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AN APPLICATION OF PASCAL DISTRIBUTION ON SPIRALLIKE PARABOLIC STARLIKE FUNCTIONS

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Abstract: In this paper we find a necessary and sufficient conditions and inclusion relations for Pascal distribution series to be in two subclasses of uniformly spirallike and convex functions. Further, we examined an integral operator related to Pascal distribution series. Several corollaries and consequences of the main results are also investigated.

1 Introduction and Definitions

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by the conditions $f(0) = 0 = f'(0) - 1$. Also, let $\mathcal{T}$ be the subclass of $\mathcal{A}$ consisting of functions of the form,

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, a_n \geq 0.$$

For functions $f \in \mathcal{A}$ given by (1.1) and $g(z) = z + \sum\limits_{n=2}^{\infty} b_n z^n$, the Hadamard product (or convolution) is defined as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n, \qquad (z \in \mathbb{U}).$$

A function $f \in \mathcal{A}$ is said to be spirallike if

$$\Re\left(e^{-i\vartheta}\frac{zf'(z)}{f(z)}\right) > 0$$

for some ϑ with $| \vartheta | < \frac{\pi}{2}$ and for all $z \in \mathbb{D}$, this class was introduced by Spaček [38]. Also $f(z)$ is convex spirallike if $zf'(z)$ is spirallike. In [35], Selvaraj and Geetha introduced the following subclasses of uniformly spirallike and convex functions.

Definition 1.1. A function $f(z)$ of the form (1.1) is said to be in the class $\mathcal{SP}_P(\vartheta, \delta)$, if it satisfies the analytic characterization

$$\Re\left(e^{-i\vartheta}\frac{zf'(z)}{f(z)}\right) > \left|\frac{zf''(z)}{f'(z)} - 1\right| + \delta, \ \ (| \vartheta | < \pi/2; 0 \leq \delta < 1)$$

and $f \in \mathcal{UCV}_P(\vartheta, \delta)$ if and only if $zf' \in \mathcal{SP}_P(\vartheta, \delta)$.

We write

$$\mathcal{TSP}_P(\vartheta,\delta) = \mathcal{SP}_P(\vartheta,\delta) \cap \mathcal{T}$$

and

$$\mathcal{UCT}_P(\vartheta,\delta) = \mathcal{UCV}_P(\vartheta,\delta) \cap \mathcal{T}.$$

In particular, $\mathcal{SP}_p(\vartheta,0) = \mathcal{SP}_P(\vartheta)$ and $\mathcal{UCV}_P(\vartheta,0) = \mathcal{UCV}_P(\vartheta)$, the classes of uniformly spirallike and uniformly convex were introduced by Ravichandran et al. [32]. For $\vartheta = 0$, the classes $\mathcal{UCV}_P(\vartheta)$ and $\mathcal{SP}_P(\vartheta)$, respectively, reduces to the classes $\mathcal{UCV}$ and $\mathcal{SP}$ introduced and studied by Rønning [34]. For more thought-provoking advancements of some classes related to subclasses of uniformly spirallike and uniformly convex spirallike, the readers may be referred to the works of Frasin [17, 6], Goodman [20, 19], Al-Hawary and Frasin [1], Kanas and Wisniowska [21, 22] and Rønning [33, 34].

A variable X is said to be *Pascal distribution* if it takes the values $0, 1, 2, 3, \ldots$ with probabilities
$(1-q)^m$, $\dfrac{qm(1-q)^m}{1!}$, $\dfrac{q^2m(m+1)(1-q)^m}{2!}$, $\dfrac{q^3m(m+1)(m+2)(1-q)^m}{3!}, \ldots$
respectively, where q and m are called the parameters, and thus

$$P(X=k) = \binom{k+m-1}{m-1} q^k (1-q)^m, \qquad k = 0, 1, 2, 3, \cdots.$$

Very recently, El-Deeb et al. [5] provided a power series whose coefficients are probabilities of Pascal distribution

$$\Psi_q^m(z) = z + \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1}(1-q)^m z^n, \qquad z \in \mathbb{D},$$

where $m \geq 1; 0 \leq q \leq 1$ and one can easily verify that the radius of convergence of above series is infinity by ratio test. This series laid the path to many young researches. We also define the series

$$\Phi_q^m(z) = 2z - \Psi_q^m(z) = z - \sum_{n=2}^{\infty} \binom{n+m-2}{m-1} q^{n-1}(1-q)^m z^n, \qquad z \in \mathbb{D}.$$

In this article we investigate the mapping properties of the function $\mathfrak{F}(z)$ which is the linear combination of $\Phi_q^m(z)$ and its derivative such that

$$\mathfrak{F}(z) = (1-\mu)\Phi_q^m(z) + \mu z(\Phi_q^m(z))', \qquad \mu \geq 0$$

$$\mathfrak{F}(z) = z - \sum_{n=2}^{\infty} (1-\mu+n\mu) \binom{n+m-2}{m-1} q^{n-1}(1-q)^m z^n, \qquad z \in \mathbb{D}.$$

Now, we define the linear operator

$$\mathcal{L}_q^m(z) : \mathcal{A} \to \mathcal{A}$$

defined by the convolution or Hadamard product

$$\mathcal{L}_q^m f(z) = \mathfrak{F}(z) * f(z) = z + \sum_{n=2}^{\infty} (1-\mu+n\mu) \binom{n+m-2}{m-1} q^{n-1}(1-q)^m a_n z^n, \qquad z \in \mathbb{D}.$$

Inspired by noticeable earlier results on relations between different subclasses of analytic and univalent functions by using hypergeometric functions, special functions (see for example, [2, 18, 23, 36, 37, 30]) and by the recent investigations (see for example, ([4], [7]-[16], [24, 25, 27, 28, 29, 31]), in the present paper we determine the necessary and sufficient conditions for $\mathfrak{F}(z)$ to be in our classes $\mathcal{TSP}_p(\vartheta,\delta)$ and $\mathcal{UCT}_p(\vartheta,\delta)$ and relations of these subclasses with $\mathcal{R}^\tau(\eta,\nu)$ introduced by Swaminathan [39]. Finally, we provide conditions for the integral operator $\mathcal{G}_q^m(z) = \int_0^z \frac{\mathfrak{F}(t)}{t} dt$ belonging to the above classes.

Lemma 1.2. *[35] A function $f(z)$ of the form (1.1) is in $\mathcal{TSP}_p(\vartheta, \delta)$ if and only if it satisfies*

$$\sum_{n=2}^{\infty}(2n - \cos\vartheta - \delta)\,|a_n| \le \cos\vartheta - \delta, \qquad (|\vartheta| < \pi/2; 0 \le \delta < 1).$$

In particular, when $\delta = 0$, we obtain a necessary and sufficient condition for a function $f(z)$ of the form (1.1) to be in the class $\mathcal{TSP}_p(\vartheta)$ is that

$$\sum_{n=2}^{\infty}(2n - \cos\vartheta)\,|a_n| \le \cos\vartheta, \qquad (|\vartheta| < \pi/2).$$

Lemma 1.3. *[35] A function $f(z)$ of the form (1.1) is in $\mathcal{UCT}_p(\vartheta, \delta)$ if and only if it satisfies*

$$\sum_{n=2}^{\infty} n(2n - \cos\vartheta - \delta)\,|a_n| \le \cos\vartheta - \delta, \qquad (|\vartheta| < \pi/2; 0 \le \delta < 1).$$

In particular, when $\delta = 0$, we obtain a necessary and sufficient condition for a function $f(z)$ of the form (1.1) to be in the class $\mathcal{UCT}_p(\vartheta)$ is that

$$\sum_{n=2}^{\infty} n(2n - \cos\vartheta)\,|a_n| \le \cos\vartheta, \qquad (|\vartheta| < \pi/2).$$

2 The necessary and sufficient conditions

For convenience throughout in the sequel, we use the following identities that hold for $m \ge 1$ and $0 \le q < 1$:

$$\sum_{n=0}^{\infty}\binom{n+m-1}{m-1}q^n = \frac{1}{(1-q)^m}, \qquad \sum_{n=0}^{\infty}\binom{n+m-2}{m-2}q^n = \frac{1}{(1-q)^{m-1}},$$
$$\sum_{n=0}^{\infty}\binom{n+m}{m}q^n = \frac{1}{(1-q)^{m+1}}, \qquad \sum_{n=0}^{\infty}\binom{n+m+1}{m+1}q^n = \frac{1}{(1-q)^{m+2}}.$$

By simple calculations we derive the following relations:

$$\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1} = \sum_{n=0}^{\infty}\binom{n+m-1}{m-1}q^n - 1 = \frac{1}{(1-q)^m} - 1, \tag{2.1}$$

$$\sum_{n=2}^{\infty}(n-1)\binom{n+m-2}{m-1}q^{n-1} = qm\sum_{n=0}^{\infty}\binom{n+m}{m}q^n = \frac{qm}{(1-q)^{m+1}}, \tag{2.2}$$

$$\begin{aligned}\sum_{n=3}^{\infty}(n-1)(n-2)\binom{n+m-2}{m-1}q^{n-1} &= q^2m(m+1)\sum_{n=0}^{\infty}\binom{n+m+1}{m+1}q^n \\ &= \frac{q^2m(m+1)}{(1-q)^{m+2}}.\end{aligned} \tag{2.3}$$

and

$$\begin{aligned}\sum_{n=4}^{\infty}(n-1)(n-2)(n-3)\binom{n+m-2}{m-1}q^{n-1} &= q^3m(m+1)(m+2)\sum_{n=0}^{\infty}\binom{n+m+2}{m+2}q^n \\ &= \frac{q^3m(m+1)(m+2)}{(1-q)^{m+3}}.\end{aligned} \tag{2.4}$$

Unless otherwise mentioned, we shall assume through out this paper that $|\vartheta| < \pi/2$, $0 \le \delta < 1$ and $0 \le q < 1$.

First we obtain the necessary and sufficient conditions for $\mathfrak{F}(z)$ to be in $\mathcal{TSP}_P(\vartheta, \delta)$ and $\mathcal{UCT}_p(\vartheta, \delta)$.

Theorem 2.1. *If* $m \geq 1$*, then* $\mathfrak{F}(z)$ *is in* $\mathcal{TSP}_p(\vartheta, \delta)$ *if and only if*

$$\frac{2\mu m(m+1)q^2}{(1-q)^2} + \frac{(2+\mu(4-\cos\vartheta-\delta))mq}{1-q} + (2-\cos\vartheta-\delta)\left[1-(1-q)^m\right] \leq \cos\vartheta-\delta. \quad (2.5)$$

Proof. Since

$$\mathfrak{F}(z) = z - \sum_{n=2}^{\infty}(1-\mu+n\mu)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m z^n.$$

Using the Lemma 1.2, it suffices to show that

$$\sum_{n=2}^{\infty}(2n-\cos\vartheta-\delta)(1-\mu+n\mu)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \leq \cos\vartheta-\delta. \quad (2.6)$$

From (2.6) we let

$$\begin{aligned}
\mathfrak{L}_1(m,n,\vartheta,\delta) &= \sum_{n=2}^{\infty}(2n-\cos\vartheta-\delta)(1-\mu+n\mu)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\
&= 2\mu\sum_{n=2}^{\infty}n^2\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\
&\quad +[2(1-\mu)-\mu(\cos\vartheta+\delta)]\sum_{n=2}^{\infty}n\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\
&\quad -(\cos\vartheta+\delta)(1-\mu)\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m.
\end{aligned}$$

Writing

$$n = (n-1)+1$$

and

$$n^2 = (n-1)(n-2)+3(n-1)+1,$$

we get

$$\begin{aligned}
\mathfrak{L}_1(m,n,\vartheta,\delta) &= 2\mu\sum_{n=2}^{\infty}(n-1)(n-2)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\
&\quad +[2(1+2\mu)-\mu(\cos\vartheta+\delta)]\sum_{n=2}^{\infty}(n-1)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\
&\quad +[2-(\cos\vartheta+\delta)]\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m
\end{aligned}$$

Now using (2.1) - (2.3), we get

$$\begin{aligned}
\mathfrak{L}_1(m,n,\vartheta,\delta) &= \frac{2\mu m(m+1)q^2}{(1-q)^2} + \frac{(2+\mu(4-\cos\vartheta-\delta))mq}{1-q} \\
&\quad +(2-\cos\vartheta-\delta)\left[1-(1-q)^m\right].
\end{aligned}$$

Hence $\mathfrak{L}_1(m,n,\vartheta,\delta)$ is bounded above by $\cos\vartheta-\delta$ if and only if (2.5) holds. □

Theorem 2.2. *If* $m \geq 1$*, then* $\mathfrak{F}(z)$ *is in* $\mathcal{UCT}_p(\vartheta, \delta)$ *if and only if*

$$\begin{aligned}
&\frac{2\mu m(m+1)(m+2)q^3}{(1-q)^3} + \frac{m(m+1)q^2(2+\mu[12-\cos\vartheta-\delta])}{(1-q)^2} \\
&\quad +\frac{q\,m}{1-q}(6-\cos\vartheta-\delta+\mu[14-2\cos\vartheta-2\delta]) \\
&\quad +(2\mu+2-\cos\vartheta-\delta)\left[1-(1-q)^m\right] \leq \cos\vartheta-\delta. \quad (2.7)
\end{aligned}$$

Proof. In view of Lemma 1.3, we have to show that

$$\sum_{n=2}^{\infty} n(2n-\cos\vartheta-\delta)(1-\mu+n\mu)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \leq \cos\vartheta-\delta. \tag{2.8}$$

From (2.8), consider the expression

$$\begin{aligned}
\mathfrak{L}_2(m,n,\vartheta,\delta) &= \sum_{n=2}^{\infty} n(2n-\cos\vartheta-\delta)(1-\mu+n\mu)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\
&= 2\mu\sum_{n=2}^{\infty} n^3\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\
&\quad +(2-\mu\,[2+\cos\vartheta+\delta])\sum_{n=2}^{\infty} n^2\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \\
&\quad +(1-\mu)(-\cos\vartheta-\delta)\sum_{n=2}^{\infty} n\binom{n+m-2}{m-1}q^{n-1}(1-q)^m
\end{aligned}$$

Writing

$$n=(n-1)+1$$

$$n^2=(n-1)(n-2)+3(n-1)+1$$

and

$$n^3=(n-1)(n-2)(n-3)+6(n-1)(n-2)+7(n-1)+1$$

and using (2.1) - (2.4), we get

$$\begin{aligned}
\mathfrak{L}_2(m,n,\vartheta,\delta) &= \frac{2\mu m(m+1)(m+2)q^3}{(1-q)^3}+\frac{m(m+1)q^2[2+\mu(12-\cos\vartheta-\delta)]}{(1-q)^2} \\
&\quad +\frac{mq}{1-q}[6-\cos\vartheta-\delta+2\mu(12-\cos\vartheta-\delta)] \\
&\quad +(2\mu+2-\cos\vartheta-\delta)\,[1-(1-q)^m]\,.
\end{aligned}$$

Hence, $\mathfrak{L}_2(m,n,\vartheta,\delta)$ is bounded above by $\cos\vartheta-\delta$ if (2.7) is satisfied. □

3 Inclusion Properties

A function $f\in\mathcal{A}$ is said to be in the class $\mathcal{R}^{\tau}(\eta,\upsilon)$, $(\tau\in\mathbb{C}\backslash\{0\},\ 0<\eta\leq 1;\upsilon<1)$, if it satisfies the inequality

$$\left|\frac{(1-\eta)\frac{f(z)}{z}+\eta f'(z)-1}{2\tau(1-\upsilon)+(1-\eta)\frac{f(z)}{z}+\eta f'(z)-1}\right|<1, \qquad (z\in\mathbb{D}).$$

The class $\mathcal{R}^{\tau}(\eta,\upsilon)$ was introduced earlier by Swaminathan [39] (for special cases see the references cited there in) and obtained the following estimate.

Lemma 3.1. *[39] If $f\in\mathcal{R}^{\tau}(\eta,\upsilon)$ is of form (1.1), then*

$$|a_n|\leq\frac{2\,|\tau|\,(1-\upsilon)}{1+\eta(n-1)}, \quad n\in\mathbb{N}\setminus\{1\}. \tag{3.1}$$

The bounds given in (3.1) are sharp.

Making use of the Lemma 3.1, we will focus the influence of the Pascal distribution series on the classes $\mathcal{TSP}_p(\vartheta,\delta)$ and $\mathcal{UCT}_p(\vartheta,\delta)$.

Theorem 3.2. *Let $m > 1$ and $f \in \mathcal{R}^{\tau}(\eta, \upsilon)$. Then $\mathcal{L}_q^m f(z)$ is in the class $\mathcal{TSP}_p(\vartheta, \delta)$ if*

$$\frac{2\,|\tau|\,(1-\upsilon)}{\eta}\left[(2-\mu[2+\cos\vartheta+\delta])\,(1-(1-q)^m)+\frac{2\mu mq}{1-q}\right.$$
$$\left.-\frac{(1-\mu)(\cos\vartheta+\delta)}{q(m-1)}\,(1-q-(1-q)^m\,[1+q(m-1)])\right] \quad\leq\quad \cos\vartheta-\delta. \tag{3.2}$$

Proof. In view of Lemma 1.2, it is required to show that

$$\sum_{n=2}^{\infty}(2n-\cos\vartheta-\delta)(1-\mu+n\mu)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\,|a_n|\leq\cos\vartheta-\delta.$$

Since $f \in \mathcal{R}^{\tau}(\eta, \upsilon)$ then by Lemma 3.1 we have

$$|a_n|\leq\frac{2\,|\tau|\,(1-\upsilon)}{1+\eta(n-1)},\qquad n\in\mathbb{N}\setminus\{1\}.$$

Thus, we have

$$\begin{aligned}
&\mathfrak{L}_3(m,n,\vartheta,\delta)\\
&=\sum_{n=2}^{\infty}(2n-\cos\vartheta-\delta)(1-\mu+n\mu)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\,|a_n|\\
&\leq 2\,|\tau|\,(1-\upsilon)\left[\sum_{n=2}^{\infty}\frac{1}{1+\eta(n-1)}(2n-\cos\vartheta-\delta)(1-\mu+n\mu)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\right].
\end{aligned}$$

Since $1+\eta(n-1)\geq n\eta$, we get

$$\begin{aligned}
&\mathfrak{L}_3(m,n,\vartheta,\delta)\\
&\leq\frac{2\,|\tau|\,(1-\upsilon)}{\eta}\left[\sum_{n=2}^{\infty}\frac{1}{n}(2n-\cos\vartheta-\delta)(1-\mu+n\mu)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\right]\\
&=\frac{2\,|\tau|\,(1-\upsilon)}{\eta}\left[(2-\mu[\cos\vartheta+\delta])\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\right.\\
&\quad+2\mu\sum_{n=2}^{\infty}(n-1)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m+2\mu\sum_{n=2}^{\infty}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\\
&\quad\left.-(1-\mu)(\cos\vartheta+\delta)\sum_{n=2}^{\infty}\frac{1}{n}\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\right]\\
&\leq\frac{2\,|\tau|\,(1-\upsilon)}{\eta}\left[(2-\mu[\cos\vartheta+\delta])\,(1-q)^m\sum_{n=0}^{\infty}\binom{n+m-1}{m-1}q^n\right.\\
&\quad+2\mu(1-q)^m\sum_{n=0}^{\infty}\binom{n+m}{m}q^n+2\mu(1-q)^m\left(\sum_{n=0}^{\infty}\binom{n+m-1}{m-1}q^{n-1}-1\right)\\
&\quad\left.-(1-\mu)(\cos\vartheta+\delta)(1-q)^m\frac{1}{q(m-1)}\left(\sum_{n=0}^{\infty}\binom{n+m-2}{m-2}q^n-1-(m-1)q\right)\right]\\
&=\frac{2\,|\tau|\,(1-\upsilon)}{\eta}\left[(2-\mu[\cos\vartheta+\delta])\,(1-(1-q)^m)+\frac{2\mu mq}{1-q}\right.\\
&\quad\left.-\frac{(1-\mu)(\cos\vartheta+\delta)}{q(m-1)}\,(1-q-(1-q)^m\,[1+q(m-1)])\right].
\end{aligned}$$

But the above equation is bounded by $\cos\vartheta-\delta$, if (3.2) holds. This completes the proof of Theorem 3.2. □

Theorem 3.3. *Let $f \in \mathcal{R}^{\tau}(\eta, \upsilon)$. Then $\mathcal{L}_q^m f(z)$ is in the class $\mathcal{UCT}_p(\vartheta, \delta)$ if*

$$\frac{2\,|\tau|\,(1-\upsilon)}{\eta}\left[\frac{2\mu m(m+1)q^2}{(1-q)^2} + (2-\cos\vartheta-\delta)\,[1-(1-q)^m] + \frac{(2+\mu(4-\cos\vartheta-\delta))mq}{1-q}\right] \leq \cos\vartheta - \delta.$$

Proof of Theorem 3.3 is omitted because it can be made similar to the proof of Theorem 3.2.

Remark 3.4. For the special case $\mu = 1$, Theorems 2.1 - 2.2 and Theorems 3.2 - 3.3 provide similar results obtained by Murugusundaramoorthy et al. [26].

4 An integral operator

Theorem 4.1. *If the function $\mathcal{G}_q^m(z)$ is given by*

$$\mathcal{G}_q^m(z) = \int_0^z \frac{\mathfrak{F}(t)}{t}dt, \qquad z \in \mathbb{D} \tag{4.1}$$

then $\mathcal{G}_q^m(z) \in \mathcal{UCT}_p(\vartheta, \delta)$ if and only if

$$\frac{2\mu m(m+1)q^2}{(1-q)^2} + \frac{(2+\mu(4-\cos\vartheta-\delta))mq}{1-q} + (2-\cos\vartheta-\delta)\,[1-(1-q)^m] \leq \cos\vartheta - \delta.$$

Proof. Since

$$\mathcal{G}_q^m(z) = z - \sum_{n=2}^{\infty}(1-\mu+n\mu)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m\frac{z^n}{n}$$

then by Lemma 1.3, we need only to verify that

$$\sum_{n=2}^{\infty} n(2n-\cos\vartheta-\delta) \times \frac{1}{n}(1-\mu+n\mu)\binom{n+m-2}{m-1}\,q^{n-1}(1-q)^m \leq \cos\vartheta - \delta,$$

or, equivalently

$$\sum_{n=2}^{\infty}(2n-\cos\vartheta-\delta)(1-\mu+n\mu)\binom{n+m-2}{m-1}q^{n-1}(1-q)^m \leq \cos\vartheta - \delta.$$

The rest part of the proof of Theorem 4.1 is similar to that of Theorem 2.1, and so we omit the details. □

Theorem 4.2. *If $m > 1$, then the integral operator $\mathcal{G}_q^m(z)$ given by (4.1) is in the class $\mathcal{TSP}_p(\vartheta, \delta)$ if and only if*

$$\begin{aligned}&(2-\mu[\cos\vartheta+\delta])\,(1-(1-q)^m) + \frac{2\mu qm}{1-q}\\ &-\frac{(1-\mu)(\cos\vartheta+\delta)}{q(m-1)}\,(1-q-(1-q)^m\,[1+q(m-1)]) \quad \leq \quad \cos\vartheta - \delta.\end{aligned}$$

The proof of Theorem 4.2 is omitted because it can be made similar to the proof of Theorem 3.2.

Remark 4.3. For $\mu = 0$, the Theorems 4.1 - 4.2 provide similar results with those recently attained by Murugusundaramoorthy et al. [26].

5 Corollaries and consequences

By giving the special value to the parameter $\delta = 0$ in Theorems 2.1 - 2.2, 3.2 - 3.3 and 4.1 - 4.2 we obtain the following corollaries.

Corollary 5.1. *If* $m \geq 1$, *then* $\mathfrak{F}(z) \in \mathcal{TSP}_p(\vartheta)$ *if and only if*

$$\frac{2\mu m(m+1)q^2}{(1-q)^2} + \frac{(2+\mu(4-\cos\vartheta))q\,m}{1-q} + (2-\cos\vartheta)\left[1-(1-q)^m\right] \leq \cos\vartheta.$$

Corollary 5.2. *If* $m \geq 1$, *then* $\mathfrak{F}(z)$ *is in* $\mathcal{UCT}_p(\vartheta)$ *if and only if*

$$\frac{2\mu q^3 m(m+1)(m+2)}{(1-q)^3} + \frac{q^2\, m(m+1)(2+\mu[12-\cos\vartheta])}{(1-q)^2}$$
$$+\frac{q\,m}{1-q}(6-\cos\vartheta+\mu[14-2\cos\vartheta]) + (2\mu+2-\cos\vartheta)\left[1-(1-q)^m\right] \quad \leq \quad \cos\vartheta.$$

Corollary 5.3. *Let* $m > 1$ *and* $f \in \mathcal{R}^\tau(\eta,\upsilon)$. *Then* $\mathcal{L}_q^m f(z)$ *is in the class* $\mathcal{TSP}_p(\vartheta)$ *if*

$$\frac{2\,|\tau|\,(1-\upsilon)}{\eta}\left[(2-\mu\cos\vartheta)\left(1-(1-q)^m\right) + \frac{2\mu q m}{1-q}\right.$$
$$\left.-\frac{(1-\mu)\cos\vartheta}{q(m-1)}\left(1-q-(1-q)^m\left[1+q(m-1)\right]\right)\right] \quad \leq \quad \cos\vartheta.$$

Corollary 5.4. *Let* $f \in \mathcal{R}^\tau(\eta,\upsilon)$. *Then* $\mathcal{L}_q^m f(z)$ *is in the class* $\mathcal{UCT}_p(\vartheta)$ *if*

$$\frac{2\,|\tau|\,(1-\upsilon)}{\eta}\left[\frac{2\mu m(m+1)q^2}{(1-q)^2} + (2-\cos\vartheta)\left(1-(1-q)^m\right) + \frac{(2+\mu(4-\cos\vartheta))q\,m}{1-q}\right] \leq \cos\vartheta.$$

Corollary 5.5. *If* $m \geq 1$, *then the integral operator* $\mathcal{G}_q^m(z)$ *given by* (4.1) *is in the class* $\mathcal{UCT}_p(\vartheta)$ *if and only if*

$$\frac{2\mu m(m+1)q^2}{(1-q)^2} + \frac{(2+\mu(4-\cos\vartheta))q\,m}{1-q} + (2-\cos\vartheta)\left[1-(1-q)^m\right] \leq \cos\vartheta.$$

Corollary 5.6. *If* $m > 1$, *then the integral operator* $\mathcal{G}_q^m(z)$ *given by* (4.1) *is in the class* $\mathcal{TSP}_p(\vartheta)$ *if and only if*

$$(2-\mu[\cos\vartheta])\left(1-(1-q)^m\right) + \frac{2\mu q m}{1-q} - \frac{(1-\mu)(\cos\vartheta)}{q(m-1)}\left(1-q-(1-q)^m\left[1+q(m-1)\right]\right) \leq \cos\vartheta.$$

Remark 5.7. For $\mu = 0$, the Corollaries 5.1 - 5.6 which correspond the results very recently reached by Murugusundaramoorthy et al. [26].

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