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Arithmetic Function Graph of a Finite Group

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Abstract Let h be an arithmetical function. We define arithmetic function graph $G_h(\mathfrak{G})$ of a given finite group $\mathfrak{G}$ with respect to h, as a graph with vertex set $V(G_h(\mathfrak{G})) = \mathfrak{G}$ and any two distinct vertices a and b are adjacent in $G_h(\mathfrak{G})$ if and only if $h(|a||b|) = h(|a|)h(|b|)$. It is observed that the order prime graph of a finite group is nothing but the arithmetic function graph with respect to the Euler's ϕ-function. In this paper, we investigate some results regrading arithmetic function graphs of finite groups.

1 Introduction

For standard terminology and notion in group theory, number theory, graph theory and matrices, we refer the reader to the text-books of Herstein [8], Apostol [2], Harary [7] and Bapat [3]. The non-standard will be given in this paper as and when required.

Throughout this paper, $\mathfrak{G}$ denotes a finite group and we denote the identity element of $\mathfrak{G}$ by e. The order of an element a in a group $\mathfrak{G}$ is denoted by $|a|$ and order of $\mathfrak{G}$ is denoted by $|\mathfrak{G}|$. The group of residue classes modulo n is denoted by $\mathbb{Z}_n$. The Klein's 4-group is denoted by V_4.

For a graph G, $V(G)$ and $E(G)$ denote vertex set and edge set of G, respectively. If the adjacency matrix A_G of a graph G is an $n \times n$ matrix and $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the eigenvalues of A_G, the energy of G is defined as

$$\mathcal{E}(G) = \sum_{i=1}^{n} |\lambda_i|.$$

The spectrum of the graph G is the collection of eigen values of the adjacency matrix of G.

The greatest common divisor of two numbers r and s is denoted by gcd(r, s). A function $f : \mathbb{N} \to \mathbb{C}$ is called an arithmetical function. An arithmetical function h is multiplicative if it is not is not identically zero and $h(rs) = h(r)h(s)$ whenever gcd$(r, s) = 1$. We say that a multiplicative function h is completely multiplicative if $h(rs) = h(r)h(s)$ for all $r, s \in \mathbb{N}$. Throughout this paper, h denotes an arithmetical function that is not identically zero.

In [16], M. Sattanathan and R. Kala defined the order prime graphs of finite groups and studied some properties of order prime graphs. Further, Ma et al. [10], Dorbidi [6] and Rajendra et al. [14, 15] have studied order prime graphs of finite groups, but Ma et al. and Dorbidi have called order prime graphs as coprime graphs. Rajendra et al. defined the general order prime graphs of finite groups and studied some properties in [11, 12, 14]. Rajendra et al. [13] defined the set-prime graph of a finite group $\mathfrak{G}$ of order n with respect to a non-empty set S of positive integers.

In [1], the authors defined the prime coprime graph $\Theta(\mathfrak{G})$ of a finite group $\mathfrak{G}$ defined as: The vertex set of $\Theta(\mathfrak{G})$ is $\mathfrak{G}$, and any two vertices x, y in $\Theta(\mathfrak{G})$ are adjacent if and only if gcd$(|x|, |y|)$ is equal to 1 or a prime number. The general order prime graph and the prime coprime graph

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coincide for all finite groups except for the cyclic groups of prime order. Also, the prime coprime graph of a finite group which coincides with the concepts of the set-prime graph $G_S(\mathfrak{G})$ (see [13]) of a finite group $\mathfrak{G}$ for some particular set S.

In this paper, we define the arithmetic function graph of a finite group with respect to an arithmetical function. It is observed that the order prime graph of a finite group is nothing but the arithmetic function graph with respect to the Euler's ϕ-function. We investigate some results related to diameter, dominating sets, planarity and isomorphism of arithmetic function graphs of finite groups.

2 Definitions

Let $\mathfrak{G}$ be a finite group of order n and let S be a non-empty set of positive integers. We recall the definition of order prime graph and we define arithmetic function graph of $\mathfrak{G}$ with respect to an arithmetical function. For standard terminology and notion in group theory, number theory, graph theory and matrices, we refer the reader to the text-books of Herstein [8], Apostol [2], Harary [7] and Bapat [3]. The non-standard will be given in this paper as and when required.

Throughout this paper, $\mathfrak{G}$ denotes a finite group and we denote the identity element of $\mathfrak{G}$ by e. The order of an element a in a group $\mathfrak{G}$ is denoted by $|a|$ and order of $\mathfrak{G}$ is denoted by $|\mathfrak{G}|$. The group of residue classes modulo n is denoted by $\mathbb{Z}_n$. The Klein's 4-group is denoted by V_4.

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The greatest common divisor of two numbers r and s is denoted by $\gcd(r,s)$. A function $f : \mathbb{N} \to \mathbb{C}$ is called an arithmetical function. An arithmetical function h is multiplicative if it is not is not identically zero and $h(rs) = h(r)h(s)$ whenever $\gcd(r,s) = 1$. We say that a multiplicative function h is completely multiplicative if $h(rs) = h(r)h(s)$ for all $r, s \in \mathbb{N}$. Throughout this paper, h denotes an arithmetical function that is not identically zero.

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In this paper, we define the arithmetic function graph of a finite group with respect to an arithmetical function. It is observed that the order prime graph of a finite group is nothing but the arithmetic function graph with respect to the Euler's ϕ-function. We investigate some results related to diameter, dominating sets, planarity and isomorphism of arithmetic function graphs of finite groups.

3 Definitions

Let $\mathfrak{G}$ be a finite group of order n and let S be a non-empty set of positive integers. We recall the definition of order prime graph and we define arithmetic function graph of $\mathfrak{G}$ with respect to an arithmetical function.

Definition 3.1. [16] The order prime graph $OP(\mathfrak{G})$ of $\mathfrak{G}$ is defined as a graph with the vertex set $V(OP(\mathfrak{G})) = \mathfrak{G}$ and any two vertices a and b are adjacent in $OP(\mathfrak{G})$ if and only if $\gcd(|a|, |b|) = 1$.

Definition 3.2. The Arithmetic function graph $G_h(\mathfrak{G})$ of $\mathfrak{G}$ with respect to an arithmetical function h is defined as a graph with vertex set $V(G_h(\mathfrak{G})) = \mathfrak{G}$ and two vertices a and b are adjacent in $G_h(\mathfrak{G})$ if and only if $h(|a||b|) = h(|a|)h(|b|)$.

By the Definition 3.2, it is clear that, for any arithmetical function h and any finite group $\mathfrak{G}$, the arithmetic function graph $G_h(\mathfrak{G})$ is a simple graph. Rani Jose and D. Susha have introduced the μ-graph of a finite group [9], which serves as an example of an arithmetic function graph with $h = \mu$. In the next section, it is shown that the order prime graph $OP(\mathfrak{G})$ is nothing but the arithmetic function graph $G_\phi(\mathfrak{G})$ with respect to the Euler's ϕ-function (see Corollary 4.11).

4 Results

Proposition 4.1. *Let h be an arithmetical function. If $\mathfrak{G}_1$ is a subgroup of a finite group $\mathfrak{G}_2$, then $G_h(\mathfrak{G}_1)$ is a subgraph of $G_h(\mathfrak{G}_2)$.*

Proof. Suppose that $\mathfrak{G}_1$ is a subgroup of the finite group $\mathfrak{G}_2$. Then $V(G_h(\mathfrak{G}_1)) \subset V(G_h(\mathfrak{G}_2))$ and the order of an element x in $\mathfrak{G}_1$ remains same in both $\mathfrak{G}_1$ and $\mathfrak{G}_2$. Hence, if $x, y \in V(G_h(\mathfrak{G}_1))$ are adjacent in $G_h(\mathfrak{G}_1)$, then they remain adjacent $G_h(\mathfrak{G}_2)$. So, $G_h(\mathfrak{G}_1)$ is a subgraph of $G_h(\mathfrak{G}_2)$. □

Proposition 4.2. *Let $\mathfrak{G}$ be a finite group. If h is a completely multiplicative function, then $G_h(\mathfrak{G})$ is a complete graph.*

Proof. Suppose that h is a completely multiplicative function. Then $h(rs) = h(r)h(s)$ for all $r, s \in \mathbb{N}$. Hence $h(|a||b|) = h(|a|)h(|b|)$ for any two elements a and b in $\mathfrak{G}$. Therefore, any two vertices in $G_h(\mathfrak{G})$ are adjacent and consequently, $G_h(\mathfrak{G})$ □

Proposition 4.3. *Let $\mathfrak{G}_1$ and $\mathfrak{G}_2$ be two finite groups and let h be a multiplicative function. If $\mathfrak{G}_1 \cong \mathfrak{G}_2$, then $G_h(\mathfrak{G}_1) \cong G_h(\mathfrak{G}_2)$.*

Proof. Let $\eta : \mathfrak{G}_1 \to \mathfrak{G}_2$ be a group isomorphism. Clearly, η is a bijective mapping of $V(G_h(\mathfrak{G}_1))$ onto $V(G_h(\mathfrak{G}_2))$. Let x and y be two vertices in $G_h(\mathfrak{G}_1)$. Since $|a| = |\eta(a)|, \forall a \in \mathfrak{G}$, it follows that, $xy \in E(G_h(\mathfrak{G}_1))$ if and only if $\eta(x)\eta(y) \in E(G_h(\mathfrak{G}_2))$. Thus, η is a graph isomorphism of $G_h(\mathfrak{G}_1)$ onto $G_h(\mathfrak{G}_2)$. □

Remark 4.4. The converse of the Proposition 4.3 is not true in general. For instance, for a completely multiplicative function h, we have $G_h(V_4) \cong G_h(\mathbb{Z}_4) \cong K_4$, but the groups V_4 and $\mathbb{Z}_4$ are not isomorphic.

Proposition 4.5. *Let $\mathfrak{G}$ be a finite group of order n. If h is an arithmetical function such that $h(1) = 1$, then*

(1) $G_h(\mathfrak{G})$ *is a connected graph,*

(2) $\{e\}$ *is a dominating set,*

(3) $\deg(e) = n - 1$ *and* $\Delta(G_h(\mathfrak{G})) = n - 1$,

(4) $diam(G_S(\mathfrak{G})) \leq 2$.

(5) $(n-1) \leq |E(G_h(\mathfrak{G}))| \leq \binom{n}{2}$.

Proof. Suppose that h is an arithmetical function such that $h(1) = 1$. Then $h(|e||a|) = h(1 \cdot |a|) = h(|a|) = h(1)h(|a|) = h(|e|)h(|a|)$ for any element a in $\mathfrak{G}$. Therefore, e is adjacent to every other vertex in $G_h(\mathfrak{G})$ and $\deg(e) = n - 1$. Therefore, $G_h(\mathfrak{G})$ is a connected graph, $\{e\}$ is a dominating set, $\Delta(G_h(\mathfrak{G})) = n - 1$ and $diam(G_S(\mathfrak{G})) \leq 2$; proving (1)-(4). Since $G_h(\mathfrak{G})$ is a simple graph with $\deg(e) = n - 1$, (5) follows. □

Corollary 4.6. *Let $\mathfrak{G}$ be a finite group order n. If h is multiplicative, then*

(1) $G_h(\mathfrak{G})$ *is a connected graph,*

(2) $\{e\}$ *is a dominating set,*

(3) $\deg(e) = n - 1$ *and* $\Delta(G_h(\mathfrak{G})) = n - 1$,

(4) $diam(G_S(\mathfrak{G})) \leq 2$.

(5) $(n - 1) \leq |E(G_h(\mathfrak{G}))| \leq \binom{n}{2}$.

Proof. If h is multiplicative, then $h(1) = 1$ (see [2, Theorem 2.12]) and hence by Proposition 4, the corollary follows. □

Remark 4.7. If $h(1) \neq 1$, then h is not multiplicative and the arithmetic function graph of a finite group may or may not be connected. For instance, consider the Mangoldt's function $\Lambda(n)$:

$$\Lambda(n) := \begin{cases} \log p, & \text{if } n = p^m \text{ for some prime } p \text{ and some } m \geq 1, \\ 0, & \text{otherwise.} \end{cases}$$

Note that Mangoldt's function is not multiplicative. We have $\Lambda(1) = 0$, $\Lambda(2) = \log 2$ and $\Lambda(1 \cdot 2) \neq \Lambda(1)\Lambda(2)$. Therefore, in the Λ-graph $G_\Lambda(V_4)$ of the Klein-4 group $V_4 = \{e, a, b, c\}$, a, b and c are not adjacent to e. Hence $G_\Lambda(V_4)$ is disconnected. But the Λ-graph $G_\Lambda(\mathbb{Z}_6)$ is a connected graph given in Figure 1.

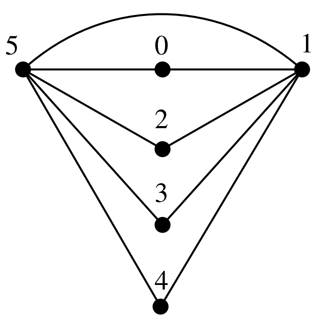

Figure 1. Λ-graph $G_\Lambda(\mathbb{Z}_6)$ of $\mathbb{Z}_6$

Proposition 4.8. *Let $\mathfrak{G}$ be a finite group of order n and h be a multiplicative function. If $x \in \mathfrak{G}$ such that $|x|$ is relatively prime to $|y|$ for all $y \neq x$ in $\mathfrak{G}$, then* $\deg(x) = n - 1$ *and* $\{x\}$ *is a dominating set of* $G_h(\mathfrak{G})$.

Proof. Suppose that $x \in \mathfrak{G}$ such that $|x|$ is relatively prime to $|y|$ for all $y \neq x$ in $\mathfrak{G}$. Then $h(|x||y|) = h(|x|)h(|y|)$, for all $y \neq x$ in $\mathfrak{G}$. Hence, x is adjacent to every other vertex in $G_h(\mathfrak{G})$ and consequently, $\deg(x) = n - 1$ and $\{x\}$ is a dominating set of $G_h(\mathfrak{G})$. □

Theorem 4.9. *Let $\mathfrak{G}$ be a finite group. If h is a multiplicative function, then $OP(\mathfrak{G})$ is a subgroup of $G_h(\mathfrak{G})$.*

Proof. Suppose that h is a multiplicative function. Let $a, b \in \mathfrak{G}$. Then, we have

$$\begin{aligned} a \text{ and } b \text{ are adjacent in } OP(\mathfrak{G}) &\implies \gcd(|a|, |b|) = 1 \\ &\implies h(|a||b|) = h(|a|)h(|b|) \\ &\implies a \text{ and } b \text{ are adjacent in } G_h(\mathfrak{G}). \end{aligned}$$

Therefore, $OP(\mathfrak{G})$ is a subgroup of $G_h(\mathfrak{G})$. □

The Theorem 4.9 together with the Definitions 3.1 and 3.2, gives the following result:

Corollary 4.10. *Let h be a multiplicative function and $\mathfrak{G}$ be a finite group. Then*

$$OP(\mathfrak{G}) = G_h(\mathfrak{G}) \text{ if and only if for any } a, b \in \mathfrak{G},\ h(|a||b|) = h(|a|)h(|b|) \text{ implies } \gcd(|a|, |b|) = 1.$$

Corollary 4.11. *Let $\mathfrak{G}$ be a finite group. For the Euler's ϕ-function,*

$$G_\phi(\mathfrak{G}) = OP(\mathfrak{G}).$$

Proof. The Euler's ϕ-function is a multiplicative function and we have the identity

$$\phi(mn) = \phi(m)\phi(n)\frac{d}{\phi(d)},$$

where $d = \gcd(m, n)$. We have $\varphi(d) < d$ whenever $d > 1$. Therefore, if $\gcd(m, n) > 1$, then $\phi(mn) > \phi(m)\phi(n)$. So,

$$\phi(mn) = \phi(m)\phi(n) \text{ if and only if } \gcd(m, n) = 1.$$

Thus, for any $a, b \in \mathfrak{G}$, $\phi(|a||b|) = \phi(|a|)\phi(|b|)$ implies $\gcd(|a|, |b|) = 1$. Hence, by the Corollary 4.10, $G_\phi(\mathfrak{G}) = OP(\mathfrak{G})$. □

Corollary 4.12. *Let $\mathfrak{G}$ be a finite group and F be the set of all multiplicative functions. Then*

$$\bigcap_{f \in F} G_h(\mathfrak{G}) = OP(\mathfrak{G}).$$

Proof. By Theorem 4.9, we have

$$OP(\mathfrak{G}) \subset G_h(\mathfrak{G}), \quad \forall f \in F \tag{4.1}$$

From the Corollary 4.11, we have

$$G_\phi(\mathfrak{G}) = OP(\mathfrak{G}) \tag{4.2}$$

Since $\phi \in F$, combining (4.1) and (4.2), we get $\bigcap_{f \in F} G_h(\mathfrak{G}) = OP(\mathfrak{G})$. □

Observation: We observe that, if h is an arithmetical function with $h(1) = 1$, then

(i) $G_h(\mathbb{Z}_2) \cong K_2$,

(ii) $G_h(\mathbb{Z}_3) \cong K_3$ if and only if $h(3^2) = h(3)^2$,

(iii) $G_h(\mathbb{Z}_3) \cong K_{1,2}$ if and only if $h(3^2) \neq h(3)^2$.

Theorem 4.13. *Let $\mathfrak{G}$ be a finite group of order $p > 2$, where p is a prime, and h be a multiplicative function. Then*

(i) $G_h(\mathfrak{G})$ is a star if and only if $h(p^2) \neq h(p)^2$,

(ii) $G_h(\mathfrak{G})$ is a complete graph if and only if $h(p^2) = h(p)^2$.

Proof. (i) Suppose that $G_h(\mathfrak{G})$ is a star. Since h is multiplicative, $h(1) = 1$ and so e is adjacent to every other vertex a in $G_h(\mathfrak{G})$. Hence, it follows that, any two vertices a, b different from e are not adjacent in $G_h(\mathfrak{G})$. This implies that $h(|a||b|) \neq h(|a|)h(|b|)$, which gives $h(p^2) \neq h(p)^2$ ($\because |\mathfrak{G}| = p$ is a prime, $|a| = p, \forall a \neq e$ in $\mathfrak{G}$).

Conversely, suppose that $h(p^2) \neq h(p)^2$. Since h is multiplicative, $h(1) = 1$ and so e is adjacent to every other vertex a in $G_h(\mathfrak{G})$. Let $a, b \in \mathfrak{G}$, $a \neq b$ and $a, b \neq e$. Since $|\mathfrak{G}| = p$ is a prime, $|a| = |b| = p$. Therefore, we have

$$h(|a||b|) = h(p^2) \neq h(p)^2 = h(p)h(p) = h(|a|)h(|b|),$$

and consequently, a and b are not adjacent in $G_h(\mathfrak{G})$. Since a, b are arbitrary elements of $\mathfrak{G}$ different from e, it follows that, $G_h(\mathfrak{G})$ is a star isomorphic to $K_{1,p-1}$.

(ii) Suppose that $G_h(\mathfrak{G})$ is a complete graph. Since $p > 2$, there are at least two elements in $\mathfrak{G}$ different from e. Let $a, b \in \mathfrak{G}$, $a \neq b$ and $a, b \neq e$. Since $|\mathfrak{G}| = p$ is a prime, $|a| = |b| = p$. Since $G_h(\mathfrak{G})$ is a complete graph, a and b are adjacent in $G_h(\mathfrak{G})$ and so

$$\begin{aligned} & h(|a||b|) = h(|a|)h(|b|) \\ \Longrightarrow \quad & h(p^2) = h(p)h(p) = h(p)^2. \end{aligned}$$

Conversely, suppose that $h(p^2) = h(p)^2$. Since h is multiplicative, $h(1) = 1$ and so e is adjacent to every other vertex a in $G_h(\mathfrak{G})$. Let $a, b \in \mathfrak{G}$, $a \neq b$ and $a, b \neq e$. Since $|\mathfrak{G}| = p$ is a prime, $|a| = |b| = p$. Therefore, we have

$$h(|a||b|) = h(p^2) = h(p)^2 = h(p)h(p) = h(|a|)h(|b|),$$

and consequently, a and b are adjacent in $G_h(\mathfrak{G})$. Thus, it follows that, any two vertices in $G_h(\mathfrak{G})$ are adjacent and consequently, $G_h(\mathfrak{G})$ is a complete graph isomorphic to K_p. □

The following two corollaries are immediate from the Theorem 4.13:

Corollary 4.14. *Let $\mathfrak{G}$ be a finite group of order $p > 2$, where p is a prime, and h be a multiplicative function. Then*

(i) $diam(G_h(\mathfrak{G})) = 2$ and $\chi(G_h(\mathfrak{G})) = 2$, if $h(p^2) \neq h(p)^2$;

(ii) $diam(G_h(\mathfrak{G})) = 1$ and $\chi(G_h(\mathfrak{G})) = n$, if $h(p^2) = h(p)^2$.

Corollary 4.15. *Let $\mathfrak{G}$ be a finite group of order $p > 2$, where p is a prime, and h be a multiplicative function.*

(i) If $h(p^2) \neq h(p)^2$, the graph $G_h(\mathfrak{G})$ has spectrum

$$\begin{matrix} \textit{eigen value} \rightarrow \\ \textit{multiplicity} \rightarrow \end{matrix} \begin{pmatrix} \sqrt{p-1} & 0 & -\sqrt{p-1} \\ 1 & p-2 & 1 \end{pmatrix}$$

and its energy is $\mathcal{E}(G_h(\mathfrak{G})) = 2\sqrt{n-1}$.

(ii) If $h(p^2) = h(p)^2$, the graph $G_h(\mathfrak{G})$ has spectrum (eigen values of adjacency matrix)

$$\begin{matrix} \textit{eigen value} \rightarrow \\ \textit{multiplicity} \rightarrow \end{matrix} \begin{pmatrix} p-1 & -1 \\ 1 & p-1 \end{pmatrix}$$

and its energy is $\mathcal{E}(G_h(\mathfrak{G})) = 2(p-1)$.

Theorem 4.16. *Let $\mathfrak{G}$ be a finite group of order n and h be a multiplicative function. If n is a composite number that is not a power of a prime, then $G_h(\mathfrak{G})$ is not a star.*

Proof. Suppose that n is a composite number that is not a power of a prime. Then there exist two distinct prime divisors of n, say p and q . By Cauchy's theorem for finite abelian groups, there exist elements $a, b \in \mathfrak{G}$ such that $|a| = p$ and $|b| = q$. Clearly, $\gcd(p, q) = 1$ and since h is multiplicative, we have

$$h(|a||b|) = h(pq) = h(p)h(q) = h(|a|)h(|b|).$$

Therefore, a and b are adjacent in $G_h(\mathfrak{G})$. Since h is multiplicative, $h(1) = 1$ and so e is adjacent to every other vertex in $G_h(\mathfrak{G})$. Therefore, there is a triangle (cycle of length 3) with vertices e, a, b in $G_h(\mathfrak{G})$. Hence, $G_h(\mathfrak{G})$ is not a star. □

By taking contrapositive of the Theorem 4.16, we have the following result:

Corollary 4.17. *Let $\mathfrak{G}$ be a finite group of order n and h be a multiplicative function. If $G_h(\mathfrak{G})$ is a star, then $n = p^k$, where p is a prime and $k \geq 1$.*

By the proof of the Theorem 4.16, we have the following result:

Corollary 4.18. *Let $\mathfrak{G}$ be a finite group of order n and h be a multiplicative function. If n is a composite number that is not a power of a prime, then*

(i) $G_h(\mathfrak{G})$ *has at least one triangle;*

(ii) $\chi(G_h(\mathfrak{G})) \geq 3$.

Theorem 4.19. *Let h be a multiplicative function such that $h(r^2) \neq h(r)^2$ for any prime r. If $\mathfrak{G}$ is a non-abelian group of order pq, where p and q are distinct primes with $p < q$, then*

(i) $G_h(\mathfrak{G})$ *is a complete tripartite graph isomorphic to $K_{1,r,s}$, where r and s are the number of elements of order p and q, respectively;*

(ii) $\chi(G_h(\mathfrak{G})) = 3$*;*

(iii) $G_h(\mathfrak{G})$ *is non-planar of diameter* 2.

Proof. Suppose that $\mathfrak{G}$ is a non-abelian group of order pq, where p and q are distinct primes $p < q$. Then $p|(q-1)$, the number of elements of G of order p is a multiple of q and the number of elements of G of order q is a multiple of p. Also, the order of any element other than e is either p or q. There exists elements x, y in $\mathfrak{G}$ with $|x| = p$ and $|y| = q$. Since, p, q are distinct primes, they are relatively prime. Since h is multiplicative, it follows that e, x, y are mutually adjacent in $G_h(\mathfrak{G})$ and form a triangle. Let $P = \{x \in \mathfrak{G} \ : \ |x| = p\}$ and $Q = \{y \in \mathfrak{G} \ : \ |y| = q\}$. Note that P and Q are non-empty sets. Also, $\{e\}, P, Q$ are mutually disjoint maximal independent sets of $G_h(\mathfrak{G})$. We see that every vertex of each set of these three sets is adjacent to every vertex in the other two sets. Therefore, $G_h(\mathfrak{G})$ is a complete tripartite graph isomorphic $K_{1,r,s}$, where $r = |P|$ and $s = |Q|$, and consequently, $\chi(G_h(\mathfrak{G})) = 3$.

Since p and q are distinct primes with $p < q$, it follows that $p \geq 2$ and $q \geq 3$. Since $r = |P|$ is a multiple of q and $s = |Q|$ is a multiple of p, we have $r \geq 3$ and $s \geq 2$. Hence, the complete tripartite graph $K_{1,r,s}$ is a non-planar graph of diameter 2. Therefore, $G_h(\mathfrak{G})$ is non-planar of diameter 2. □

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