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r-Almost Yamabe solitons in Lorentzian manifolds

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Abstract In this paper we establish the new concept of r-almost Yamabe soliton for spacelike hypersurfaces immersed into a Lorentzian manifold, which involves the Newton tensors and extends in a natural way the notion of immersed almost Yamabe solitons introduced by Barbosa and Ribeiro [3]. In this setting, we exhibit some examples and prove nonexistence and rigidity results of these geometric objects under suitable constraints on the potential and soliton functions.

1 Introduction

A *Yamabe soliton* is a Riamannian manifold (Σ^n, g) that admits a smooth vector field X on Σ^n such that

$$\frac{1}{2}\mathcal{L}_X g = (\mathrm{Scal} - \lambda)\mathrm{g}, \tag{1.1}$$

where $\mathcal{L}_X$ denotes the Lie derivative in the direction of the vector field X, Scal is the scalar curvature of (Σ^n, g) and λ is a real number. In the particular case of X being a gradient, that is, there exists a smooth function $f : \Sigma^n \to \mathbb{R}$ such that $X = \nabla f$, (Σ^n, g) is said a *gradient Yamabe soliton* and f the potential function. Equation (1.1) then becomes

$$\mathrm{Hess} f = (\mathrm{Scal} - \lambda)\mathrm{g}, \tag{1.2}$$

where Hess$f$ stands for the Hessian of f.

It is well known that Yamabe solitons correspond to self-similar solutions of the Yamabe flow,

$$\partial_t g(t) = -\mathrm{Scal}(t) g(t),$$

which was introduced by Hamilton [17, 18]. Motivated by the results on the context of Ricci solitons, the geometry of Yamabe solitons has been subject of great highlight and started to be investigated in the last few years (see, for instance, [14, 6, 16, 21, 12, 7, 9, 10, 20, 3, 15, 29, 5, 19, 24, 30] and the references therein). Among others, a remarkable result is due to Chow et al. [12] where the authors proved that every compact Yamabe soliton must have constant scalar curvature (see also [20, 15]).

In a recent work, Barbosa and Ribeiro [3] extended the definition of Yamabe solitons by adding the condition on the parameter λ to be a real smooth function on Σ^n, attracting a lot of attention in the mathematical community. They defined an *almost Yamabe soliton* to be a Riemannian manifold (Σ^n, g) satisfying (1.1) with $\lambda : \Sigma^n \to \mathbb{R}$ being a smooth function, called of soliton function. When $X = \nabla f$ the almost Yamabe soliton is said a *gradient almost Yamabe soliton*, and equation (1.1) agrees with (1.2). In particular, an almost Yamabe soliton is called *expanding*, *steady*, or *shrinking*, respectively, if $\lambda < 0$, $\lambda = 0$, or $\lambda > 0$.

Based on the ideas of [13], our aim in this paper is to introduce the concept of r-almost Yamabe soliton immersed into a Lorentzian manifold and to study the properties of this new geometric object. As we are going to explain, it will give a natural extension of the concept of gradient almost Yamabe soliton, which appears to be natural and meaningful. In order to do this, our approach is based in the use of the so-called Newton tensors P_r and their associated second order differential operators L_r (see Section 3 for more details).

Precisely, let Σ^n be an oriented and connected spacelike hypersurface immersed into a $(n+1)$-dimensional Lorentzian manifold L^{n+1}, which means that the induced metric g on Σ^n via the immersion is a Riemannian metric. We say that Σ^n is an *r-almost Yamabe soliton*, for some $0 \leq r \leq n$, if there exists a smooth function $f : \Sigma^n \to \mathbb{R}$ such that the following equation holds:

$$P_r \circ \text{Hess} f = (\text{Scal} - \lambda) g, \tag{1.3}$$

for some smooth function $\lambda : \Sigma^n \to \mathbb{R}$.

Here, we interpret the term on the left-hand side of (1.3) as being the tensor given by

$$P_r \circ \text{Hess} f(X, Y) = \langle P_r \nabla_X \nabla f, Y \rangle,$$

for tangent vector fields $X, Y \in \mathfrak{X}(\Sigma)$. In particular, when $r = 0$ we have that $P_0 = I$ is the identity operator and, consequently, we recover the definition of a gradient almost Yamabe soliton. When the potential f is constant, an r-almost Yamabe soliton will be called *trivial*, otherwise it will be a *nontrivial* r-almost Yamabe soliton.

Having realized the above conceptual point, in the rest of the paper we shall study isometric immersions of r-almost Yamabe solitons into a Loretzian manifold and provide some obstruction results in order to obtain a maximal immersion as well as rigidity results. Actually, the paper is organized as follow: In Section 2 some examples of nontrivial r-almost Yamabe solitons are given. Next, in Section 3 we recall some basic facts and notations which will appear throughout this paper. Afterwards, in Section 4 we prove nonexistence and rigidity results for r-almost Yamabe solitons. Finally, in Section 5 we consider the case of 1-almost Yamabe solitons immersed into a locally symmetric Einstein manifold.

2 Examples

In order to give examples of hypersurfaces satisfying the structure of r-almost Yamabe solitons let us denote by $\mathbb{L}^{n+1}_c$ the standard model of an $(n+1)$-dimensional Lorentzian space form with constant sectional curvature c, where $c \in \{0, 1, -1\}$. Actually, $\mathbb{L}^{n+1}_c$ denotes the Lorentz-Minkowski space $\mathbb{R}^{n+1}_1$ when $c = 0$, that is, the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ endowed with the Lorentzian metric

$$\langle\ ,\ \rangle_1 = -dx_1 + \ldots + dx_{n+1}, \tag{2.1}$$

the $(n+1)$-dimensional de Sitter space

$$\mathbb{S}^{n+1}_1 = \{p \in \mathbb{R}^{n+2}_1\ ;\ \langle p, p\rangle_1 = 1\} \subset \mathbb{R}^{n+2}_1$$

endowed with the Lorentzian metric induced from $\mathbb{R}^{n+2}_1$ when $c = 1$, and the $(n+1)$-dimensional anti-de Sitter space

$$\mathbb{H}^{n+1}_1 = \{p \in \mathbb{R}^{n+2}_2\ ;\ \langle p, p\rangle_2 = -1\} \subset \mathbb{R}^{n+2}_2$$

endowed with the Lorentzian metric induced from $\mathbb{R}^{n+2}_2$ when $c = -1$. Here, $\mathbb{R}^{n+2}_2$ stands for the $(n+2)$-dimensional Euclidean space $\mathbb{R}^{n+2}$ endowed with the semi-Riemannian metric

$$\langle\ ,\ \rangle_2 = -dx_1 - dx_2 + \ldots + dx_{n+2}. \tag{2.2}$$

In order to simplify the notation, when $c = \pm 1$ we agree to denote by $\langle\ ,\ \rangle$ without distinction, both the Lorentzian metric (2.1) on $\mathbb{R}^{n+2}_1$ and the semi-Riemannian metric (2.2) on $\mathbb{R}^{n+2}_2$. Also we agree to denote by $\langle\ ,\ \rangle$ the corresponding Lorentzian metric induced on $\mathbb{L}^{n+1}_c \hookrightarrow \mathbb{R}^{n+2}$. In this context, we are ready to provide some examples.

Example 2.1. Every totally geodesic spacelike hypersurface of a Lorentzian manifold is an r-almost
Yamabe soliton with $1 \leq r \leq n$. In particular, all gradient almost Yamabe soliton Σ^n viewed as a totally geodesic hypersurface in standard product space $\Sigma^n \times \mathbb{R}_1$ is an r-almost Yamabe soliton.

Moreover, denoting by Σ^n a totally geodesic spacelike hypersurface of $\mathbb{L}_c^{n+1}$, we know that: (i) when $c = 0$, $\Sigma^n = \mathbb{R}^n$; (ii) when $c = 1$, by Montiel [23] the only totally geodesic spacelike hypersurface in the de Sitter space $\mathbb{S}_1^{n+1}$ is the unit sphere $\mathbb{S}^n$; (iii) when $c = -1$, it is well known that the hyperbolic space $\mathbb{H}^n$ is the only totally geodesic spacelike hypersurface immersed into the anti-de Sitter space $\mathbb{H}_1^{n+1}$ (see, for instance, Section 4 of [1]).

Example 2.2. Let us consider the standard immersion of the n-dimensional hyperbolic space $\mathbb{H}^n$ into the Lorentz-Minkowski space $\mathbb{R}_1^{n+1}$ endowed with induced metric $g = \langle\, ,\, \rangle$. It is well known that $\mathbb{H}^n$ can be represented as the warped product $\mathbb{R} \times_{\cosh t} \mathbb{H}^{n-1}$ (for more details we refer to Montiel [22]). In particular, according to Proposition 2.2 of [26], by choosing the functions

$$\lambda(t,p) = a\sinh t - (n-1) \quad \text{and} \quad f(t,p) = a\sinh t + b,$$

for some constants $a, b \in \mathbb{R}$, we have that $\mathbb{H}^n$ satisfies $\mathrm{Ric} + \mathrm{Hess}\, f = \lambda g$ (that is, it is an almost gradient Ricci soliton). On the other hand, we have that $\mathbb{H}^n$ is a totally umbilical spacelike hypersurface with r-th Newton tensor given by $P_r = \alpha_r I$, for every $0 \le r \le n-1$ and some $\alpha_r \in \mathbb{R}$. Hence, taking the smooth functions $\tilde{f} = \alpha^{-1} f$ and $\tilde{\lambda} = -\lambda - (n-1)(n+1)$ we get that $\mathbb{H}^n$ is a nontrivial r-almost Yamabe soliton.

Example 2.3. Let $a \in \mathbb{R}_1^{n+1}$ be a fixed nonzero vector with $|a|^2 \in \{0, 1, -1\}$. For every $\tau \in \mathbb{R}$, with $\tau^2 > |a|^2$, we define the spacelike hypersurface Σ_τ isometrically immersed into the de Sitter space $\mathbb{S}_1^{n+1}$ by setting

$$\Sigma_\tau = \{p \in \mathbb{S}_1^{n+1}\ ;\ \langle p, a\rangle = \tau\}.$$

By Montiel [23] (see Example 1 of [23]), it is well known that Σ_τ is a totally umbilical spacelike hypersurface with r-th mean curvature given by $H_r = \left(\frac{\tau}{\sqrt{\tau^2 - |a|^2}}\right)^r$ and the corresponding Newton tensor $P_r = \alpha_r I$ for some suitable constant $\alpha_r \in \mathbb{R}$. Let us analyze three cases.

First, when $|a|^2 = 1$ we know that Σ_τ is isometric to an n-dimensional hyperbolic space $\mathbb{H}^n(-\sqrt{\tau^2 - 1})$ and we can reason as in Example 2.2 to conclude that Σ_τ is a nontrivial r-almost Yamabe soliton for every $0 \le r \le n-1$.

In the case $|a| = 0$, we have Σ_τ is isometric to the n-dimensional Euclidean space $\mathbb{R}^n$. On the other hand, let us recall that the Euclidean space $\mathbb{R}^n$ endowed with its standard metric and potential function $f(x) = \frac{1}{4}|x|^2$ has structure of almost gradient Ricci soliton, called of Gaussian soliton (see, for instance, Section 4.2 of [12]). Hence, reasoning once more as in Example 2.2 it is not difficult to verify that Σ_τ is a nontrivial r-almost Yamabe soliton for every $0 \le r \le n-1$.

Finally, when $|a|^2 = -1$, we obtain that Σ_τ is isometric to an n-dimensional Euclidean sphere $\mathbb{S}^n(\sqrt{\tau^2 + 1})$. By using Example 1 of [4], it follows that $\mathbb{S}^n(\sqrt{\tau^2 + 1})$ has structure of almost gradient Ricci soliton. Therefore, the analogous of Example 2.2 applies here again to $\mathbb{S}^n(\sqrt{\tau^2 + 1})$ and in this case Σ_τ can also be endowed as a nontrivial r-almost Yamabe soliton for every $0 \le r \le n-1$, provided that $\tau \neq 0$. The case $\tau = 0$ is contained in Example 2.1.

Example 2.4. Let Σ_τ be the spacelike hypersurface immersed into the anti-de Sitter space $\mathbb{H}_1^{n+1}$ given by

$$\Sigma_\tau = \{p \in \mathbb{H}_1^{n+1}\ ;\ \langle p, a\rangle = \tau\},$$

where $a \in \mathbb{R}_2^{n+1}$ is a unit timelike vector, that is, $\langle a, a\rangle = -1$, and $\tau^2 < 1$. It is well known that Σ_τ is a totally umbilical hypersurface which is isometric to an n-dimensional hyperbolic space $\mathbb{H}^n(-\sqrt{1 - \tau^2})$ (see, for instance, Section 4 of [1]). Then, as aforementioned we can check that Σ_τ has the structure of a nontrivial r-almost Yamabe soliton for every $0 \le r \le n-1$.

3 preliminaries

Let Σ^n be an oriented and connected spacelike hypersurface immersed into an $(n+1)$-dimensional Lorentzian manifold L^{n+1}, which means that the induced metric on Σ^n via the immersion is a Riemannian metric. Let $A : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ be the second fundamental form of Σ^n in L^{n+1} with respect to a fixed orientation. Then by the Gauss equation the scalar curvature of Σ^n is given by

$$\mathrm{Scal} = \sum_{i,j}^{n} \langle \overline{R}(e_i, e_j)e_j, e_i\rangle - n^2H^2 + |A|^2, \tag{3.1}$$

where $\overline{R}$ denotes the curvature tensor of L^{n+1}, $\{e_1, \dots, e_n\}$ is an orthonormal frame on $T\Sigma$, $H = -\frac{1}{n}\mathrm{tr}(A)$ is the mean curvature of Σ^n and $|\cdot|$ denotes the Hilbert-Schmidt norm. When $L^{n+1} = \mathbb{L}_c^{n+1}$ is a Lorentzian space form of constant sectional curvature c, we have the identity

$$\mathrm{Scal} = n(n-1)c - n^2H^2 + |A|^2. \tag{3.2}$$

For each $0 \le r \le n$, the r-th mean curvature H_r of the immersion is defined by

$$H_0 = 1 \;\text{ and }\; \binom{n}{r}H_r = (-1)^r \sum_{i_1<\ldots<i_r} k_{i_1}\cdots k_{i_r}.$$

In particular, when $r = 1$ we have $H_1 = H$ the mean curvature of Σ^n. We also recall that a hypersurface is said *maximal* when its mean curvature vanishes identically, $H \equiv 0$.

The r-th *Newton tensor* $P_r : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$ is defined by setting $P_0 = I$ (the identity operator) and, for $1 \le r \le n$, via the recurrence relation

$$P_r = \sum_{j=0}^{r} \binom{n}{j} H_j A^{r-j}. \tag{3.3}$$

Proceeding, associated to each Newton tensor P_r one has the second order linear differential operator $L_r : C^\infty(\Sigma) \to C^\infty(\Sigma)$ defined by

$$L_r u = \mathrm{tr}(P_r \circ \mathrm{Hess}\, u).$$

When $r = 0$, we note that $L_0 = \Delta$ is just the Laplacian operator. Moreover, it is not difficult to see that

$$\begin{aligned} \mathrm{div}_\Sigma(P_r\nabla u) &= \sum_{i=1}^{n}\langle(\nabla_{e_i}P_r)(\nabla u), e_i\rangle + \sum_{i=1}^{n}\langle P_r(\nabla_{e_i}\nabla u), e_i\rangle \\ &= \langle \mathrm{div}_\Sigma P_r, \nabla u\rangle + L_r u, \end{aligned} \tag{3.4}$$

where the divergence of P_r on Σ^n is given by

$$\mathrm{div}_\Sigma P_r = \mathrm{tr}(\nabla P_r) = \sum_{i=1}^{n}(\nabla_{e_i}P_r)(e_i).$$

In particular, when the ambient space has constant sectional curvature, equation (3.4) reduces to $L_r u = \mathrm{div}_\Sigma(P_r\nabla u)$, because $\mathrm{div}_\Sigma P_r = 0$ (see [2] and [27] for more details).

To close this section we quote two very useful maximum principles. The former one due to Caminha et al. [8] (for more details, see Proposition 1 of [8]).

Lemma 3.1. *Let X be a smooth vector field on the n-dimensional, complete, noncompact, oriented Riemannian manifold Σ^n, such that* $\mathrm{div}_\Sigma X$ *does not change sign on Σ^n. If $|X| \in L^1(\Sigma)$, then* $\mathrm{div}_\Sigma X = 0$.

Here, for each $p \ge 1$, we use the notation $L^p(\Sigma) = \{u : \Sigma^n \to \mathbb{R}\, ;\, \int_\Sigma |u|^p d\Sigma < +\infty\}$. The next key lemma is due to Yau and corresponds to Theorem 3 of [31].

Lemma 3.2. *Let u be a nonnegative smooth subharmonic function on a complete Riemannian manifold Σ^n. If $u \in L^p(\Sigma)$, for some $p > 1$, then u is constant.*

4 Main results

We start this section by proving a result which gives conditions for nonexistence of maximal immersions of a (spacelike) r-almost Yamabe soliton immersed into a Lorentzian space form.

Theorem 4.1. *Let Σ^n be a complete r-almost Yamabe soliton immersed into a Lorentzian space form $\mathbb{L}_c^{n+1}$, with bounded second fundamental form and potential function such that $|\nabla f| \in L^1(\Sigma)$. Then:*

(i) If $c \geq 0$ and $\lambda < 0$, then Σ^n cannot be maximal;

(ii) If $c > 0$ and $\lambda \leq 0$, then Σ^n cannot be maximal;

(iii) If $c = 0$, $\lambda \leq 0$ and Σ^n is maximal, then Σ^n is isometric to the Euclidean space $\mathbb{R}^n$.

Proof. To prove the first and second claim we shall proceed by contradiction, that is, assuming that Σ^n is maximal. In this case, follows directly from equation (3.2) jointly with the assumption $c \geq 0$ $(c > 0)$ that the scalar curvature of Σ^n satisfies Scal ≥ 0 (Scal > 0). Hence, contracting equation (1.3) we have $L_r f = n(\text{Scal} - \lambda) > 0$ in both cases.

On the other hand and for further reference, we observe that since the ambient space has constant sectional curvature, by equation (3.4) the operator L_r is a divergent type operator. Now observe that since Σ^n has bounded second fundamental form it follows from (3.3) that the Newton tensor P_r has bounded norm. In particular,

$$|P_r \nabla f| \leq |P_r||\nabla f| \in L^1(\Sigma).$$

So taking into account Lemma 3.1, we must have $L_r f = 0$, arriving to a contradiction.

For the last assertion, let us observe that $c = 0$ and Σ^n be maximal imply jointly with equation (3.2) that Scal $= |A|^2 \geq 0$. Then $L_r(f) = n(\text{Scal} - \lambda) \geq 0$. By using that $L_r u = \text{div}_\Sigma(P_r \nabla u)$ and $|P_r \nabla f| \in L^1(\Sigma)$ we have once more from Lemma 3.1 that $L_r f = 0$ on Σ^n. Hence, we conclude that Scal $= |A|^2 = 0$. Therefore, the r-almost Yamabe soliton Σ^n must be totally geodesic and flat. □

Next, we are in condition to prove the following result, which holds when the ambient space is an arbitrary Lorentzian manifold.

Theorem 4.2. *Let Σ^n be a complete r-almost Yamabe soliton immersed into a Lorentzian manifold L^{n+1} of sectional curvature K, such that P_r is bounded from above (in the sense of quadratic forms) and the potential function is nonnegative with $f \in L^p(\Sigma)$ for some $p > 1$. Then:*

(i) If $K \geq 0$ and $\lambda < 0$, then Σ^n cannot be maximal;

(ii) If $K > 0$ and $\lambda \leq 0$, then Σ^n cannot be maximal;

(iii) If $K \geq 0$, $\lambda \leq 0$ and Σ^n is maximal, then Σ^n is flat and totally geodesic.

Proof. Regarding to (i) and (ii), let us assume once more by contradiction that Σ^n is maximal. So our assumption on the sectional curvature of the ambient space and equation (3.1) imply that Scal ≥ 0 (Scal > 0). Hence, contracting equation (1.3) we have $L_r f = n(\text{Scal} - \lambda) > 0$. Thus, since we are assuming that P_r is bounded from above, there exists a positive constant β such that $L_r f \leq \beta \Delta f$, so that $\Delta f > 0$. In particular, from Lemma 3.2 we get that f must be constant, which gives a contradiction. Finally, reasoning as in the proof of Theorem 4.1 we can also prove (iii). □

Remark 4.3. Let us recall that a Riemannian manifold Σ^n is said to be *parabolic* if the only subharmonic functions $f \in C^\infty(\Sigma)$ with $\sup_\Sigma f < +\infty$ are the constant ones. In Theorem 4.2, we can replace the hypothesis that f is nonnegative and $f \in L^p(\Sigma)$, for some $p > 1$, by assuming that f is bounded from above and Σ^n being parabolic. More generally, when L_r is eliptic, we said that Σ^n is *L_r-parabolic* if the only functions $f \in C^\infty(\Sigma)$ with $L_r f \geq 0$ and $\sup_\Sigma f < +\infty$ are the constant ones. In particular, the assumption that f is nonnegative and $f \in L^p(\Sigma)$ can be replaced by f is bounded from above and Σ^n is L_r-parabolic.

In our next result we give conditions for an immersed r-almost Yamabe soliton to be totally umbilical since it has bounded second fundamental form. In particular, such an r-almost Yamabe soliton must have constant scalar curvature. More precisely, we prove the following:

Theorem 4.4. *Let Σ^n be a complete r-almost Yamabe soliton immersed into a Lorentzian space form $\mathbb{L}_c^{n+1}$, with bounded second fundamental form and potential function such that $|\nabla f| \in L^1(\Sigma)$. Then:*

(i) If $\lambda \leq n(n-1)(c - H^2)$, then Σ^n is totally umbilical. In particular, the scalar curvature Scal $= n(n-1)K_\Sigma$ *is constant, where $K_\Sigma = \frac{\lambda}{n(n-1)}$ is the sectional curvature of Σ^n;*

(ii) If $\lambda \leq n(n-1)c - n^2H^2$*, then* Σ^n *is totally geodesic, with* $\lambda = n(n-1)c$ *and scalar curvature* Scal $= n(n-1)c$*.*

Proof. To prove (i), we observe that by definition of r-almost Yamabe soliton jointly with equation (3.2) we find

$$L_r f = n\left(n(n-1)c - n^2H^2 + |A|^2 - \lambda\right). \tag{4.1}$$

Let $\Phi = A + HI$ be the traceless second fundamental form of the hypersurface. Then $|\Phi|^2 = \mathrm{tr}(\Phi^2) = |A|^2 - nH^2 \geq 0$, with equality if and only if Σ^n is totally umbilical. It follows that equation (4.1) can be rewrite as

$$L_r f = n|\Phi|^2 + n\left(n(n-1)(c - H^2) - \lambda\right). \tag{4.2}$$

Hence from our assumption on λ we get that $L_r f$ is a nonnegative function on Σ^n. In particular, by Lemma 3.1 we obtain that $L_r f$ vanishes identically. Therefore from equation (4.2) we conclude that Σ^n is totally umbilical. In particular, the principal curvature κ of Σ^n is constant and Σ^n has constant sectional curvature given by $K_\Sigma = c - \kappa^2$. This jointly with equation (4.2) assures that

$$\lambda = n(n-1)(c - H^2) = n(n-1)(c - \kappa^2) = n(n-1)K_\Sigma,$$

which implies that Scal $= n(n-1)K_\Sigma$, proving the claim (i).

For the second assertion, let us begin observing that equation (4.1) and our hypothesis on λ give

$$L_r f \geq 0.$$

Then, by applying Lemma 3.1 once more we have $L_r f = 0$. This implies that $|A|^2 = 0$, that is, Σ^n is a totally geodesic and $\lambda = n(n-1)c$. Moreover, it is clear from equation (3.2) that Scal $= n(n-1)c$, as desired. □

As a consequence from Theorem 4.4 we have the following result.

Corollary 4.5. *Let* Σ^n *be a complete* r*-almost Yamabe soliton immersed into a Lorentzian space form* $\mathbb{L}_c^{n+1}$*, with bounded second fundamental form and potential function such that* $|\nabla f| \in L^1(\Sigma)$*. Suppose that* $\lambda \leq n(n-1)(c - H^2)$*. Then:*

(i) If Σ^n *is compact, then* $c = 1$ *and* Σ^n *is isometric to a Euclidean sphere* $\mathbb{S}^n$*;*

(ii) If $c = 0$*, then* Σ^n *is isometric either to the Euclidean space* $\mathbb{R}^n$ *or to a hyperbolic space* $\mathbb{H}^n$*;*

(iii) If $c = -1$*, then* Σ^n *is isometric to a hyperbolic space* $\mathbb{H}^n$*.*

Proof. The proof follows from classification of the totally umbilical spacelike hypersurfaces of $\mathbb{L}_c^{n+1}$. □

As another application of Lemma 3.2 we also get.

Theorem 4.6. *Let* Σ^n *be a complete* r*-almost Yamabe soliton immersed into a Lorentzian space form* $\mathbb{L}_c^{n+1}$*, such that* P_r *is bounded from above (in the sense of quadratic forms) and the potential function is nonnegative with* $f \in L^p(\Sigma)$ *for some* $p > 1$*. Then:*

(i) If $\lambda \leq n(n-1)(c - H^2)$*, then* Σ^n *is totally umbilical. In particular, the scalar curvature* Scal $= n(n-1)K_\Sigma$ *is constant, where* $K_\Sigma = \frac{\lambda}{n(n-1)}$ *is the sectional curvature of* Σ^n*;*

(ii) If $\lambda \leq n(n-1)c - n^2H^2$*, then* Σ^n *is totally geodesic, with* $\lambda = n(n-1)c$ *and scalar curvature* Scal $= n(n-1)c$*.*

Proof. Let us begin observing that by equation (4.2) and assumption on λ we get

$$L_r f = n|\Phi|^2 + n\left(n(n-1)(c - H^2) - \lambda\right) \geq 0. \tag{4.3}$$

Since we are assuming that P_r is bounded from above, $\Delta f \geq 0$. By Lemma 3.2, we have that f must be constant. Therefore $L_r f = 0$, and by equation (4.3) we conclude that Σ^n is totally umbilical and Scal $= n(n-1)K_\Sigma$ is constant with $K_\Sigma = \frac{\lambda}{n(n-1)}$, proving item (i). Finally, reasoning as in Theorem 4.4, it is not difficult to prove item (ii). □

5 Further results

In this section we will extend our previous results to the case of 1-almost Yamabe soliton immersed into a locally symmetric Lorentzian manifold. To this end, we will work with locally symmetric spaces obeying a standard curvature constraint in the setting of Nishikawa [25], Choi et al. [11] and Suh et al. [28].

Before stating the main results, let us recall that a Lorentzian manifold is said to be *locally symmetric* when all the covariant derivative components of its curvature tensor vanish identically. In this setting, such spaces consist in an interesting generalization of constant curvature spaces. Hence, it is a natural question to revisit in this ambient spaces the known results of constant curvature spaces.

In what follows we introduce our curvature constraint on the ambient space, which will be assumed in the results of this section. So, let L^{n+1} be a locally symmetric Lorentzian manifold. Following ideas of [25, 11, 28], in this section we will assume that there exists a constant c_1 such that the sectional curvature K of L^{n+1} satisfies, for any timelike vector η and any spacelike vector v, the following equality.

$$K(\eta, v) = -\frac{c_1}{n}, \tag{5.1}$$

As mentioned above, a Lorentzian space form $\mathbb{L}^{n+1}_c$ of constant sectional curvature c is a locally symmetric space and it is easy to see that the curvature condition (5.1) is satisfied with $-c_1/n = c$. Therefore, in some sense our assumption is a natural generalization of the case where the ambient space has constant sectional curvature. We observe that the standard static spacetime $\mathbb{S}^n \times \mathbb{R}_1$ also satisfies (5.1). On the other hand, Choi et al. [11] exhibited examples of Lorentzian manifolds which are not Lorentzian space forms satisfying condition (5.1).

Let L^{n+1} be a locally symmetric Lorentzian manifold satisfying condition (5.1). Let us choice a local orthonormal frame $\{e_1, \dots, e_{n+1}\}$ on TL such that $e_1, \dots, e_n$ are spacelike vectors and e_{n+1} is a timelike vector. Then, writing $\varepsilon_i = \langle e_i, e_i\rangle$ for $1 \leq i \leq n+1$, we get that the scalar curvature $\overline{\mathrm{Scal}}$ of L^{n+1} is given by

$$\begin{aligned}\overline{\mathrm{Scal}} &= \sum_{i=1}^{n+1} \varepsilon_i \overline{\mathrm{Ric}}(e_i, e_i)\\ &= \sum_{i,j=1}^{n} \langle \overline{R}(e_i, e_j)e_i, e_j\rangle - 2\sum_{i=1}^{n} \langle \overline{R}(e_{n+1}, e_i)e_{n+1}, e_i\rangle\\ &= \sum_{i,j=1}^{n} \langle \overline{R}(e_i, e_j)e_i, e_j\rangle + 2c_1.\end{aligned}$$

Moreover, it is well known that the scalar curvature of a locally symmetric Lorentzian manifold is constant. Thus, the term $\sum_{i,j=1}^{n} \langle \overline{R}(e_i, e_j)e_i, e_j\rangle$ is a constant naturally attached to a locally symmetric Lorentzian manifold satisfying condition (5.1). So, for the sake of simplicity, we will adopt the following notation

$$\mathcal{S} := \frac{1}{n(n-1)} \sum_{i,j=1}^{n} \langle \overline{R}(e_i, e_j)e_i, e_j\rangle.$$

In addition, it is worth pointing out that when L^{n+1} is a constant sectional curvature space, then the constant $\mathcal{S}$ agrees, up to a multiplicative constant, with its sectional curvature.

The following result extends Theorem 4.1 for the context of 1-almost Yamabe soliton immersed into a locally symmetric Einstein Lorentzian manifold.

Theorem 5.1. *Let L^{n+1} be a locally symmetric Einstein Lorentzian manifold satisfying the curvature condition (5.1). Let Σ^n be a complete 1-almost Yamabe soliton immersed into L^{n+1} with bounded second fundamental form and potential function such that $|\nabla f| \in L^1(\Sigma)$. Then:*

(i) If $\mathcal{S} \geq 0$ and $\lambda < 0$, then Σ^n cannot be maximal;

(ii) If $\mathcal{S} > 0$ and $\lambda \leq 0$, then Σ^n cannot be maximal;

(iii) If $\mathcal{S} = 0$, $\lambda \le 0$ and Σ^n is maximal, then Σ^n is totally geodesic.

Proof. To prove (i), let us reason as in the proof of Theorem 4.1 assuming by contradiction that Σ^n is maximal. Then, by our hypothesis on the constant $\mathcal{S}$ we get from equation (3.1) that the scalar curvature of Σ^n satisfies $\mathrm{Scal} \ge 0$, which implies $L_1 f = n(\mathrm{Scal} - \lambda) > 0$.

On the other hand, we recall from the discussion in Section 3 that the differential operator L_1 satisfies

$$L_1 f = \mathrm{div}_\Sigma(P_1 \nabla f) - \langle \mathrm{div}_\Sigma P_1, \nabla f\rangle. \tag{5.2}$$

In particular, by taking an orthonormal frame $\{e_1, \dots, e_n\}$ on $T\Sigma$ and denoting by N the orientation of Σ^n, it follows from Lemma 3.1 of [2] that

$$\langle \mathrm{div}_\Sigma P_1, \nabla f\rangle = -\sum_{i=1}^{n} \langle \overline{R}(N, e_i)\nabla f, e_i\rangle = -\overline{\mathrm{Ric}}(N, \nabla f).$$

Hence, since L^{n+1} was assumed to be Einstein we conclude by equation (5.2) jointly with the previous identity that

$$L_1 f = \mathrm{div}_\Sigma(P_1 \nabla f).$$

Moreover, as was observed in Theorem 4.1 we get from our assumption on second fundamental form that $|P_1 \nabla f| \in L^1(\Sigma)$. Therefore, we are in position to apply Lemma 3.1 to conclude that $L_1 f = 0$, which gives a contradiction.

Finally, reasoning as above it is not difficult to prove (ii) and (iii). □

We continue by obtaining an analogous result to Theorem 4.4 in the case that $r = 1$ and the ambient space is a locally symmetric Lorentzian space. More precisely, we get the following.

Theorem 5.2. *Let L^{n+1} be a locally symmetric Einstein Lorentzian manifold satisfying the curvature condition (5.1). Let Σ^n be a complete 1-almost Yamabe soliton immersed into L^{n+1} with bounded second fundamental form and potential function such that $|\nabla f| \in L^1(\Sigma)$. Then:*

(i) If $\lambda \le n(n-1)(\mathcal{S} - H^2)$, then Σ^n is totally umbilical. In particular, the scalar curvature $\mathrm{Scal} = n(n-1)(\mathcal{S} - \kappa^2)$, where κ is the principal curvature of Σ^n;

(ii) If $\lambda \le n(n-1)\mathcal{S} - n^2 H^2$, then Σ^n is totally geodesic, with $\lambda = n(n-1)\mathcal{S}$ and scalar curvature $\mathrm{Scal} = n(n-1)\mathcal{S}$.

Proof. The result follows as in the proof of Theorem 4.4. For the sake of completeness, we give the following argument which proves (i). By taking trace in (1.3) and using the definition of the constant $\mathcal{S}$, we obtain from equation (3.1) that

$$L_1 f = n|\Phi|^2 + n\left(n(n-1)(\mathcal{S} - H^2) - \lambda\right), \tag{5.3}$$

which implies that $L_1 f \ge 0$ because our hypothesis on λ. Then, by Lemma 3.1 we get that $L_1 f = 0$. Therefore, we conclude from equation (5.3) that Σ^n is totally umbilical with $\mathrm{Scal} = n(n-1)(\mathcal{S} - \kappa^2)$, proving the result. □

We close our paper quoting the following result, which can be obtained with similar arguments to used in the proofs of Theorems 4.6 and 5.2.

Theorem 5.3. *Let L^{n+1} be a locally symmetric Einstein Lorentzian manifold satisfying the curvature condition (5.1). Let Σ^n be a complete 1-almost Yamabe soliton immersed into L^{n+1} such that P_r is bounded from above (in the sense of quadratic forms) and the potential function is nonnegative with $f \in L^p(\Sigma)$ for some $p > 1$. Then:*

(i) If $\lambda \le n(n-1)(\mathcal{S} - H^2)$, then Σ^n is totally umbilical. In particular, the scalar curvature $\mathrm{Scal} = n(n-1)(\mathcal{S} - \kappa^2)$, where κ is the principal curvature of Σ^n;

(ii) If $\lambda \le n(n-1)\mathcal{S} - n^2 H^2$, then Σ^n is totally geodesic, with $\lambda = n(n-1)\mathcal{S}$ and scalar curvature $\mathrm{Scal} = n(n-1)\mathcal{S}$.

Remark 5.4. We observe that in Theorems 5.1, 5.2 and 5.3 we can replace the hypothesis that the ambient space L^{n+1} is Einstein by the weaker assumption that the tensor $\overline{\mathrm{Ric}}_N : \mathfrak{X}(\Sigma) \to \mathfrak{X}(\Sigma)$, given by $\overline{\mathrm{Ric}}_N(X) = \overline{\mathrm{Ric}}(N, X)$, is identically zero.

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