal{PDF}(L)$. Then, we get $\langle x \rangle_D \in (I^c)^e \Leftrightarrow x \in I^c \Leftrightarrow \langle x \rangle_D \in I$. Therefore $(I^c)^e = I$. □

Theorem 5.7. *Let L be a lattice. Then there is a one-to-one correspondence between the set of all D-filters of L and the set of all ideals of $\mathcal{PDF}(L)$.*

Proof. It is an immediate consequence of the above results. □

Theorem 5.8. *For any filter F of a lattice L, the following assertions are equivalent:*

(1) *F is a D-filter;*
(2) *For any $x \in L, x \in F$ implies $\langle x \rangle_D \subseteq F$;*
(3) *For any $x, y \in L$, $\langle x \rangle_D = \langle y \rangle_D$ and $x \in F$ imply that $y \in F$;*
(4) $F = \bigcup_{x \in F} \langle x \rangle_D$.

Proof. (1) $\Rightarrow$ (2): Assume that F is a D-filter of L. Let $x \in F$ and $t \in \langle x \rangle_D = [x) \vee D$. The $t \geq x \wedge d$ for some $d \in D \subseteq F$. Since $x \in F$ and $d \in F$, we get $t \in F$. Therefore $\langle x \rangle_D \subseteq F$.
(2) $\Rightarrow$ (3): Let $x, y \in L$ be such that $\langle x \rangle_D = \langle y \rangle_D$. Suppose $x \in F$. Then by (2), we get $\langle x \rangle_D \subseteq F$. Hence $y \in \langle y \rangle_D = \langle x \rangle_D \subseteq F$. Therefore $y \in F$
(3) $\Rightarrow$ (4): Assume the condition (3). Clearly $[x) \subseteq \langle x \rangle_D$ for all $x \in F$. Hence $F = \bigcup_{x \in F} [x) \subseteq \bigcup_{x \in F} \langle x \rangle_D$. Therefore $F \subseteq \bigcup_{x \in F} \langle x \rangle_D$. Conversely, let $x \in F$ and $y \in \langle x \rangle_D$. Then $\langle y \rangle_D \subseteq \langle x \rangle_D$. Therefore $\langle y \vee x \rangle_D = \langle y \rangle_D \cap \langle x \rangle_D = \langle y \rangle_D$. Since $x \in F$ and F is a filter, we get $y \vee x \in F$. Since $\langle y \vee x \rangle_D = \langle y \rangle_D$ and $y \vee x \in F$, by the assumed condition (3), we get $y \in F$. Hence $\langle x \rangle_D \subseteq F$ for all $x \in F$. Therefore $\bigcup_{x \in F} \langle x \rangle_D \subseteq F$. □

We now introduce the notion of perfect lattices.

Definition 5.9. A lattice L is called a *perfect lattice* if $\langle x \rangle_D = \langle y \rangle_D$ then $x = y$ for all $x, y \in L$.

Example 5.10. Every Boolean algebra is a perfect lattice. Indeed, if L is a Boolean algebra, then L has a unique dense element. Hence every filter is a D-filter. Let $x, y \in L$ be such that $\langle x \rangle_D = \langle y \rangle_D$. Suppose $x \neq y$. Then there exists a prime filter P of L such that $x \in P$ and $y \notin P$. Then by the hypothesis, P is a D-filter. Since $\langle x \rangle_D = \langle y \rangle_D$ and $x \in P$, we get $y \in P$, which is contradiction. Hence $x = y$. Therefore L is a perfect lattice.

The converse of the above statement is not true. i.e. every perfect lattice need not be a Boolean algebra. However, in the following theorem, we give some equivalent conditions for a perfect lattice to become a Boolean algebra. For this, we need the following lemma:

Theorem 5.11. *Let L be a quasicomplemented lattice. Then the following assertions are equivalent:*

(1) *L is a Boolean algebra;*
(2) *L is a perfect lattice;*
(3) *L contains a unique dense element.*

Proof. (1) $\Rightarrow$ (2): From the above example, it is clear.

(2) $\Rightarrow$ (3): Assume that L is a perfect lattice. Suppose a and b are two different dense elements in L. Then by Lemma 4.1(1), we get $\langle a\rangle_D = D = \langle b\rangle_D$. Since L is perfect lattice, we must have $a = b$. Therefore L has a unique dense element.

(3) $\Rightarrow$ (1): Assume that L has a unique dense element, precisely 1. Let $x \in L$. Since L is quasi-complimented, there exists $x' \in L$ such that $x \wedge x' = 0$ and $x \vee x'$ is dense. Hence $x \vee x' = 1$. Thus x' is the complement of x in L. Therefore L is a Boolean algebra. □

Theorem 5.12. *The following assertions are equivalent in a lattice L:*

(1) *L is a perfect lattice;*
(2) *Every filter is a D-filter;*
(3) *Every proper filter is a D-filter;*
(4) *Every prime filter is a D-filter.*

Proof. (1) $\Rightarrow$ (2): Assume that L is a perfect lattice. Let F be a filter of L. Let $x, y \in L$ be such that $\langle x\rangle_D = \langle y\rangle_D$ and $x \in F$. Since L is perfect, we get $y = x \in F$. Thus F is a D-filter of L.

(2) $\Rightarrow$ (3): It is clear.

(3) $\Rightarrow$ (4): It is clear.

(4) $\Rightarrow$ (1): Let $x, y \in L$ be such that $\langle x\rangle_D = \langle y\rangle_D$. Suppose $x \neq y$. Then by well-known Stone's theorem, there exist a prime filter P such that $x \in P$ and $y \notin P$. By the hypothesis, P is a D-filter. Since $x \in P$, we must have $y \in P$, which is contradiction. Hence $x = y$ and therefore L is a perfect lattice. □

6 Congruences and D-filters

In this section, we introduce two congruences: one in terms of $\bar{D}$ and the other in terms of the principal D-filters of a lattice L. A necessary and sufficient condition is given for any filter of a lattice L to become a D-filter.

Definition 6.1. Let L be a lattice and $x, y \in L$. Define a binary relation θ on L by $(x, y) \in \theta$ if and only if $(x)^+ = (y)^+$.

Lemma 6.2. *For any lattice L, the relation θ defined above is a congruence on L.*

Proof. Clearly θ is an equivalence relation on L. Let $x, y \in L$ be such that $(x, y) \in \theta$. Then $(x)^+ = (y)^+$. For any $a \in L$, we have $(x\wedge a)^{++} = (x)^{++} \vee (a)^{++} = (y)^{++} \vee (a)^{++} = (y\wedge a)^{++}$. Hence $(x \wedge a)^{+++} = (y \wedge a)^{+++}$, which gives $(x \wedge a)^+ = (y \wedge a)^+$. Therefore $(x \wedge a, y \wedge a) \in \theta$. For any $t \in L$, we get $t \in (x \vee a)^+ \Leftrightarrow t \vee (x \vee a) = 1 \Leftrightarrow t \vee (a \vee x) = 1 \Leftrightarrow (t \vee a) \vee x = 1 \Leftrightarrow (t \vee a) \in (x)^+ = (y)^+ \Leftrightarrow t \vee a \vee y = 1 \Leftrightarrow t \vee (y \vee a) = 1 \Leftrightarrow t \in (y \vee a)^+$. Thus $(x \vee a)^+ = (y \vee a)^+$. Hence $(x \vee a, y \vee a) \in \theta$. Therefore θ is a congruence on L. □

Definition 6.3. Let L be a lattice and $x, y \in L$. Define a binary relation $\theta_{\bar{D}}$ on L by $(x, y) \in \theta_{\bar{D}}$ if and only if $x \vee d = y \vee d$ for some $d \in \bar{D}$.

Lemma 6.4. *Let L be a lattice. Then the relation $\theta_{\bar{D}}$ defined above is a congruence on L.*

Proof. Clearly $\theta_{\bar{D}}$ is an equivalence relation on L. Let $(x, y) \in \theta_{\bar{D}}$. Then $x \vee d = y \vee d$ for some $d \in \bar{D}$. For $a \in L$, $(x \vee a) \vee d = (a \vee x) \vee d = a \vee (x \vee d) = a \vee (y \vee d) = (a \vee y) \vee d$. Hence $(x \vee a) \vee d = (y \vee a) \vee d$. Therefore $(x \vee a, y \vee a) \in \theta_{\bar{D}}$. For $a \in L, (x \wedge a) \vee d = (x \vee d) \wedge (a \vee d) = (y \vee d) \wedge (a \vee d) = (y \wedge a) \vee d$. Hence $(x \wedge a, y \wedge a) \in \theta_{\bar{D}}$. Therefore $\theta_{\bar{D}}$ is a congruence on L. □

Theorem 6.5. *Let L be a lattice such that for each $x \in L$, there exists $x' \in L$ such that $(x)^{++} = (x')^+$. Let θ and $\theta_{\bar{D}}$ be the congruences on L as defined above. Then $\theta = \theta_{\bar{D}}$.*

Proof. Let $x, y \in L$. Suppose $(x, y) \in \theta$. Then $(x)^+ = (y)^+$. By the assumption, there exists $x' \in L$ such that $(x)^{++} = (x')^+$. Since $(x, y) \in \theta$, we get $(x, x \vee y) \in \theta$. Hence $(x)^+ = (x \vee y)^+$. Put $d = (x \vee y) \wedge x'$. Now $x \vee d = x \vee [(x \vee y) \wedge x'] = [x \vee (x \vee y)] \wedge (x \vee x') = (x \vee y) \wedge 1 = x \vee y$

and $y\vee d=y\vee[(x\vee y)\wedge x']=[y\vee(x\vee y)]\wedge(y\vee x')=(x\vee y)\wedge(y\vee x')=(x\vee y)\wedge 1=x\vee y$. Hence $x\vee d=y\vee d$. Now $(d)^+=((x\vee y)\wedge x')^+=(x\vee y)^+\cap(x')^+=(x)^+\cap(x')^+=(x)^+\cap(x)^{++}=\{1\}$. Hence $d\in\bar{D}$. Then $(x,y)\in\theta_{\bar{D}}$. Therefore $\theta\subseteq\theta_{\bar{D}}$.

Conversely, let $(x,y)\in\theta_{\bar{D}}$. Then $x\vee d=y\vee d$ for some $d\in\bar{D}$. Hence $(x)^{++}=(x)^{++}\cap L=(x)^{++}\cap(d)^{++}=(x\vee d)^{++}=(y\vee d)^{++}=(y)^{++}\cap(d)^{++}=(y)^{++}\cap L=(y)^{++}$, which gives $(x)^+=(y)^+$. Hence $(x,y)\in\theta$. Therefore $\theta_{\bar{D}}\subseteq\theta$. □

Definition 6.6. For any filter F of a lattice L, define a binary relation θ_F on L by $(x,y)\in\theta_F$ if and only if $[x)\vee\langle a\rangle_D=[y)\vee\langle a\rangle_D$ for some $a\in F$.

Lemma 6.7. *For any filter F of a lattice L, the relation θ_F defined above is a congruence on L.*

Proof. Clearly θ_F is reflexive and symmetric. Let $(x,y),(y,z)\in\theta_F$. Then $[x)\vee\langle a\rangle_D=[y)\vee\langle a\rangle_D$ and $[y)\vee\langle b\rangle_D=[z)\vee\langle b\rangle_D$ for some $a,b\in F$. Since $a,b\in F$, we get $a\wedge b\in F$. Now

$$\begin{aligned}[x)\vee\langle a\wedge b\rangle_D &= [x)\vee\{\langle a\rangle_D\vee\langle b\rangle_D\}\\ &= \{[x)\vee\langle a\rangle_D\}\vee\langle b\rangle_D\\ &= \{[y)\vee\langle a\rangle_D\}\vee\langle b\rangle_D\\ &= \{[y)\vee\langle b\rangle_D\}\vee\langle a\rangle_D\\ &= \{[z)\vee\langle b\rangle_D\}\vee\langle a\rangle_D\\ &= [z)\vee\{\langle b\rangle_D\vee\langle a\rangle_D\}\\ &= [z)\vee\langle a\wedge b\rangle_D.\end{aligned}$$

Hence $(x,z)\in\theta_F$. Thus θ_F is an equivalence relation on L. Let $(x,y),(z,w)\in\theta_F$. Then there exist $a,b\in F$ such that $[x)\vee\langle a\rangle_D=[y)\vee\langle a\rangle_D$ and $[z)\vee\langle b\rangle_D=[w)\vee\langle b\rangle_D$. Since F is a filter, we get $a\wedge b\in F$. Now, we have

$$\begin{aligned}[x\vee z)\vee\langle a\wedge b\rangle_D &= \{[x)\cap[z)\}\vee\ \{\langle a\rangle_D\vee\langle b\rangle_D\}\\ &= \{[x)\vee(\langle a\rangle_D\vee\langle b\rangle_D)\}\cap\ \{[z)\vee(\langle a\rangle_D\vee\langle b\rangle_D)\}\\ &= \{([x)\vee\langle a\rangle_D)\vee\langle b\rangle_D\}\cap\ \{([z)\vee\langle b\rangle_D)\vee\langle a\rangle_D\}\\ &= \{([y)\vee\langle a\rangle_D)\vee\langle b\rangle_D\}\cap\ \{([w)\vee\langle b\rangle_D)\vee\langle a\rangle_D\}\\ &= \{[y)\vee(\langle a\rangle_D\vee\langle b\rangle_D)\}\cap\ \{[w)\vee(\langle a\rangle_D\vee\langle b\rangle_D)\}\\ &= \{[y)\cap\ [w)\}\vee\{\langle a\rangle_D\vee\langle b\rangle_D\}\\ &= [y\vee w)\vee\langle a\wedge b\rangle_D.\end{aligned}$$

Thus $(x\vee z,y\vee w)\in\theta_F$. Similarly, we can prove that $(x\wedge z,y\wedge w)\in\theta_F$. Therefore θ_F is a congruence on L. □

For any filter F of a lattice L, the co-kernel of the congruence θ_F is given by $Coker\ \theta_F=\{x\in L\mid(x,1)\in\theta_F\}=\{x\in L\mid[x)\vee\langle a\rangle_D=\langle a\rangle_D\text{ for some }a\in F\}$.

Lemma 6.8. *For any filter F of a lattice L, $Coker\ \theta_F$ is a filter of L.*

Proof. Clearly $1\in Coker\ \theta_F$. Let $x,y\in\theta_F$. Then $(x,1)\in\theta_F$ and $(y,1)\in\theta_F$. Hence $[x)\vee\langle a\rangle_D=\langle a\rangle_D$ for some $a\in F$ and $[y)\vee\langle b\rangle_D=\langle b\rangle_D$ for some $b\in F$. Now

$$\begin{aligned}[x\wedge y)\vee\langle a\wedge b\rangle_D &= [x\wedge y)\vee(\langle a\rangle_D\vee\langle b\rangle_D)\\ &= ([x)\vee[y))\vee(\langle a\rangle_D\vee\langle b\rangle_D)\\ &= ([x)\vee\langle a\rangle_D)\vee([y)\vee\langle b\rangle_D)\\ &= \langle a\rangle_D\vee\langle b\rangle_D\\ &= \langle a\wedge b\rangle_D\end{aligned}$$

Since $a\wedge b\in F$, we get $(x\wedge y,1)\in\theta_F$. Therefore $x\wedge y\in Coker\ \theta_F$. Let $x\in Coker\ \theta_F$ and $x\leq y$. Then $(x,1)\in Coker\ \theta_F$. Since $x\leq y$, we get $[y)\subseteq[x)$. Then $[y)\vee\langle a\rangle_D\subseteq[x)\vee\langle a\rangle_D=\langle a\rangle_D$ for some $a\in F$. Hence $(y,1)\in\theta_F$, and so $y\in Coker\ \theta_F$. Thus $Coker\ \theta_F$ is a filter of L. □

Lemma 6.9. *Let F be a filter of a lattice L. Then $F \subseteq Coker\ \theta_F$.*

Proof. Let $x \in L$ and $x \in F$. Then $[x) \vee \langle x\rangle_D = \langle x\rangle_D = [1) \vee \langle x\rangle_D$ and $x \in F$. Hence $(x, 1) \in \theta_F$. Thus $x \in Coker\ \theta_F$. Therefore $F \subseteq Coker\ \theta_F$. □

Theorem 6.10. *Let F be a filter of a lattice L. Then F is a D-filter of L if and only if $F = Coker\ \theta_F$.*

Proof. Assume that F is a D-filter of L. By Lemma 5.9, we get $F \subseteq Coker\ \theta_F$. Again, let $x \in Coker\ \theta_F$. Then $(x, 1) \in \theta_F$. Hence $[x) \vee \langle a\rangle_D = \langle a\rangle_D$ for some $a \in F$. Since F is a D-filter and $a \in F$, we get $\langle a\rangle_D \subseteq F$. Thus $x \in [x) \subseteq \langle x\rangle_D \vee \langle a\rangle_D = \langle a\rangle_D \subseteq F$. Hence $Coker\ \theta_F \subseteq F$. Therefore $F = Coker\ \theta_F$.

Conversely, assume that $F = Coker\ \theta_F$. Let $x, y \in L$ be such that $\langle x\rangle_D = \langle y\rangle_D$. Suppose $x \in F$. Since $x \in F = Coker\ \theta_F$, we get $[x) \vee \langle a\rangle_D = \langle a\rangle_D$ for some $a \in F$. Hence $[x) \subseteq \langle a\rangle_D$, which gives $\langle x\rangle_D \subseteq \langle a\rangle_D$. Since $\langle x\rangle_D = \langle y\rangle_D$, we get $[y) \subseteq \langle y\rangle_D \subseteq \langle a\rangle_D$. Hence $[y) \vee \langle a\rangle_D = \langle a\rangle_D$. Thus $y \in Coker\ \theta_F = F$. Hence, by Theorem 2.8, F is a D-filter of L. □

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