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A general optimal inequality for warped product submanifolds in Lorentzian paracosymplectic manifolds

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Abstract In this article, we first recognize the presence of pointwise semi-slant warped product submanifolds M by illustrating a numerical example, and then we develop an optimal inequality for the squared length of the second fundamental form concerning the gradient of warping function on M in Lorentzian paracosymplectic manifolds. The equality case for the inequality and the condition for such warped product to be minimal is also discussed.

1 Introduction

There is a long and fascinating history behind the warped product of (pseudo)-Riemannian manifolds, which have many applications in mathematics and physics, especially in harmonic maps, Ricci soliton, general relativity theory, and black holes. Some examples are surface of revolution $C \times_f \mathbb{S}^1$, which realizes the importance in the construction of different models of some relativistic theories. $\mathbb{S}^{m-1}_+$: open hemisphere and $\mathbb{S}^1$ a circle, for some warping function f on $\mathbb{S}^{m-1}_+$ is also a warped product $\mathbb{S}^{m-1}_+ \times_f \mathbb{S}^1$ that play a crucial part in the research of harmonic maps, Ricci solitons and Einstein manifolds [1, 9, 18, 19, 25, 33].
Bishop-O'Neill [10], originated the geometry of warped product manifolds carrying non-positive curvature as, let B and F be two pseudo-Riemannian manifolds of dimension m and n respectively. Let f be a positive C^∞ function on B. The *warped product* $M = B \times_f F$ of B and F is the product manifold $B \times F$ endowed with metric of the form $g = \pi^* g_B + (f \circ \pi)^2 \sigma^* g_F$ such that π and σ denoted the natural projections on $B \times F$ to B and F, respectively. Hence in $B \times_f F$, B is labeled the *base manifold* while F is named the *fiber* and f warping function. However, the theory gained popularity after Chen examined the existence of Cauchy-Riemann warped structure in an even-dimensional manifold admit the Riemann metric and deriving inequalities for the warped product exists. Also in the same, he proved that the non-trivial warped product Cauchy-Riemann submanifolds in the form $N_\perp \times_f N_T$ doesn't exist for Kaehler ambient [14]. Subsequently, Hasegawa [17] and Munteanu [23] carried out the idea for Sasakian manifolds. Afterward, Sahin [29] and Uddin[20] extended the geometry of Cauchy-Riemann warped product to warped product semi-slant submanifolds and presented nonexistence results for such warped products in complex and contact settings, respectively. Thereafter Yuksel [35] proved that there doesn't exist warped products as semi-slant of Lorentzian paracosymplectic manifolds $\overline{M}$. Later, Sahin [30] continued the study by introducing a new generalized class of warped product semi-slant submanifolds called warped product pointwise semi-slant submanifolds in Kählerian manifolds. Thenceforth, many differential geometers contributed to the theory of warped products as pointwise slant, semi (pseudo) slant submanifolds with applications viewpoint in different ambients. [2, 3, 4, 31]. Motivation to present study is due to specifically two reasons, one its numerous applications, and hence extensively studying nowadays (see, [5, 6, 24, 26]). Second, to extend the study in the Lorentz setting. Moreover, in continuation to [32], in the current manuscript, we first verify the existence of the warped product $M_T \times_f M_\theta$ by illustrating a numerical example and then analyze the *problems* imposed by Chen in [13], by

deriving the relationship between the extrinsic and intrinsic quantities for such warped products of a Lorentzian paracosymplectic manifold.
The following is an overview of the paper. In section 2, we review some valuable notions related to Lorentzian almost paracontact and Lorentzian paracosymplectic manifolds. Section 3, contains a few fundamental formulas and characterization results for (pointwise) slant submanifolds and a preliminary lemma for further use. Finally, in section 4, we first build a sharp generic inequality involving squared length and wrapping function, and then deduce a minimal theorem for warped product submanifolds of the kind $M_T \times_f M_\theta$ in a Lorentzian paracosymplectic manifold. The equality case for inequality is also examined.

2 Preliminaries

A quadruple (ϕ, ξ, η, g) of tensors is called an almost paracontact Lorentz metric structure on a manifold $\overline{M}$ of dimension $2m+1$, such that ϕ is a $(1,1)$ type endomorphism tensor, ξ is a characteristics vector field, η a differential 1-form, and g is a Lorentz metric of type $(0,2)$ satisfying

$$\phi^2 = id + \eta \otimes \xi, \quad \eta(\xi) = -1 \quad \text{and} \quad g(\phi E_1, \phi E_2) = g(E_1, E_2) + \eta(E_1)\eta(E_1), \tag{2.1}$$

where id is an identity map. As a consequence, we can conveniently derive from the Eq. (2.1), that

$$\phi\xi = 0, \quad \eta \circ \phi = 0, \quad g(E_1, \xi) = \eta(E_1) \quad \text{and} \quad g(\phi E_1, E_2) = g(E_1, \phi E_2). \tag{2.2}$$

Now, the differential 2-form Ω on $\overline{M}$ is provided by

$$g(E_1, \phi E_2) = \Omega(E_1, E_2) \quad \text{and} \quad (\overline{\nabla}_{E_1}\Omega)(E_2, E_3) = g((\overline{\nabla}_{E_1}\phi)E_2, E_3) \tag{2.3}$$

$\forall E_i \in \mathfrak{X}(\overline{M})$ where, $i \in \{1, 2, \cdots\}$, $\mathfrak{X}(\overline{M})$ and $\overline{\nabla}$ denotes the tangent bundle and the Levi-Civita connection on $\overline{M}$ respectively. The Lorentz metric g allows ξ a timelike unit vector field, this means, $g(\xi, \xi) = -1$. Thus, the manifold $\overline{M}$ admitted structure (ϕ, ξ, η, g) is called a Lorentzian almost paracontact manifold [in short LAP-manifold] [21, 22].
Let $\overline{M}$ be an LAP-manifold. Then it is called

- Lorentzian paracontact manifold [briefly LP-manifold] if

$$g(E_1, \phi E_2) = \frac{1}{2}((\overline{\nabla}_{E_1}\eta)E_2 + (\overline{\nabla}_{E_2}\eta)E_1).$$

- Lorentzian para-Saskian manifold [in short LP-Sasakian manifold] if

$$(\overline{\nabla}_{E_1}\phi)E_2 = g(\phi E_1, \phi E_2)\xi + \eta E_2 \phi^2 E_1.$$

Definition 2.1. A LAP-manifold $\overline{M}$ is called [28] *Lorentzian paracosymplectic* briefly LP-cosymplectic $\overline{M}$, if $\overline{\nabla}\phi = 0$, that is, ϕ is parallel.

Because of the immediate result of Eq. (2.2), definition 2.1 and covariant differentiation formula, we get

$$\overline{\nabla}_{E_1}\xi = 0, \quad \forall\, E_1, \xi \in \mathfrak{X}(\overline{M}). \tag{2.4}$$

Suppose M be an isometrically immersed submanifold in a LP-cosymplectic manifold $\overline{M}$. The tangent space $\mathfrak{X}p(\overline{M})$ of $\overline{M}$ at a point $p \in M$ can be expressed into the direct sum $\mathfrak{X}p(\overline{M}) = \mathfrak{X}p(M) \oplus \mathfrak{X}p(M)^\perp$, $\forall\ p \in M$, where $\mathfrak{X}p(M)$ represent the tangent subspace of $\mathfrak{X}p(\overline{M})$ and $\mathfrak{X}p(M)^\perp$ denotes the normal space of M. We employ for the induced metric the same notation g on M, as we have for $\overline{M}$. Then the Gauss and Weingarten formulas are defined respectively by

$$\overline{\nabla}_{E_1}E_2 = \nabla_{E_1}E_2 + \sigma(E_1, E_2), \tag{2.5}$$

$$\overline{\nabla}_{E_1}E^\perp = -A_{E^\perp}E_1 + \nabla^\perp_{E_1}E^\perp, \tag{2.6}$$

$\forall E_1, E_2 \in \mathfrak{X}(M)$ and $E^{\perp} \in \mathfrak{X}(M^{\perp})$, where ∇ (resp., $\nabla^{\perp}$) is the induced connection on tangent bundle $\mathfrak{X}(M)$ (resp., $\mathfrak{X}(M^{\perp})$, σ is the second fundamental form [in short, SFF], and the Weingarten operator $A_{E^{\perp}}$ endowed with the normal section $E^{\perp}$ is given in [15] by

$$g\left(A_{E^{\perp}} E_1, E_2\right) = g\left(\sigma(E_1, E_2), E^{\perp}\right). \tag{2.7}$$

Now, from Eqs. (2.4) and (2.7), we have got the following observation(s) as lemma for future use;

Lemma 2.2. *We have for a submanifold M of a LP-cosymplectic manifold $\overline{M}$ with $\xi \in \mathfrak{X}(M)$ that $\nabla_E \xi$, $\nabla_\xi E$, $\nabla_\xi \xi$, $\sigma(E, \xi)$, $A_{E^{\perp}} \xi$ vanishes and $A_{E^{\perp}} E \perp \xi$ for any $E \in \mathfrak{X}(M)$ and $E^{\perp} \in \mathfrak{X}(M^{\perp})$.*

If p be any point in M and $\{e_1, \cdots, e_{d+1}, e_{d+2}, \cdots, e_{2m+1}\}$ be an orthonormal frame of the tangent space $\mathfrak{X}_p\overline{M}$ such that $\{e_1, \cdots, e_{d+1}\}$ tangent to M at p and $\{e_{d+2}, \cdots, e_{2m+1}\}$ normal to M. Then the mean curvature vector $\mathcal{H}$ of M is given by $\mathcal{H}(p) = \frac{1}{d}\text{trace}\, \sigma$, and the squared length of the SFF i.e., $||\sigma||^2$ is defined by

$$||\sigma||^2 = \sum_{x,y=1}^{d} g(\sigma(e_x, e_y), \sigma(e_x, e_y)). \tag{2.8}$$

By setting $\sigma_{xy}^k = g(\sigma(e_x, e_y), e_k),\ x, y \in \{e_1 \cdots, e_{d+1}\},\ k \in \{e_{d+2}, \cdots, e_{2m+1}\}$, Eq. (2.8) can be represented as,

$$||\sigma||^2 = \sum_{k=d+2}^{2m+1} \sum_{x,y=1}^{d+1} g(\sigma(e_x, e_y), e_k). \tag{2.9}$$

Next, A submanifold M is totally umbilical (resp., geodesic) if $\sigma(E_1, E_2)$ equals $g(E_1, E_2)\mathcal{H}$ (resp., 0). M is *minimal* If $\mathcal{H}$ vanishes [11]. If we admit, for all $E \in \mathfrak{X}(M)$ and $E^{\perp} \in \mathfrak{X}(M^{\perp})$ that

$$\phi E = tE + nE, \tag{2.10}$$

$$\phi E^{\perp} = t' E^{\perp} + n' E^{\perp}, \tag{2.11}$$

where tE ($t'E^{\perp}$) and nE ($n'E^{\perp}$) are tangential (normal) components of $\phi\nu$ ($\phi E^{\perp}$). Then the submanifold M is called *invariant* (resp., *anti-invariant*), if n (resp., t) is identically zero. Furthermore, from Eqs. (2.2) and (2.10), we obtain that

$$g(E_1, tE_2) = g(tE_1, E_2). \tag{2.12}$$

3 Pointwise semi-slant submanifolds

It was Chen-Garay who first suggested the notion of a pointwise slant submanifold [12] as the natural extension of Chen's [15] work. They were previously investigated by Etayo [16] as a submanifold with quasi-slant in Hermitian manifolds. Furthermore, Park et al. began the investigation for (para) contact manifolds [8, 27, 32]. In this section, from above-mentioned papers, we revise some useful findings for subsequent use.

Definition 3.1. A submanifold M of a LAP-manifold $\overline{M}$ is

- *pointwise slant* if for any given point $p \in M$ and non-zero vector field $E \in \mathfrak{X}(M)$ with $g(E, \xi)_p = 0$, the *slant angle* $\theta = \theta(E)$ between $\phi(E)$ and the tangent space $\mathfrak{X}_pM/\{0\}$ is independent of the choice of the vector field E tangent to M. In this context, the angle θ is interpreted as a function on M, called *slant function*. In addition, for θ is constant pointwise slant is simply *slant*.
- *pointwise semi-slant* [in brief PSS] if it is paired with a complimentary orthogonal invariant $\mathfrak{D}_T$ and pointwise slant distributions $\mathfrak{D}_\theta$) with slant function θ satisfying $\mathfrak{X}(M) = \mathfrak{D}_T \oplus \mathfrak{D}_\theta \oplus \{\xi\}$, such that $\phi(\mathfrak{D}_T) \subseteq \mathfrak{D}_T$.

Now, the normal bundle $\mathfrak{X}(M^{\perp})$ of M in a LAP-manifold can be decomposed as $\mathfrak{X}(M^{\perp}) = \phi(\mathfrak{D}_\theta) \oplus \mu$, where μ is normal sub-bundle orthogonal to $\phi(\mathfrak{D}_\theta)$ and invariant under ϕ.

Remark 3.2. A PSS submanifold is *proper* if $\mathfrak{D}_T, \mathfrak{D}_\theta \neq \{0\}$ and θ is not constant, *mixed geodesic* if σ on M satisfies $\sigma(\mathfrak{D}_T, \mathfrak{D}_\theta) = 0$.

In particular, we have the following submanifolds if :

(i). $\mathfrak{D}_T = \{0\}$ and $\theta = \pi/2$, then M is anti-invariant [7, 35].

(ii). $\mathfrak{D}_\theta = \{0\}$, then M is invariant [7, 35].

(iii). $\mathfrak{D}_T = \{0\}$ and $\mathfrak{D}_\theta \neq \{0\}$ with θ globally constant such that $\theta \in (0, \pi/2)$, then M is a proper slant [7].

(iv.) $\mathfrak{D}_T \neq \{0\}$ and $\mathfrak{D}_\theta \neq \{0\}$ such that slant angle $\theta = \pi/2$, then M is a semi-invariant [34].

(v). $\mathfrak{D}_T \neq \{0\}$ and $\mathfrak{D}_\theta \neq \{0\}$ such that slant angle θ satisfies that $\theta \in (0, \pi/2)$ is independent of point and vector fields on M, then M is a proper semi-slant [35].

(vi). $\mathfrak{D}_T = \{0\}$ and θ is a slant function, then M is a pointwise slant [8].

Proposition 3.3. *On submanifold M of a LAP-manifold with $\xi \in \mathfrak{X}(M)$ and θ defined as real valued function, M is a pointwise slant $\iff t^2 = \cos^2\theta\phi^2$.*

The subsequent corollary is a natural deduction of the previous proposition:

Corollary 3.4. *For distribution $\mathfrak{D}_\theta$ and P_θ the orthogonal projection of $\mathfrak{D}_\theta$ on M, $\mathfrak{D}_\theta$ is pointwise slant $\iff \exists$ a function θ with $(tP_\theta)^2 E_3 = \cos^2\theta\, E_3$ for $E_3 \in \mathfrak{X}(\mathfrak{D}_\theta)$.*

If the projections on $\mathfrak{D}_T$ and $\mathfrak{D}_\theta$ are represented by $\mathcal{P}_T$ and $\mathcal{P}_\theta$, respectively. Then we can write for any $E_3 \in \mathfrak{X}(M)$ that

$$E_3 = \mathcal{P}_T E_3 + \mathcal{P}_\theta E_3 + \eta(E_3)\xi. \tag{3.1}$$

Previous equation by operating ϕ and Eqs. (2.2), (2.10), becomes $\phi E_3 = t\mathcal{P}_T E_3 + t\mathcal{P}_\theta E_3 + n\mathcal{P}_\theta E_3$. Thus, from previous expression, we conclude that $t\mathcal{P}_T E_3 \in \mathfrak{X}(\mathfrak{D}_T)$, $n\mathcal{P}_T E_3 = 0$, and $t\mathcal{P}_\theta E_3 \in \mathfrak{X}(\mathfrak{D}_\theta)$, $n\mathcal{P}_\theta E_3 \in \mathfrak{X}(M^{\perp})$. Using Eq. (2.10) and above expressions in Eq. (3.1), we deduce that $tE_3 = t\mathcal{P}_T E_3 + t\mathcal{P}_\theta E_3, nE_3 = n\mathcal{P}_\theta E_3$, for any $E_3 \in \mathfrak{X}(M)$. Since, $\mathfrak{D}_\theta$ is pointwise slant distribution, by the consequences of Corollary 3.4, we obtain that

$$t^2 E_3 = (\cos^2\theta) E_3, \tag{3.2}$$

for any $E_3 \in \mathfrak{X}(\mathfrak{D}_\theta)$.

4 Inequality

Srivastava [32] accompany [35], for the results that there doesn't exist warped products in the form $M_\theta \times_f M_T$ and $M_T \times_f M_\theta$ in the LP-cosymplectic manifolds $\overline{M}$ with ξ belongs to the base, second factor and second factor respectively. But, the authors in the same presented the presence of the warped product of the form $M_T \times_f M_\theta$ when ξ tangent to the first factor along with a numerical example to support the argument which is contrary to semi-slant in [35]. Now, in this section, we continue the study by first giving more stronger support to [32] by presenting a numerical example and then deriving a general sharp geometric inequality and a minimal theorem for M of the form $M_T \times_f M_\theta$ in an LP-cosymplectic manifold $\overline{M}$ with the structure vector field $\xi \in \mathfrak{X}(M_T)$. Hence we have that,

Definition 4.1. A proper PSS submanifold M in a LP-cosymplectic manifold $\overline{M}$ is called a *pointwise semi-slant submanifold as warped product* [in short PSSWP] if it is a warped product of the form: $M_T \times_f M_\theta$, where M_T and M_θ are invariant and pointwise slant integral submanifolds of $\mathfrak{D}_T$ and $\mathfrak{D}_\theta$ on M respectively, and f is a non-constant positive smooth function on M_T. If the warping function f is constant then PSSWP is called *pointwise semi-slant product or trivial product* [in short PSSP].

If we labled ∇ as the Levi-Civita connection on $M_T \times_f M_\theta$, then for any vector fields E_1, E_2 tangent to M_T and E_3, E_4 tangent to M_θ, we have that

$$\nabla_{E_1} E_2 \in \mathfrak{X}(M_T), \quad \nabla_{E_1} E_3 = \nabla_{E_3} E_1 = \left(\frac{E_1 f}{f}\right) E_3, \quad \nabla_{E_3} E_4 = \frac{-g(E_3, E_4)}{f} \nabla f, \tag{4.1}$$

where ∇f is the gradient of f defined by $g(\nabla f, E_1) = E_1 f$ [10].

Remark 4.2. Since *lift* is of the utmost use in computation on product manifold, therefore, for simplicity, we will examine on $M = M_T \times_f M_\theta$ the E_1 vector field on M_T with the *lift* $\tilde{E}_1$ and the E_3 vector field on M_θ with the *lift* $\tilde{E}_3$. Moreover, it is indeed worth remembering that for the warped product $M_T \times_f M_\theta$; M_T is totally geodesic and M_θ is totally umbilical in M [10].

Here, we recall some major findings of [32] for further extension to study;

Proposition 4.3. *There doesn't exist any non-trivial warped product submanifolds $B \times_f F$ in a LP-cosymplectic manifold $\overline{M}$ such that $\xi \in \mathfrak{X}(F)$.*

Proposition 4.4. *If $M = B \times_f F$ is simply a non-trivial warped product submanifold of a $\overline{M}$ with $\xi \in \mathfrak{X}(B)$, then $\xi(\ln f)E_1$ vanishes for all non-null vector field $E_1 \in \mathfrak{X}(F)$.*

Theorem 4.5. *Suppose M be a submanifold of a LP-cosymplectic manifold $\overline{M}$. Then, a if and only if condition for PSS submanifold to be locally warped product in the form $M_T \times_f M_\theta$ is that the Weingraten operator of M satisfies $A_{nE_3} tE_1 - A_{ntE_3} E_1 = (sin^2\theta) E_1(\nu) E_3$, $\forall E_1 \in \mathfrak{X}(\mathfrak{D}_T \oplus \{\xi\})$, $E_3 \in \mathfrak{X}(\mathfrak{D}_\theta)$ and function ν on M satisfying $E_4(\nu) = 0$, $E_4 \in \mathfrak{X}(\mathfrak{D}_\theta)$.*

Next, we first present a example that shows the existence of PSSWP of the form $M = M_T \times_f M_\theta$ when $\xi \in (M_T)$ in a Lorentzian paracosymplectic manifold and then prove an important lemma for later use;

Example 4.6. Suppose $\overline{M} = \mathbb{R}^4 \times \mathbb{R}_- \subseteq \mathbb{R}^5$ be a manifold with the standard cartesian coordinates $(\overline{x}_1, \overline{x}_2, \overline{x}_3, \overline{x}_4, \overline{x}_5)$. Define the Lorentz metric structure (ϕ, ξ, η, g) on $\overline{M}$ by

$$\phi e_1 = e_1, \ \phi e_2 = e_2, \ \phi e_3 = -e_3, \ \phi e_4 = -e_4, \ \phi e_5 = 0, \tag{4.2}$$

$$where \ \xi = e_5, \ \eta = -d\overline{x}_5, \ g = \sum_{i=1}^{4} (d\overline{x}_i)^2 - \eta \otimes \eta. \tag{4.3}$$

Here, $\{e_1, e_2, e_3, e_4, e_5\}$ is a local orthonormal frame for $\mathfrak{X}(\overline{M})$, given by $e_i = \dfrac{\partial}{\partial \overline{x}_i}$ and $e_5 = \dfrac{\partial}{\partial \overline{x}_5}$. Let M be a submanifold of a $\overline{M}$ defined by $\Omega(x, y, z) = (\ x\cos(y), x\cos(y), x\sin(y), x\sin(y), z \ )$. Then the tangent bundle (M) of M is spanned by the vectors

$$\begin{aligned} E_x &= \cos(y)e_1 + \cos(y)e_2 + \sin(y)e_3 + \sin(y)e_4, \\ E_y &= -x\sin(y)e_1 - x\sin(y)e_2 + x\cos(y)e_3 + x\cos(y)e_4, \\ E_z &= e_5. \end{aligned} \tag{4.4}$$

From Eq. (4.4), we obtain that, the sets $\{E_x, E_z\}$ and $\{E_y\}$ spans the invariant distribution $\mathfrak{D}_T$ and pointwise slant distribution $\mathfrak{D}_\theta$ respectively, where $\xi = E_z$ for $\phi(E_z) = 0$ and $\eta(E_z) = -1$ with slant function $\cos^{-1}(\cos(2y))$. Therefore, M becomes a proper PSS submanifold. Now, the induced Lorentz metric tensor g of M is defined by

$$[g_{ij}] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2x^2 & 0 \\ 0 & 0 & -1 \end{bmatrix},$$

that is,

$$g = 2dx^2 - dz^2 + (2x^2)\{dy^2\} = g_{M_T} +_{(2x^2)} g_{M_\theta}.$$

Thus, M is a non-trivial 3-dimensional PSSWP submanifold in $\overline{M}$ with warping function $f = \sqrt{2}x$.

Next, we continue the study by deriving a general optimal geometric inequality and a minimal theorem for M of the form $M_T \times_f M_\theta$ in an LP-cosymplectic manifold $\overline{M}$ with the structure vector field $\xi \in \mathfrak{X}(M_T)$.

Lemma 4.7. *If $M = M_T \times_f M_\theta$ is a PSSWP in a LP-cosymplectic manifold $\overline{M}$ with the structure vector field $\xi \in \mathfrak{X}(\mathfrak{D}_T)$, then*

1). $g(\sigma(E_1, E_2), nE_3) = 0,$

2). $g(\sigma(E_1, E_3), nE_4) = tE_1 \ln f g(E_3, E_4) - E_1 \ln f g(E_3, tE_4),$

for any $E_1, E_2 \in \mathfrak{X}(\mathfrak{D}_T)$ and $E_3, E_4 \in \mathfrak{X}(\mathfrak{D}_\theta)$.

Proof. We yields by virtue of Eqs. (2.2), (2.10) and definition 2.1, that $g(\sigma(E_1, E_2), nE_3) = g(\overline{\nabla}_{E_1}\phi E_2, E_3) - g(\overline{\nabla}_{E_1} E_2, tE_3)$. Hence, by employing the fact that the pair of distribution $(\mathfrak{D}_T, \mathfrak{D}_\theta)$ are orthogonal, Eqs. (2.12) and (4.1) in above expression, we derive the formula-1. Formula-2, can be achieved by employing Eqs. (2.5) and (4.1) in LHS of formula-2. This completes the proof. □

Theorem 4.8. *Let $M_T \times_f M_\theta \to \overline{M}$ be an isometric immersion of a PSSWP into a LP-cosymplectic manifold. such that M_T is an invariant submanifold tangent to ξ and M_θ is a pointwise slant submanifold of M. Then the squared length of the SFF $||\sigma||^2$ of M satisfies*

$$||\sigma||^2 \geq 2\beta(1 + 2\cot^2\theta)||\nabla(\ln f)||^2 \tag{4.5}$$

where $\nabla(\ln f)$ is the gradient of $\ln f$.

Proof. Let the metrics on M_T and M_θ are denoted by g_T and g_θ respectively, then the warped metric on M is defined by $g = g_T +_f g_\theta$. Now, we choose the local frame;

- on M_T: $\{e_0 = \xi, e_i, e_{i^\star} = \phi(e_i)\}$, $\forall i = \{1, \cdots, \alpha\}$ with $\epsilon_0 = g(e_0, e_0) = -1$, $\epsilon_i = g(e_i, e_i) = 1$ consequently $g(e_{i^\star}, e_{i^\star}) = 1$ for all i,
- on M_θ: by $\{\overline{e}_a = \sec\theta t(\overline{e}_a), \overline{e}_{a^\star} = \phi(\overline{e}_a) = \csc\theta n(\overline{e}_a)\}$ for any $a = \{1, \cdots, \beta\}$ such that $\overline{\epsilon}_a = g(\overline{e}_a, \overline{e}_a) = 1\ \forall\ a$, where θ is a slant function, and
- on μ: the orthonormal frame can be assumed as $\{e_z^\perp\}$, for any $z = \{1, \cdots, q\}$ such that $\epsilon_z = g(e_z^\perp, e_z^\perp) = 1$ for all z.

By the definition of $||\sigma||^2$ we obtain that

$$||\sigma||^2 = ||\sigma(\mathfrak{D}_T, \mathfrak{D}_T)||^2 + 2||\sigma(\mathfrak{D}_T, \mathfrak{D}_\theta)||^2 + ||\sigma(\mathfrak{D}_\theta, \mathfrak{D}_\theta)||^2. \tag{4.6}$$

Using first part of the r.h.s. of Eq. (4.6), we get

$$||\sigma(\mathfrak{D}_T, \mathfrak{D}_T)||^2 = \sum_{i,j=0}^{\alpha} g\big(\sigma(e_i, e_j), \sigma(e_i, e_j)\big) = \sum_{k=1}^{2m+1}\sum_{i,j=0}^{\alpha} g\big(\sigma(e_i, e_j), e_k\big)^2. \tag{4.7}$$

Applying local orthonormal frame on Eq. (4.7), we arrive at

$$||\sigma(\mathfrak{D}_T, \mathfrak{D}_T)||^2 = \sum_{l=1}^{\alpha}\sum_{i,j=0}^{\alpha} g\big(\sigma(e_i, e_j), e_{l^\star}\big)^2 + \sum_{a=1}^{\beta}\sum_{i,j=0}^{\alpha} g\big(\sigma(e_i, e_j), \overline{e}_{a^\star}\big)^2 + \sum_{z=1}^{q}\sum_{i,j=0}^{\alpha} g\big(\sigma(e_i, e_j), e_z^\perp\big)^2. \tag{4.8}$$

Eq. (4.8) in light of Lemma 2.2 and the fact that $\mathfrak{D}_T$ is invariant, can be written as

$$||\sigma(\mathfrak{D}_T, \mathfrak{D}_T)||^2 = \sum_{a=1}^{\beta}\sum_{i,j=1}^{\alpha} g\big(\sigma(e_i, e_j), \overline{e}_{a^\star}\big)^2 + \sum_{z=1}^{q}\sum_{i,j=1}^{\alpha} g\big(\sigma(e_i, e_j), e_z^\perp\big)^2. \tag{4.9}$$

Furthermore, we derive from second factor of r.h.s. of Eq. (4.6), that

$$||\sigma(\mathfrak{D}_T,\mathfrak{D}_\theta)||^2=\sum_{i=0}^{\alpha}\sum_{a=1}^{\beta}g\big(\sigma(e_i,\overline{e}_a),\sigma(e_i,\overline{e}_a)\big)=\sum_{k=1}^{2m+1}\sum_{i=0}^{\alpha}\sum_{a=1}^{\beta}g\big(\sigma(e_i,\overline{e}_a),e_k\big)^2. \tag{4.10}$$

Eq. (4.10) by the use of defined local orthonormal frame results in

$$\begin{aligned}||\sigma(\mathfrak{D}_T,\mathfrak{D}_\theta)||^2=&\sum_{l=1}^{\alpha}\sum_{i=0}^{\alpha}\sum_{a=1}^{\beta}g\big(\sigma(e_i,\overline{e}_a),e_{l^\star}\big)^2+\sum_{i=0}^{\alpha}\sum_{a,c=1}^{\beta}g\big(\sigma(e_i,\overline{e}_a),\overline{e}_{c^\star}\big)^2\\&+\sum_{z=1}^{q}\sum_{i=0}^{\alpha}\sum_{a=1}^{\beta}g\big(\sigma(e_i,\overline{e}_a),e_z^{\perp}\big)^2.\end{aligned} \tag{4.11}$$

Using Lemma 2.2 and frame for the orthogonal pair $(\mathfrak{D}_T,\mathfrak{D}_\theta)$ in Eq. (4.11), we arrive at

$$||\sigma(\mathfrak{D}_T,\mathfrak{D}_\theta)||^2=\sum_{i=1}^{\alpha}\sum_{a,c=1}^{\beta}g\big(\sigma(e_i,\overline{e}_a),\csc\theta n\overline{e}_c\big)^2+\sum_{z=1}^{q}\sum_{i=1}^{\alpha}\sum_{a=1}^{\beta}g\big(\sigma(e_i,\overline{e}_a),e_z^{\perp}\big)^2. \tag{4.12}$$

Thus, from the third factor of the r.h.s. of Eq. (4.6), we have that

$$||\sigma(\mathfrak{D}_\theta,\mathfrak{D}_\theta)||^2=\sum_{a,b=1}^{\beta}g\big(\sigma(\overline{e}_a,\overline{e}_b),\sigma(\overline{e}_a,\overline{e}_b)\big)=\sum_{k=1}^{2m+1}\sum_{a,b=1}^{\beta}g\big(\sigma(\overline{e}_a,\overline{e}_b),e_k\big)^2. \tag{4.13}$$

Above equation in light of local orthonormal frame become

$$\begin{aligned}||\sigma(\mathfrak{D}_\theta,\mathfrak{D}_\theta)||^2=&\sum_{i=1}^{\alpha}\sum_{a,b=1}^{\beta}g\big(\sigma(\overline{e}_a,\overline{e}_b),e_{i^\star}\big)^2+\sum_{c=1}^{\beta}\sum_{a,b=1}^{\beta}g\big(\sigma(\overline{e}_a,\overline{e}_b),\overline{e}_{c^\star}\big)^2\\&+\sum_{z=1}^{q}\sum_{a,b=1}^{\beta}g\big(\sigma(\overline{e}_a,\overline{e}_b),e_z^{\perp}\big)^2.\end{aligned} \tag{4.14}$$

Eq. (4.14), by virtue of Lemma 2.2 and some computation, reduced to

$$||\sigma(\mathfrak{D}_\theta,\mathfrak{D}_\theta)||^2=\sum_{a,b,c=1}^{\beta}g\big(\sigma(\overline{e}_a,\overline{e}_b),\csc\theta n\overline{e}_c\big)^2+\sum_{z=1}^{q}\sum_{a,b=1}^{\beta}g\big(\sigma(\overline{e}_a,\overline{e}_b),e_z^{\perp}\big)^2. \tag{4.15}$$

Therefore from equations (4.6), (4.9), (4.12), (4.15), we can conclude that

$$||\sigma||^2\geq 2\sum_{i=1}^{\alpha}\sum_{a,c=1}^{\beta}\csc^2\theta g\big(\sigma(e_i,\overline{e}_a),n\overline{e}_c\big)^2. \tag{4.16}$$

Equation (4.16) in light of chosen frame and formula-2 of Lemma 4.7 can be expressed as

$$||\sigma||^2\geq 2\beta\sum_{i=1}^{\alpha}\csc^2\theta\big(te_i\ln fg(\overline{e}_a,\overline{e}_c)-e_i\ln fg(\overline{e}_a,t\overline{e}_c)\big)^2. \tag{4.17}$$

Hence by direct computation we completes the proof of the theorem. □

In light of the Theorem 4.8, we conclude the following result as remark;

Remark 4.9. *Equality sign of Eq. (4.5) holds identically, if $\sigma(\mathfrak{D}_T,\mathfrak{D}_T)$, $\sigma(\mathfrak{D}_\theta,\mathfrak{D}_\theta)$ vanishes and $\sigma(\overline{e}_a,\overline{e}_b)\oplus_\perp e_z^{\perp}$ in $\overline{M}$.*

Next, by the use of Theorem 4.7, Lemmas 3.4, 3.6 of [32] and Remark 4.9, we conclude the following important result as a Theorem:

Theorem 4.10. *Let $M=M_T\times_f M_\theta$ be an isometrically immersed PSSWP in a LP-cosymplectic manifold $\overline{M}$ such that $\xi\in\mathfrak{X}(M_T)$. If the integral submanifolds M_T, M_θ of the distributions $\mathfrak{D}_T$, $\mathfrak{D}_\theta$ are totally geodesic, totally umbilical in $\overline{M}$ respectively, and $\sigma(\mathfrak{D}_\theta,\mathfrak{D}_\theta)\oplus_\perp\mu$. Then M is Minimal.*

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