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A characterization of Jordan (α,β)-higher *-derivations

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Abstract Let R be a ring with involution $'*'$. Next, let $\mathbb{N}_0$ be the set of all nonnegative integers, and $D=(d_n)_{n\in\mathbb{N}_0}$ a family of additive mappings of a $*$-ring R such that $d_0=id_R$. D is called a Jordan (α,β)-higher $*$-derivation (respectively, a Jordan triple (α,β)-higher $*$-derivation) of R if $D_n(x^2)=\sum_{i+j=n} d_i(\beta^j(x))d_j(\alpha^i(x^{*^i}))$ (respectively, $d_n(xyx)=\sum_{i+j+k=n} d_i$ $(\beta^{j+k}(x))d_j(\beta^k(\alpha^i(y^{*^i})))d_k(\alpha^{i+j}(x^{*^{i+j}})))$ for all $x,y\in R$ and each $n\in\mathbb{N}_0$. The main aim of this paper is to characterize Jordan triple (α,β)-higher $*$-derivation of semiprime rings with involution. As an application, we prove that every Jordan triple (α,β)-higher $*$-derivation onto a 6-torsion free semiprime ring is a Jordan higher $*$-derivation.

1 Introduction

This research is motivated by the recent work of Alhazmi et al. [1] and Ezzat [15]. Throughout this paper, unless otherwise mentioned, R will denote an associative ring. Following [20], an additive mapping, $d: R\to R$, is called a derivation (respectively, Jordan derivation) if $d(xy)=d(x)y+xd(y)$ (respectively, $d(x^2)=d(x)x+xd(x)$) holds for all $x,y\in R$. Following Brešar [12], an additive mapping $F: R\to R$ is said to be a generalized derivation (respectively, generalized Jordan derivation) on R if there exists a derivation $d: R\to R$ such that $F(xy)=F(x)y+xd(y)$ (correspondingly, $F(x^2)=F(x)x+xd(x)$) holds for all $x,y\in R$.

For given endomorphisms α and β, an additive mapping $d: R\to R$ is said to be an (α,β)-derivation (respectively, Jordan (α,β)-derivation) if $d(xy)=d(x)\alpha(y)+\beta(x)d(y)$ (respectively, $d(x^2)=d(x)\alpha(x)+\beta(x)d(x)$) holds for all $x,y\in R$. According to Ashraf et al. [8], an additive mapping $F: R\to R$ is called a generalized (α,β)-derivation (correspondingly, generalized Jordan (α,β)-derivation) on R if there exists an (α,β)-derivation, $d: R\to R$, such that $F(xy)=F(x)\alpha(y)+\beta(x)d(y)$ (correspondingly, $F(x^2)=F(x)\alpha(x)+\beta(x)d(x)$) holds for all $x,y\in R$. It is obvious to see that every generalized (α,β)-derivation on a ring is a generalized Jordan (α,β)-derivation, but the converse need not be true in general ([8], Example 3.1). A number of authors have studied this problem in the setting of prime and semiprime rings. Recently, Ali and Haetinger [5], proved that every generalized Jordan (α,β)-derivation on a 2-torsion free semiprime ring is a generalized (α,β)-derivation (see also [9] for more related results).

The concept of derivations was extended to higher derivations by Hasse and Schmidt [19]. Let $D=\{d_n\}_{n\in\mathbb{N}_0}$ be a family of additive mappings on R. D is said to be a higher derivation (correspondingly, Jordan higher derivation) on R if $d_0=id_R$ (where id_R is the identity map on R) and $d_n(xy)=\sum_{i+j=n} d_i(x)d_j(y)$ (correspondingly, $d_n(x^2)=\sum_{i+j=n} d_i(x)d_j(x)$) for all $x,y\in R$. A family $D=(d_n)_{n\in\mathbb{N}_0}$ of additive mappings of a ring R, where $d_0=id_R$, is called a *Jordan triple higher derivation* if $d_n(xyx)=\sum_{i+j+k=n} d_i(x)d_j(y^i)d_k(x^{i+j})$ holds for all $x,y\in R$. Ferrero and Haetinger [16] proved that in a 2-torsion free ring every Jordan higher derivation is a Jordan triple higher derivation. They also showed that in a 2-torsion free semiprime ring every Jordan triple higher derivation is a higher derivation. It is easy to see that the first member of each higher derivation is itself a derivation. More related results can be found in Haetinger [18]. Later on, Cortes and Haetinger [13] defined generalized higher derivations: a family $F=(f_n)_{n\in\mathbb{N}_0}$ of additive mappings of a ring R, such that $f_0=id_R$, is said to be a generalized higher derivation (correspondingly, generalized Jordan higher derivation) of R if there exists a higher derivation (correspondingly, Jordan higher derivation) $D = \{d_n\}_{n\in\mathbb{N}_0}$ and for each $n \in \mathbb{N}_0, f_n(xy) = \sum_{i+j=n} f_i(x)d_j(y)$ (correspondingly, $f_n(x^2) = \sum_{i+j=n}) f_i(x)d_j(x))$ holds for all $x, y \in R$. Obviously, every generalized higher derivation is a generalized Jordan higher derivation, but the converse need not be true. The converse has already been proved for by Cortes and Haetinger [5] for square closed Lie ideals of a prime ring R. Later, Wei and Xao [24] established this result for a 2-torsion free semiprime ring. In 2010, Ashraf et al. [7] introduced the concept of (α, β)-higher derivations as follows: a family D of additive mappings d_n on R is said to be an (α, β)-higher derivation (correspondingly, Jordan (α, β)-higher derivation) of R if $d_0 = id_R$ and $d_n(xy) = \sum_{i+j=n} d_i\big(\beta^{n-i}(x)\big)d_j\big(\alpha^{n-j}(y)\big)$ (correspondingly, $d_n(x^2) = \sum_{i+j=n} d_i\big(\beta^{n-i}(x)\big)d_j\big(\alpha^{n-j}(x)\big)\big)$ for all $x, y \in R$ and for each $n \in \mathbb{N}_0$. For given endomorphisms α and β, a family $F = (f_n)_{n\in\mathbb{N}_0}$ of additive mappings $f_n : R \to R$ is said to be a generalized (α, β)-higher derivation (correspondingly, generalized Jordan (α, β)-higher derivation) of R if there exists an (α, β)-higher derivation $D = \{d_n\}_{n\in\mathbb{N}_0}$ and for each $n \in \mathbb{N}_0, f_n(xy) = \sum_{i+j=n} f_i\big(\beta^{n-i}(x)\big)d_j\big(\alpha^{n-j}(y)\big)$ (correspondingly, $f_n(x^2) = \sum_{i+j=n} f_i\big(\beta^{n-i}(x)\big)d_j\big(\alpha^{n-j}(x)\big)\big)$ holds for all $x, y \in R$. It is straightforward to check that any generalized (α, β)-higher derivation is a generalized Jordan (α, β)-higher derivation. However, the converse statement need not be true. Ashraf and Khan [6] proved that every generalized Jordan (α, β)-higher derivation is a generalized (α, β)-higher derivation on Lie ideals of a prime ring R. Some more related results can be found in [1], and [6].

Motivated by the recent work's Alhazmi et al. [1] and Ezzat [15], we introduce the following notions:

Definition 1.1. Let $\mathbb{N}_0$ be the set of all nonnegative integers, α, β be the endomorphisms of R, and let $D = (d_n)_{n\in\mathbb{N}_0}$ be a family of additive mappings of R such that $d_0 = id_R$. D said to be

(i) an (α, β)-*higher* $*$-*derivation* of R if for each $n \in \mathbb{N}_0$,

$$d_n(xy) = \sum_{i+j=n} d_i(\beta^j(x))d_j(\alpha^i(y^{*^i})) \text{ for all } x, y \in R;$$

(ii) a *Jordan* (α, β)-*higher* $*$-*derivation* of R if for each $n \in \mathbb{N}_0$,

$$d_n(x^2) = \sum_{i+j=n} d_i(\beta^j(x))d_j(\alpha^i(x^{*^i})) \text{ for all } x \in R;$$

(iii) a *Jordan triple* (α, β)-*higher* $*$-*derivation* of R if for each $n \in \mathbb{N}_0$,

$$d_n(xyx) = \sum_{i+j+k=n} d_i(\beta^{i+j}(x))d_j(\beta^k(\alpha^i(y^{*^i})))d_k(\alpha^{i+j}(x^{*^{i+j}})) \text{ for all } x, y \in R.$$

In this definition, if we take $\alpha = \beta = id_R$, the identity map on R then we obtain the notion of higher $*$-derivations, Jordan higher $*$-derivations and Jordan triple higher $*$-derivations. Also the first member of this family is an $(\alpha, \beta)^*$-derivation. Therefore, the interesting thing about this new concept is that they covers the notions of higher $*$-derivations, Jordan (α, β)-higher $*$-derivations etc.

The main objective of this paper is to characterize Jordan triple (α, β)-higher $*$-derivations and related mappings in semiprime rings with involution. As consequences of our main theorems, many known results can be either generalized or deduced.

2 Preliminaries

Throughout this section, we will use the following notations: Let $D = (d_n)_{n\in\mathbb{N}_0}$ be a Jordan triple (α, β)-higher $*$-derivation of R. For every fixed $n \in \mathbb{N}_0$ and each $x, y \in R$, we denote by $A_n(x)$ and $B_n(x, y)$ the elements of R and defined by

$$
\begin{aligned}
A_n(x) &= d_n(x^2) - \sum_{i+j=n} d_i(\beta^j(x))d_j(\alpha^i(x^{*^i})),\\
B_n(x,y) &= d_n(xy+yx) - \sum_{i+j=n} d_i(\beta^j(x))d_j(\alpha^i(y^{*^i}))\\
&- \sum_{i+j=n} d_i(\beta^j(y))d_j(\alpha^i(x^{*^i})).
\end{aligned}
$$

Then, it is straightforward to check that $A_n(x+y) = A_n(x) + A_n(y) + B_n(x,y),\ A_n(x) = A_n(-x)$ and $B_n(-x,y) = -B_n(x,y)$ for all $x, y \in R$.

Lemma 2.1. *([10], Lemma 2.1). Let R be a 2-torsion free semiprime ring. If $x, y \in R$ are such that $xry = 0$ for all $r \in R$, then $yrx = xy = yx = 0$.*

Lemma 2.2. *Let R be a 2-torsion free semiprime $*$-ring and $m, n \in \mathbb{N}_0$. Next, let $D = (d_n)_{n\in\mathbb{N}_0}$ a Jordan triple (α,β)-higher $*$-derivation of R such that β is an automorphism of R and $\alpha\beta = \beta\alpha$. If $A_m(x) = 0$ for all $x \in R$ and for each $m \leq n$, then $\beta^n(x^2)A_n(x) = A_n(x)\beta^n(x^2) = 0$ for all $x \in R$ and for each $n \in \mathbb{N}_0$.*

Proof. Compute the value of $M = d_n(x^2yx^2)$ in two different ways:

First by substitution of xyx for y in the definition of Jordan triple (α,β)-higher $*$-derivation, we find that

$$
\begin{aligned}
M &= \sum_{i+j+k=n} d_i(\beta^{j+k}(x))d_j(\beta^k(\alpha^i((xyx)^{*^i})))d_k(\alpha^{i+j}(x^{*^{i+j}}))\\
&= \sum_{i+j+k=n} d_i(\beta^{i+k}(x))\Big(\sum_{p+q+r=j} d_p(\beta^{q+r+k}(\alpha^i(x^{*^i})))\\
&\quad\times d_q(\beta^{r+k}(\alpha^{p+i}(y^{*^{p+i}})))d_r(\beta^k(\alpha^{p+q+i}(x^{*^{p+q+i}})))\Big)d_k(\alpha^{i+j}(x^{*^{i+j}}))\\
&= \sum_{i+p+q+r+k=n} d_i(\beta^{p+q+r+k}(x))d_p(\beta^{q+r+k}(\alpha^i(x^{*^i})))\\
&\quad\times d_q(\beta^{r+k}(\alpha^{p+i}(y^{*^{p+i}})))d_r(\beta^k(\alpha^{p+q+i}(x^{*^{p+q+i}})))\\
&\quad\times d_k(\alpha^{i+p+q+r}(x^{*^{i+p+q+r}}))\\
&= \sum_{i+p=n} d_i(\beta^p(x))d_p(\alpha^i(x^{*^i}))\alpha^n(y^{*^n}x^{2^{*^n}})\\
&\quad+\beta^n(x^2y)\sum_{r+k=n} d_r(\beta^r(x))d_k(\alpha^r(x^{*^r}))\\
&\quad+\sum_{\substack{i+p+q+r+k=n\\ i+p\neq n,\, r+k\neq n}} d_i(\beta^{p+q+r+k}(x))d_p(\beta^{q+r+k}(\alpha^i(x^{*^i})))\\
&\quad\times d_q(\beta^{r+k}(\alpha^{p+i}(y^{*^{p+i}})))d_r(\beta^k(\alpha^{p+q+i}(x^{*^{p+q+i}})))\\
&\quad\times d_k(\alpha^{i+p+1+r}(x^{*^{i+p+q+r}})).
\end{aligned}
$$

The second way to compute M is the substitution of x^2 for x in the definition of Jordan triple

(α,β)-higher $*$-derivation and using our assumption that $A_m(x)=0$ for $m<n$, we find that

$$\begin{aligned} M &= \sum_{i+j+k=n} d_i(\beta^{j+k}(x^2)d_j(\beta^k(\alpha^i(y)^{*^i})))d_k(\alpha^{i+j}(x^{2^{*^{i+j}}})) \\ &= d_n(x^2)\alpha^n(y^{*^n}x^{2^{*^n}} + \beta^n(x^2y)d_n(x^2) \\ &\quad + \sum_{\substack{i+j+k=n\\ i+\neq n, k\neq n}} d_i(\beta^{j+k}(x^2))d_j(\beta^k(\alpha^i(y)^{*^i}))d_k(\alpha^{i+j}(x^{2^{i+j}})) \\ &= d_n(x^2)\alpha^n(y^{*^n}x^{2^{*^n}} + \beta^n(x^2y)d_n(x^2) \\ &\quad + \sum_{\substack{i+j+k=n\\ i+\neq n, k\neq n}} \Big(\sum_{u+v=i} d_u(\beta^{v+j+k}(x))d_v(\beta^{j+k}(\alpha^u(x^{*^u})))\Big)d_j(\beta^k(\alpha^i(y)^{*^i})) \\ &\quad \times\Big(\sum_{s+t=k} d_s(\beta^t(\alpha^{i+j}(x^{*^{i+j}})))d_t(\alpha^{s+i+j}(x^{*^{i+j+s}}))\Big) \\ &= d_n(x^2)\alpha^n(y^{*^n}x^{2^{*^n}} + \beta^n(x^2y)d_n(x^2) \\ &\quad + \sum_{\substack{u+v+j+s+t=n\\ u+v\neq n, s+t\neq n}} d_u(\beta^{v+j+k}(x))d_v(\beta^{j+k}(\alpha^u(x^{*^u})))d_j(\beta^k(\alpha^{u+v}(y)^{*^{u+v}})) \\ &\quad \times d_s(\beta^t(\alpha^{u+v+j}(x^{*^{u+v+j}})))d_t(\alpha^{s+u+v+j}(x^{*^{u+v+j+s}})). \end{aligned}$$

Now, subtracting the two values so obtained for M and using our notation, we obtain

$$A_n(x)\alpha^n(y^{*^n}y^{2^{*^n}}) + \beta^n(x^2y)A_n(x) = 0. \tag{2.1}$$

In case n is even (2.1) reduces to

$$A_n(x)\alpha^n(yx^2) + \beta^n(x^2y)A_n(x) = 0 \text{ for all } x,y\in R. \tag{2.2}$$

Replacing y by $rx^2y, r\in R$ in (2.2), we get

$$A_n(x)\alpha^n(rx^2)\alpha^n(yx^2) + \beta^n(x^2r)\beta^n(x^2y)A_n(x) = 0 \text{ for all } x,y,r\in R.$$

Using (2.2) for the value of $A_n(x)\alpha^n(rx^2)$, we obtain

$$-\beta^n(x^2r)A_n(x)\alpha^n(bx^2) + \beta^n(x^2r)\beta^n(x^2y)A_n(x) = 0 \text{ for all } x,y,r\in R.$$

Again, using (2.2) for the value of $A_n(x)\alpha^n(yx^2)$ yields, in view of R is 2-torsion free, that

$$\beta^n(x^2r)\beta^n(x^2y)A_n(x) = 0 \text{ for all } x,y,r\in R.$$

Now put $r=y\beta^{-n}(A_n(x))r$ in the last expression, we reach to

$$\beta^n(x^2y)A_n(x)\beta^n(r)\beta^n(x^2y)A_n(x) = 0 \text{ for all } x,y,r\in R.$$

Since β onto, the last relation implies that

$$\beta^n(x^2y)A_n(x)R\beta^n(x^2y)A_n(x) = \{0\} \text{ for all } x,y\in R.$$

The semiprimeness of R yields $\beta^n(x^2y)A_n(x)=0$ for all $x,y\in R$. Again since β is onto we have $\beta^n(x^2)RA_n(x)=\{0\}$ for all $x\in R$, and by Lemma 2.1, we reach to $\beta^n(x^2)A_n(x) = A_n(x)\beta^n(x^2)=0$ for all $x\in R$.

In case n is odd (2.1) reduces to

$$A_n(x)\alpha^n(y^*x^{2^*}) + \beta^n(x^2y)A_n(x) = \{0\} \text{ for all } x,y\in R. \tag{2.3}$$

Putting $y = rx^2y$ gives for all $x, y, r \in R$ that

$$A_n(x)\alpha^n(y^*x^{2^*})\alpha^n(r^*x^{2^*}) + \beta^n(x^2r)\beta^n(x^2y)A_n(x) = 0.$$

Substituting the value of $A_n(x)\alpha^n(y^*x^{2^*})$ from (2.3) in the last relation gives for all $x, y, r \in R$ that

$$-\beta^n(x^2y)A_n(x)\alpha^n(r^*x^{2^*}) + \beta^n(x^2r)\beta^n(x^2y)A_n(x) = 0.$$

Again by using (2.3) for the value of $A_n(x)\alpha^n(r^*x^{2^*})$, we get for all $x, y, r \in R$

$$\beta^n(x^2)\big(\beta^n(yx^2r) + \beta^n(rx^2y)\big)A_n(x) = 0. \tag{2.4}$$

Taking $r = y$ in (2.4) leads, in view of R in 2-torsion free, to

$$\beta^n(x^2y)\beta^n(x^2y)A_n(x) = 0 \text{ for all } x, y \in R. \tag{2.5}$$

Now putting $r = y\beta^{-n}(A_n(x))r$ in (2.4) gives for all $x, y, r \in R$

$$\beta^n(x^2y)\beta^n(x^2y)A_n(x)\beta^n(r)A_n(x) + \beta^n(x^2y)\beta^n(rx^2y)A_n(x) = 0$$

But using (2.5), the first summand fo the last equation is zero. Hence, we get $\beta^n(x^2y)\beta^n(rx^2y)A_n(x) = 0$ for all $x, y, r \in R$. Surjectiveness of β leads to $\beta^n(x^2y)R\beta^n(x^2y)A_n(x) = \{0\}$ for all $x, y \in R$ and since R is semiprime we get $\beta^n(x^2)\beta^n(y)A_n(x) = 0$ for all $x, y, r \in R$. Again by using the surjectiveness of β, we find $\beta^n(x^2)RA_n(x) = \{0\}$ for all $x \in R$. Thus, since R is semiprime we get by Lemma 2.1 that $A_n(x)\beta^n(x^2) = \beta^n(x^2)A_n(x) = 0$ for all $x \in R$. □

Proposition 2.3. *Let R be a 2-torsion free $*$-ring and $n \in \mathbb{N}_0$. Then every Jordan (α, β)-higher $*$-derivation $D = (d_n)_{n\in\mathbb{N}_0}$ of R is a Jordan triple (α, β)-higher $*$-derivation of R.*

Proof. By the assumption, we have

$$d_n(x^2) = \sum_{i+j=n} d_i(\beta^j(x))d_j(\alpha^i(x^{*^i})). \tag{2.6}$$

for all $x, y \in R$. Write $w = x + y$ and using (2.6), we get

$$\begin{aligned} d_n(w^2) &= \sum_{i+j=n} d_i(\beta^j(x+y))d_j(\alpha^i((x+y)^{*^i})) \\ &= \sum_{i+j=n} (d_i\beta^j(x))d_j(\alpha^i(x^{*^i})) + d_i\beta^j(x))d_j(\alpha^i(y^{*^i})) \\ &\quad + d_i\beta^j(y))d_j(\alpha^i(x^{*^i})) + d_i\beta^j(y))d_j(\alpha^i(y^{*^i}))), \end{aligned}$$

and

$$\begin{aligned} d_n(w^2) &= d_n(x^2 + xy + yx + y^2) \\ &= d_n(x^2) + d_n(y^2) + d_n(xy + yx) \\ &= \sum_{l+m=n} d_l(\beta^m(x))d_m(\alpha^l(x^{*^i})) + \sum_{r+s=n} d_r(\beta^s(y))d_s(\alpha^r(y^{*^r})) \\ &\quad + d_n(xy + yx). \end{aligned}$$

Subtracting the last two expressions of $d_n(w^2)$ gives

$$d_n(xy + yx) = \sum_{i+j=n} \Big(d_i\beta^j(x))d_j(\alpha^i(y^{*^i})) + d_i\beta^j(y))d_j(\alpha^i(x^{*^i}))\Big). \tag{2.7}$$

Now take $c = x(xy+yx)+(xy+yx)x$. Using (2.7), we get

$$\begin{aligned}
d_n(c) &= \sum_{i+j=n} d_i(\beta^j(x))d_j(\alpha^i((xy+yx)^{*^i})) \\
&\quad + \sum_{i+j=n} d_i(\beta^j(xy+yx))d_j(\alpha^i(x^{*^i})) \\
&= \sum_{i+r+s=n} \Big(d_i(\beta^{r+s}(x))d_r(\beta^s(\alpha^i(x^{*^i})))d_s(\alpha^{i+r}(y^{*^{i+r}})) \\
&\quad + d_i(\beta^{r+s}(x))d_r(\beta^j(\alpha^i(y^{*^i})))d_s(\alpha^{i+r}(x^{*^{i+r}}))\Big) \\
&= \sum_{k+l+j=n} \Big(d_k(\beta^{l+j}(x))d_l(\beta^j(\alpha^k(y^{*^k})))d_j(\alpha^{k+l}(x^{*^{k+l}})) \\
&\quad + d_k(\beta^{l+j}(y))d_l(\beta^j(\alpha^k(x^{*^k})))d_j(\alpha^{k+l}(x^{*^{k+l}}))\Big) \\
&= \sum_{i+r+s=n} d_i(\beta^{r+s}(x))d_r(\beta^s(\alpha^i(x^{*^i})))d_s(\alpha^{i+r}(y^{*^{i+r}})) \\
&\quad + 2\sum_{i+j+k=n} d_i(\beta^{j+k}(x))d_j(\beta^k(\alpha^i(y^{*^i})))d_k(\alpha^{i+j}(x^{*^{i+j}})) \\
&\quad + \sum_{k+l+j=n} d_k(\beta^{l+j}(y))d_l(\beta^j(\alpha^k(x^{*^k})))d_j(\alpha^{k+l}(x^{*^{k+l}})).
\end{aligned}$$

Also, we have

$$\begin{aligned}
d_n(c) &= d_n(2xyx + (x^2y+yx^2)) \\
&= 2d_n(xyx) + d_n(x^2y+yx^2) \\
&= 2d_n(xyx) + \sum_{i+r+s=n} d_i(\beta^{r+s}(x))d_r(\beta^s(\alpha^i(x^{*^i})))d_s(\alpha^{i+r}(y^{*^{i+r}})) \\
&\quad + \sum_{k+l+j=n} d_k(\beta^{l+j}(y))d_l(\beta^j(\alpha^k(x^{*^k})))d_j(\alpha^{k+l}(x^{*^{k+l}})).
\end{aligned}$$

Subtracting the last two expressions of $d_n(c)$ and using the fact that R is 2-torsion free, we get

$$d_n(xyx) = \sum_{i+j+k=n} d_i(\beta^{j+k}(x))d_j(\beta^k(\alpha^i(y^{*^i})))d_k(\alpha^{i+j}(x^{*^{i+j}}))$$

for all $x, y \in R$. This proves the theorem. □

3 Main results

The main result of the present paper is the following theorem.

Theorem 3.1. *Let R be a 6-torsion free semiprime $*$-ring and β an autmorphism of R. Then every Jordan triple (α,β)-higher $*$-derivation $D = (d_n)_{n\in\mathbb{N}_0}$ of R, with $\alpha\beta = \beta\alpha$, is a Jordan (α,β)-higher $*$-derivation of R.*

Proof. We will use induction on n in our proof. We see trivially that $A_0(x) = 0$ for all $x \in R$. In case $n = 1$, we get from ([4], Theorem 2.1) that $A_1(x) = 0$ for all $x \in R$. So we suppose that $A_m(x) = 0$ for all $x \in R$ and $m < n$. In view of Lemma 2.2, we have

$$A_n(x)x^2 = 0 \text{ for all } x \in R \tag{3.1}$$

and

$$x^2A_n(x) = 0 \text{ for all } x \in R. \tag{3.2}$$

The replacement of $x+y$ for x in (3.1) gives

$$A_n(x)\beta^n(y^2)+A_n(y)\beta^n(x^2)+B_n(x,y)\beta^n(x^2+y^2)+(A_n \\ (x)+A_n(y)+B_n(x,y))\beta^n(xy+yx)=0 \text{ for all } x,y\in R. \tag{3.3}$$

By replacing x by $-x$ in (3.3) we obtain

$$A_n(x)\beta^n(y^2)+A_n(y)\beta^n(x^2)-B_n(x,y)\beta^n(x^2+y^2)-(A_n \\ (x)+A_n(y)-B_n(x,y))\beta^n(xy+yx)=0 \text{ for all } x,y\in R. \tag{3.4}$$

Adding (3.3) and (3.4) and using the fact that R is 2-torsion free, we get

$$B_n(x,y)\beta^n(x^2+y^2)+(A_n(x)+A_n(y)) \\ \beta^n(xy+yx)=0 \text{ for all } x,y\in R. \tag{3.5}$$

Substituting $2x$ for x in (3.5) gives in view of the fact that R is 2-torsion free that

$$4B_n(x,y)\beta^n(x^2)+B_n(x,y)\beta^n(y^2)+4A_n(x)\beta^n \\ (xy+yx)+A_n(y)(xy+yx)=0 \text{ for all } x,y\in R. \tag{3.6}$$

Comparing (3.5) and (3.6) we have, since R is 3-torsion free

$$B_n(x,y)\beta^n(y^2)+A_n(x)\beta^n(xy+yx)=0 \text{ for all } x,y\in R. \tag{3.7}$$

Multiply (3.7) by $A_n(A)x$ from the right and using (3.2), we arrive at

$$A_n(x)\beta^n(xy)A_n(x)\beta^n(x)+A_n(x)\beta^n(y)\beta^n(x) \\ A_n(x)\beta^n(x)=0 \text{ for all } x,y\in R. \tag{3.8}$$

Substituting y by yx in (3.8) and multiplying by x from the left we obtain using that β is onto $(\beta^n(x)A_n(x)\beta^n(x))R(\beta^n(x)A_n(x)\beta^n(x)) = \{0\}$ for all $x\in R$. But since R is semiprime $\beta^n(x)A_n(x)\beta^n(x)=0$ for all $x\in R$. So (3.8) reduces to $A_n(x)\beta^n(y)A_n(x)\beta^n(x)=0$, for all $x,y\in R$. Since β is onto, we have $A_n(x)\beta^n(x)RA_n(x)\beta^b(x)=\{0\}$ for all $x\in R$. Again, since R is semiprime, we have

$$A_n(x)\beta^n(x)=0 \text{ for all } x\in R. \tag{3.9}$$

In view of (3.9), (3.7) reduces to $B_n(x,y)\beta^n(x^2)+A_n(x)\beta^n(yx)=0$ for all $x,y\in R$. Multiplying this relation by $\beta^n(x)$ from left and by $A_n(x)$ from right we obtain for all $x,y\in R, \beta^n(x)A_n(x)\beta^n(x)A_n(x)=0$. Since β is onto we get for all $x\in R, \beta^n(x)A_n(x)R\beta^n(x)A_n(x)=\{0\}$ and by the semiprimeness of R we have

$$\beta^n(x)A_n(x)=0 \quad \text{for all } x\in R. \tag{3.10}$$

Linearizing (3.9) we have

$$A_n(x)\beta^n(y)+A_n(y)\beta^n(x)+B_n(x,y)\beta^n(x+y)=0 \text{ for all } x,y\in R. \tag{3.11}$$

Taking $x=-x$ in (3.11), we obtain

$$A_n(x)\beta^n(y)-A_n(y)\beta^n(x)+B_n(x,y)\beta^n(x-y)=0 \text{ for all } x,y\in R. \tag{3.12}$$

Adding (3.11) and (3.12) we obtain, since R is 2-torsion free

$$A_n(x)\beta^n(y)+B_n(x,y)\beta^n(x)=0 \text{ for all } x,y\in R. \tag{3.13}$$

Right multiplication (3.13) by $A_n(x)$ and using (3.10) gives for all $x,y\in R$, $A_n(x)\beta^n(y)A_n(x)=0$. Since β is onto, we get $A_n(x)RA_n(x)=0$ for all $x\in R$. By the semiprimeness of R, we conclude that $A_n(x)=0$ for all $x\in R$. Hence, every Jordan triple (α,β)-higher $*$-derivation is a Jordan (α,β)-higher $*$-derivation.

□

In view of Theorem 3.1 and Proposition 2.3, we have the following result.

Theorem 3.2. *Let R be a 6-torsion free semiprime ring with involution and α, β be the endomorphisms of R such that β is onto. If $\alpha\beta = \beta\alpha$, then the notions of Jordan (α, β)-higher $*$-derivation and Jordan triple (α, β)-higher $*$-derivation on a 6-torsion free semiprime $*$-ring are equivalent.*

The following corollaries are immediate consequences of Theorem 3.2

Corollary 3.3. *([4], Theorem 2.1) Let R be a 6-torsion free semiprime ring with involution and α, β be the endomorphisms of R such that β is onto. Then every Jordan triple $(\alpha, \beta)^*$-derivation of R is a Jordan $(\alpha, \beta)^*$-derivation.*

Corollary 3.4. *([15], Theorem 2.3) Let R be a 6-torsion free semiprime ring. Then, every Jordan triple higher $*$-derivation on R is a Jordan higher $*$-derivation.*

Corollary 3.5. *([23], Theorem 1) Let R be a 6-torsion free semiprime $*$-ring. Then every Jordan triple $*$-derivation of R is a Jordan higher $*$-derivation of R.*

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