Palestine Journal of Mathematics

Vol. 11(2)(2022) , 496–504

ON AN ARITHMETICAL FUNCTIONS INVOLVING GENERAL EXPONENTIAL

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Communicated by Ayman Badawi

MSC 2010 Classifications: Primary 11A25,11K65, 11N60; Secondary 05A17, 11P82, 33B10.

Keywords and phrases: Arithmetic function, Liouville function, Möbius function, Riemann Zeta function.

Abstract An innovative number theoretic function (arithmetic function) is introduced having analogous and additional new properties to the most commonly occurring arithmetic functions in the analytic number theory. This new function involves the general exponential term of the form $V(n) = a^{\Omega(n)}$ where $\Omega(n)$ is the total number of prime factors of n counted with multiplicity, and a is any integer. Many existing arithmetic functions become special cases of $V(n)$ as it is exciting to note that on the set of square-free positive integers and for $a = -1$, $V(n) = \mu(n)$ where $\mu(n)$ is the Möbius μ-function. Further, for $a = -1$, and for all n, $V(n) = \lambda(n)$, thus $V(n)$ reduces to the traditional Liouville λ-function. In this respect, function $V(n)$ may be thought of as an extension of the Liouville and Möbius functions. Some new results are also formulated and studied using this general exponential arithmetic function$V(n)$.

1 Introduction

Number theory is a gigantic and mesmerizing field of mathematics, also referred to as higher arithmetic, which studies the properties of numbers. Since the integers and the prime numbers have fascinated the people since the ancient times. Analytic number theory is theory of number where one makes use of the techniques of real and complex analysis to address number-theoretical problems and establishes its truth: (cf. [1], [5], [7], [8]). A feature of analytic number theory is the treatment of number-theoretical problems and provide answers to long-standing intrinsic interesting problems concerning what happens for large values of some parameter and also enumerate problems involving primes, Diophantine equations, or similar number-theoretic objects..

Also mathematical analysis tools are being used to prove various results about prime numbers and functions with integer's domain. A class of functions with any real or complex valued function with domain of the positive integers is said to be an arithmetic function. In the literature of the analytic number theory, there are definitions for a range of arithmetic functions or also refereed as number theoretic functions. There are numerous meticulous arithmetic functions, connected with some important notions, which appear in number theory, and are engaged in studies on the various properties of numbers. A foremost category of arithmetic functions which are frequently occurring are: number of divisors $\tau(n)$ or sum of divisor $\sigma(n)$ Möbius function $\mu(n)$, Euler's totient function $\phi(n)$, Liouville's function $\lambda(n)$, Von Mangoldt function $\Lambda(n)$ with their traditional symbolic notations (cf. [1], [4],[9]). Keeping in view the scattered and of autonomous interest of arithmetic functions in various fields of their study constitutes an important field of study. The definition and rules governing the variant of arithmetic functions are usually not illustrated by straight forward methods, also the asymptotic activities in terms of arithmetical functions is determined. The study of their common values is of great importance, as numerous arithmetic functions are not invariant. There are many generalizations of the arithmetic functions, as well as analogous functions described by different authors: (cf. [2], [4], [9]), they have introduced and studied properties of arithmetic functions. Articles related to a new arithmetic functions introduced and studied by authors (cf. [3], [7]).

In the present paper, a new arithmetic function $V(n)$ is introduced, which is defined $V : \mathbb{N} \to \mathbb{C}$ as follows:

$$V(n) = \begin{cases} 1 \text{ if } n = 1 \text{ or } a = 1 \\ a^{\Omega(n)} \text{ if } n > 1, a \neq 1 \text{ and } n = \prod_{i=1}^{k} p_i^{\alpha_i}. \end{cases} \tag{1.1}$$

Here is a is any integer and $\Omega(n)$ is the total number of prime factors of n counted with multiplicity. Clearly, the domain of for this function is the set of all positive integers, therefore $V(n)$ is an arithmetic function. Arithmetic function $V(n)$ defined in (1.1) is further incorporated to infinite products, partition of an integer and expressions connecting with others arithmetic functions in the field of analytic number theory. It has also been shown that Möbius function, Liouville's function are special cases of this function under certain conditions. Various other new results are established and studied using this general exponential arithmetic function.

2 Properties of the Function $V(n)$

If in the definition of an arithmetic function $V(n)$ defined in (1.1), taking $a = -1$, and n is square free, then$V(n) = (-1)^k$, $V(n)$ is equivalent to the Mobius function $\mu(n)$ on the set of square-free positive integers. If a is restricted to -1 in the definition of $V(n)$, then $V(n) = \lambda(n)$, Liouville's function. Hence Mobius function and Liouville's function can be deduced from$V(n)$under above stated conditions.

2.1 Special Cases

2.1.1 Case I

When $n = p, p$ is a prime number, then $V(n) = a, a$ is any integer.

2.1.2 Case II

When $n = p^k$, $k > 1$, p is a prime number, then $V(n) = a^k$, a is any integer.

2.1.3 Case III

When $a = p$, p is a prime number, then and $V(n) = p^{\Omega(n)}$.

3 $V(n)$is multiplicative and completely multiplicative function

Since an arithmetic function f is called multiplicative if $f(n_1n_2) = f(n_1)f(n_2)$ for all $n_1, n_2 \in N$ with condition $(n_1, n_2) = 1$. An arithmetic function f is called completely multiplicative if $f(n_1n_2) = f(n_1)f(n_2)$ for all positive integers $n_1, n_2 \in N$. Since for any positive integer$n$, the Möbius function $\mu(n)$ is defined by the following three properties:

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ 0 & \text{if } p^2/n \text{ for some prime } p \\ (-1)^k & \text{if } n = \prod_{j=1}^{k} p_j \text{ where } p_\text{j} \text{ are distinct primes} \end{cases} \tag{3.1}$$

$$\lambda(n) = \begin{cases} 1 \text{ if } n = 1 \\ (-1)^{\Omega(n)} \text{if } n > 1 \end{cases} \tag{3.2}$$

where $\Omega(n)$ is the total number of prime factors of n counted with multiplicity.

4 Some Theorems on $V(n)$

Theorem 4.1. *$V(n)$ is multiplicative function.*

Proof: Let $m, n \in \mathbb{N}$ such that $\gcd(m, n) = 1$. If $m = n = 1$, or $m \neq 1, n = 1$ or $m = 1, n \neq 1$ then nothing to prove. If $m \neq 1, n \neq 1$, using the fundamental theorem of arithmetic, m and n can be written as

$$m = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k}$$

and

$$n = q_1^{\beta_1} q_2^{\beta_2} q_3^{\beta_3} \cdots q_l^{\beta_l}$$

.

$$mn = \left(p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k}\right)\left(q_1^{\beta_1} q_2^{\beta_2} q_3^{\beta_3} \cdots q_l^{\beta_l}\right).$$

Since $\gcd(m, n) = 1$, therefore all p_i and q_j are distinct primes. Using definition of $V(n)$, we have $V(m) = a^{\alpha_1+\alpha_2+\cdots+\alpha_k}$, and $V(n) = a^{\beta_1+\beta_2+\cdots+\beta_l}$.

$$\begin{aligned}
&\Rightarrow V(mn) = a^{(\alpha_1+\alpha_2+\cdots+\alpha_k)+(\beta_1+\beta_2+\cdots+\beta_l)} \\
&= a^{\alpha_1+\alpha_2+\cdots+\alpha_k} a^{\beta_1+\beta_2+\cdots+\beta_l} \\
&= V(m)\,V(n)
\end{aligned}$$

Hence $V(mn) = V(m)\,V(n)$ with $m, n \in N$ with $(m, n) = 1$. This proves that $V(n)$ is a multiplicative function. □

Remark 4.2. $V(n)$ is completely multiplicative function, as $V(mn) = V(m)\,V(n)$ for all $m, n \in \mathbb{N}$.

Remark 4.3. $V(n)$ is not additive function i.e.$V(mn) \neq V(m) + V(n)$.

Example 4.4. Let $m = 2^3.5^2.7$and $n = 3^2.11$, here $\gcd(m, n) = 1$,therefore $V(m) = a^6$ and, $V(n) = a^3$. Now $mn = 2^3.5^2.7.3^2.11$, $V(mn) = a^9$. Thus $V(mn) \neq V(m) + V(n)$. Hence $V(n)$ is not an additive function .

Theorem 4.5. *For $n \geq 1$,$a \neq 1$ and $n = \prod_{j=1}^{k} p_j$ where p_j are distinct primes. Then*

$$\sum_{d|n} V(d) = \begin{cases} \frac{1}{(1-a)^k} \prod_{j=1}^{k} \left(1 - a^{\alpha_j+1}\right) \text{if} \quad a < 1 \\ \frac{1}{(a-1)^k} \prod_{j=1}^{k} \left(a^{\alpha_j+1} - 1\right) \text{if} \quad a > 1. \end{cases}$$

Proof. Since $V(n)$ is a multiplicative arithmetic function and let

$$F(n) = \sum_{d|n} V(d)$$

. Then $F(n)$ is also multiplicative function. Firstly let $n = p^k$, $k > 1$, p is a prime number

$$\begin{aligned}
&F\left(p^k\right) = \sum_{d|p^k} V(d) \\
&= V(1) + V(p) + V\left(p^2\right) + \cdots + V\left(p^k\right)
\end{aligned}$$

$$= 1 + a + a^2 + \cdots a^k = \sum_{j=0}^{k} a^j$$

$$= \begin{cases} \frac{a^{k+1}-1}{a-1} \text{if} a > 1 \\ \frac{1-a^{k+1}}{1-a} \text{if} a < 1 \end{cases}$$

Now when $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k}$. Using above, we obtain

$$\begin{aligned} F(n) &= F\left(p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k}\right) \\ &= F\left(p_1^{\alpha_1}\right) F\left(p_2^{\alpha_2}\right) \cdots F\left(p_k^{\alpha_k}\right) \\ &= \begin{cases} \frac{1}{(a-1)^k} \prod_{j=1}^{k} \left(a^{\alpha_j+1} - 1\right) \text{ if } a > 1 \\ \frac{1}{(1-a)^k} \prod_{j=1}^{k} \left(1 - a^{\alpha_j+1}\right) \text{ if } a < 1 \end{cases}. \end{aligned}$$

This proves the theorem. □

Corollary 4.6. *If* $a = -1$,$n > 1$, $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k}$.. *Then*

$$\sum_{d|n} V(d) = \begin{cases} 1 \textit{ if } a = -1,\ n = m^2 \textit{ for some integer } m \\ 0 \textit{ otherwise} \end{cases}$$

Proof: In the result of above theorem putting $a = -1$, we have

$$\sum_{d|n} V(d) = \frac{1}{2^k} \prod_{j=1}^{k} \left(1 - (-1)^{\alpha_j+1}\right)$$

Now if any of α_i $(1 \le i \le k)$ in $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k}$ is odd, then for that α_i, α_i+1 is even which give $(-1)^{\alpha_j+1} = 1$ consequently product on the right hand side of (4.1) is zero. Now if none of α_i $(1 \le i \le k)$ in $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k}$ is odd, then each α_i is even $(\alpha_i = 2\beta_i)$, therefore each α_i+1 is odd and $n = p_1^{2\beta_1} p_2^{2\beta_2} p_3^{2\beta_3} \cdots p_k^{2\beta_k} = m^2$. Consequently each $(-1)^{\alpha_j+1} = -1$, In this case (4.1) gives

$$\sum_{d|n} V(d) = \frac{1}{2^k} \prod_{j=1}^{k} \left(1 - (-1)^{\alpha_j+1}\right)$$

$$= \frac{1}{2^k} \prod_{j=1}^{k} (2.2.2 \cdots k \text{ times}) = \frac{1}{2^k} 2^k = 1.$$

Hence the result. □

Remark 4.7. Similar result is also established with the Liouville function$\lambda(n)$.

Theorem 4.8. *For* $n \ge 1$ *,*

$$\sum_{d|n} \mu(d) V(d) = \begin{cases} 1 \quad \text{if} \quad n = 1 \\ (1-a)^k \quad \text{if} \quad (n > 1), n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k} \end{cases}$$

where $\mu(n)$ *is the Möbius function .*

Proof. If $n = 1$, then

$$\sum_{d|1} \mu(d) V(d) = \mu(1) V(1) = 1.1 = 1$$

.

Since $V(n)$ and $\mu(n)$ are multiplicative function, therefore $F(n)$ defined by

$$F(n) = \sum_{d|n} \mu(d) V(d)$$

is also a multiplicative arithmetic function. Firstly, let $n = p^k$, $k > 1$, where p is a prime number.

$$F\left(p^k\right)=\sum_{d|p^k}\mu\left(d\right)V\left(d\right)=\mu\left(1\right)V\left(1\right)+\mu\left(p\right)V\left(p\right)+\cdots+\mu\left(p^k\right)V\left(p^k\right)$$
$$=1.1+\left(-1\right)V\left(p\right)+\cdots+0.V\left(p^k\right)$$
$$=1-V\left(p\right)=1-a.$$

Now when $n=p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_k^{\alpha_k}$, we have

$$F\left(n\right)=F\left(p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_k^{\alpha_k}\right)$$
$$=F\left(p_1^{\alpha_1}\right)F\left(p_2^{\alpha_2}\right)\cdots F\left(p_k^{\alpha_k}\right)$$
$$=\left(1-a\right)^k=\prod_{p|n}\left(1-V(a)\right).$$

This proves the theorem. □

Theorem 4.9. *For $n\geq 1$, $n=p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_k^{\alpha_k}$*

$$\sum_{d|n}V\left(\frac{n}{d}\right)2^{\omega(n)}=\prod_{j=1}^{k}\left(\frac{a^{\alpha_j+1}+a^{\alpha_j}-2}{a-1}\right)$$

where $\omega(n)$ is the number of distinct divisors of n.

Proof. Let $(n)=\sum_{d|n}V\left(\frac{n}{d}\right)2^{\omega(n)}$, where $F(n)$ is a Multiplicative function. Firstly, let $n=p^k$, $k>1$, where p is a prime number.

$$F\left(p^k\right)=\sum_{d|p^k}V\left(\frac{p^k}{d}\right)2^{\omega(p^k)}$$
$$=V\left(\frac{p^k}{1}\right)2^{\omega(1)}+V\left(\frac{p^k}{p}\right)2^{\omega(p)}+\cdots+V\left(\frac{p^k}{p^{k-1}}\right)2^{\omega(p^{k-1})}+V\left(\frac{p^k}{p^k}\right)2^{\omega(p^k)}$$
$$=a^k2^0+a^{k-1}2^1+\cdots+a2^1+1.2^1\ ,\ \left(\because\omega\left(1\right)=0,\omega(p)=\omega(p^2)=\cdots\omega(p^k)=1\right)$$
$$=a^k+2\left(a^{k-1}+a^{k-2}+\cdots a+1\right)$$
$$=a^k+2\frac{\left(a^k-1\right)}{a-1}$$
$$=\frac{a^{k+1}+a^k-2}{a-1}.$$

Now when $n=p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_k^{\alpha_k}$, we have

$$F\left(n\right)=F\left(p_1^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}\cdots p_k^{\alpha_k}\right)$$
$$=F\left(p_1^{\alpha_1}\right)F\left(p_2^{\alpha_2}\right)\cdots F\left(p_k^{\alpha_k}\right).$$

Using above results, we obtain

$$F\left(n\right)=\sum_{d|n}V\left(\frac{n}{d}\right)2^{\omega(n)}$$
$$=\prod_{j=1}^{k}\left(\frac{a^{\alpha_j+1}+a^{\alpha_j}-2}{a-1}\right).$$

Hence the theorem. □

Corollary 4.10. *If* $a = -1$,$n \geq 1$, $n = p_1^{\alpha_1} p_2^{\alpha_2} p_3^{\alpha_3} \cdots p_k^{\alpha_k}$. *Then the result of above theorem becomes the corresponding result for the Liouville function* $\lambda(n)$.
Proof. *Putting* $a = -1$, *in the result of the above theorem we have*

$$F\left(p^k\right) = \sum_{d|p^k} V\left(\frac{p^k}{d}\right) 2^{\omega(p^k)}$$
$$= \frac{a^{k+1} + a^k - 2}{a - 1}$$
$$= \frac{(-1)^{k+1} + (-1)^k - 2}{(-1) - 1}$$
$$= \begin{cases} \frac{1+(-1)-2}{(-1)-1} \text{ if } k \text{ is odd} \\ \frac{-1+1-2}{(-1)-1} \text{ if } k \text{ is even} \end{cases}$$
$$= 1.$$

Thus for $a = -1$, *we have*

$$\sum_{d|n} V\left(\frac{n}{d}\right) 2^{\omega(n)}$$
$$= \prod_{j=1}^{k} \left(\frac{a^{\alpha_j+1} + a^{\alpha_j} - 2}{a - 1}\right)$$
$$= \prod_{j=1}^{k} 1 = 1.$$

This proves the result. □

Remark 4.11. This corollary is also established with the Liouville function$\lambda(n)$.

Theorem 4.12. *Let* $R(x) = \sum_{n \leq x} V(n)$ *for a function* $V(n)$. *Then*

$$R(x) = n_0 + n_1 a + n_2 a^2 + n_3 a^3 + n_4 a^4 + n_5 a^5 + \cdots,$$

. *where* $n_0 = 1$; *then* $S_0 = \{1\}$, n_1 = *number of elements in* $S_1 = \sum_{p \leq x} 1$; $S_1 = \{2, 3, 5, \cdots\}$ *clearly* S_1 *contains all primes* $p \leq x$.*Similarly* n_2 = *number of elements in* $S_2 = \sum_{\substack{p^2 \leq x \text{ or} \\ p_1 p_2 \leq x}} 1$; *i.e.* $S_2 = \{4, 9, 25, \cdots\} \bigcup \{6, 10, 15, \cdots\}$ *which are square of prime or product of two primes* $\leq x$. *Further* n_3 = *number of elements in*

$S_3 = \sum_{p^3 \leq x \text{ or } p_1 p_2 p_3 \leq x \text{ or } p_1^2 p_2 \leq x} 1$

$S_3 = \{8, 27, 125, \cdots\} \bigcup \{30, 70, 105, \cdots\} \bigcup \{12, 18, 50, \cdots\}$

which are cubes of prime or product of three primes or product of square of prime a and another prime $\leq x$, *and so on.*

Further $\bigcup_{j=0}^{\infty} S_j = N$ *and* $\bigcap_{j=0}^{\infty} S_j = \phi$ *The polynomial* $R(x)$ *is a polynomial of degree 1, 2, 3….. in a, accordingly* $n \leq 3.99,\ n \leq 7.99,\ n \leq 15.99\ldots..\ n \leq x$.

□

Example 4.13. If $x = 100.7$, then $n_1 = 25$ for all primes $p \leq x$ $n_2 = 35$, $n_3 = 21$, $n_4 = 12$, $n_5 = 4$, $n_6 = 2$ $\bigcup_{j=0}^{6} n_j = \{1, 2, 3, 4, ..., 100\}$ and $\bigcap_{j=1}^{6} n_j = \phi$. $R(100.7) = 1 + 25a + 35a^2 + 21a^3 + 12a^4 + 4a^5 + 2a^6$ Table.1 above showing the number distribution in $n_j, j = 1, 2, 3 \cdots 6$ when $n \leq 100.7$

Table 1.

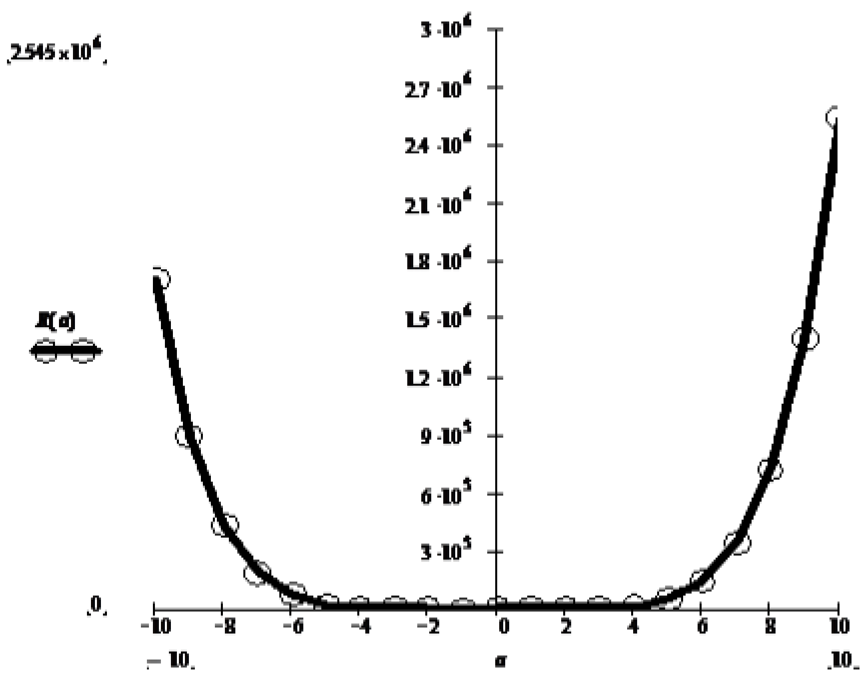
	On taking x=100.7
S1=	2,3,5,7,11,13,17,19,23,29,31,37,41,43, 47,53,59,61,67,71,73,79,83,89,97
S2=	4,6,9,10,14,15,21,22,25,26,33,34,35,38,39,46,49,51,55,57,58,62,63,65,69, 74,77,82,85,86,87,91,93,94,95
S3=	8,12,18,20,27,28,30,42,44,45,50,52,66,68,70,75,76,78,92,98,99
S4=	16,24,36,40,54,56,60,81,84,88,90,100
S5=	32,48,72,80
S6=	64, 96

In that case

$$R\,(100.7) = 1 + 25\,(-1) + 35(-1)^2 + 21(-1)^3 + 12(-1)^4 + 4(-1)^5 + 2(-1)^6$$
$$= 1 - 25 + 35 - 21 + 12 - 4 + 2$$
$$= 50 - 50 = 0$$

$$R\,(101.5) = 1 + 26\,(-1) + 35(-1)^2 + 21(-1)^3 + 12(-1)^4 + 4(-1)^5 + 2(-1)^6 \; = -1$$

Figure 1. Showing the value and sign of $R(x)$, $x \leq 100.7$ with the value of a



5 Riemann Zeta function, Dirichlet Series and $V\,(n)$

Since $f : \mathbb{N} \to \mathbb{C}$ define the Dirichlet Series of f to be $D_f\,(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$, defined for those $s \in C$ at which the series converges. Also $f : N \to C$ is multiplicative, then Riemann Zeta function

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{f(n)}{n^s}$$

$$= \prod_p \left(1 + \sum_{k \geq 1} \frac{f(p^k)}{p^{ks}}\right)$$

$$= \prod_p \left(\sum_{k \geq 0} \frac{f(p^k)}{p^{ks}}\right).$$

Theorem 5.1. *Show that* $D_V(s) = \sum_{n=1}^{\infty} \frac{V(n)}{n^s} = \prod_p \left(1 - \frac{a}{p^s}\right)^{-1}$ *where* $V(n) = a^{\Omega(n)}, a \neq 1, n > 1$, $V(n) = 1, a = 1, n = 1$; $\Omega(n)$ *being the total number of prime numbers in the factorization of* n*, including the multiplicity of the primes.*

Proof. Since

$$D_V(s) = \sum_{n=1}^{\infty} \frac{V(n)}{n^s} \qquad = \prod_p \left(V(1) + \frac{V(p)}{p^s} + \frac{V(p^2)}{p^{2s}} + \cdots \frac{V(p^k)}{p^{ks}} + \cdots\right).$$

Since $V(n)$ is a completely multiplicative function, therefore $V(p^k) = V(p)^k$.

$$= \prod_p \left(1 + \frac{a}{p^s} + \frac{a^2}{p^{2s}} + \cdots \frac{a^k}{p^{ks}} + \cdots\right)$$

$$= \prod_p \left(\frac{1}{1 - ap^{-s}}\right) = \prod_p \left(1 - \frac{a}{p^s}\right)^{-1}$$

.

For Re$(s) >$ Re(s_0) so long as $|ap^{-s}| < 1$ for all a, p.. Therefore if a > 1 and finite then

$$D_V(s) = \sum_{n=1}^{\infty} \frac{V(n)}{n^s} = \prod_p \left(1 - \frac{a}{p^s}\right)^{-1}$$

. Since the deletion or addition of finite number of terms does not effect the behavior of the series. Therefore if a$>$ 1 and finite then $D_V(s)$ in above equation will convergent for all prime $p^s > |a|, a \neq 1$. □

Deduction 1: When $a = -1$, is substituted in above obtained result we have

$$D_V(s) = \sum_{n=1}^{\infty} \frac{V(n)}{n^s}$$

$$= \prod_p \left(1 + \frac{1}{p^s}\right)^{-1}$$

$$= \prod_p \left(1 - \frac{1}{p^{2s}}\right)^{-1} \left(1 - \frac{1}{p^s}\right)$$

$$= \frac{\prod_p \left(1 - \frac{1}{p^{2s}}\right)^{-1}}{\prod_p \left(1 - \frac{1}{p^s}\right)^{-1}} = \frac{\zeta(2s)}{\zeta(s)}$$

Hence $D_V(s) = D_\lambda(s)$ (of Liouville function) when$a = -1$.

Deduction 2: When $a = 1$, is substituted in above obtained result we have $D_V(s) = \sum_{n=1}^{\infty} \frac{V(n)}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \zeta(s)$ Hence $D_V(s) = D_I(s)$ (of identity function) when $a = 1$, .

Theorem 5.2. *Let $V(n)$ is a completely multiplicative, therefore $\left|V(n)^{-1}\right| \le |V(n)|$ for all $n \ge 1$. Show that $D_V(s) = \sum_{n=1}^{\infty} \frac{V(n)}{n^s} = \prod_p \left(1 - \frac{a}{p^s}\right)^{-1}$ where $V(n) = a^{\Omega(n)}, a \ne 1, n > 1$, $V(n) = 1, a = 1, n = 1$; $\Omega(n)$ being the total number of prime numbers in the factorization of n, including the multiplicity of the primes.*

Proof: Since $V(n)$ is a completely multiplicative. Therefore

$$V(n)^{-1} = \mu(n) V(n)$$
$$= \begin{cases} (-1)^k V(p)^k, \text{if } n = p_1 p_2 \cdots p_k \\ \quad 0 \text{ otherwise} \end{cases}$$
$$\therefore \left|V(n)^{-1}\right| = |\mu(n) V(n)| \le |V(n)|.$$

Hence the result. □

6 Conclusions

As this general exponential function $V(n)$ is a newly introduced arithmetic function that appears to contains much prospective and therefore various characteristics of the analytic number theory and it can utilize as mathematical analysis tool to establish various other results. Further, $V(n)$ can initialize and institute new properties in the analytic number theory.

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Received: February 14, 2021
Accepted: September 5, 2021