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Covering properties by (a)-θ-open sets in (a)topological spaces

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Abstract We introduced the concept of (a)-θ-compactness and (a)-θ-Mengerness in (a)topological spaces. We discussed the relationship of the above notions with the other known covering properties. It is shown that the product of two (a)-θ-Menger (resp. (a)-θ-compact) spaces is (a)-θ-Menger (resp. (a)-θ-compact) if one of them is $(a)^s$-compact. If $\mathcal{X}^i$ is (a)-θ-Menger for each finite i, then (a)topological space $\mathcal{X}$ satisfies the selection principle $S_{fin}(\Theta\text{-}\Omega(\mathcal{X}), \Theta\text{-}\Omega(\mathcal{X}))$. Further, it is shown that the (a)-θ-Menger covering property is preserved under (a)-θ-continuous and (a)-strongly-θ-continuous map.

1 Introduction

Many authors [9, 10, 16, 17, 29, 36, 38, 41] investigated several covering properties extensively in topological spaces. All these covering properties are related with selection principles, introduced by Scheepers [11, 37]. The theory of selection principles is further studied by many authors. At first we recall one of the classical selection principle:

The selection principle $S_{fin}(\mathcal{P}, \mathcal{Q})$ is defined as:

For each sequence $< P_n : n \in \mathbb{N} >$ of elements of $\mathcal{P}$, there exists a finite set $\mathcal{Q}_n \subseteq \mathcal{P}_n$ (for each $n \in \mathbb{N}$) such that $\cup_{n\in\mathbb{N}}\mathcal{Q}_n \in \mathcal{Q}$ (see [17, 18]).

The property $S_{fin}(\mathcal{G}, \mathcal{G})$, where $\mathcal{G}$ is the family of all open covers of a topological space, is known as the the Menger covering property [29] (or the Menger property). Various weaker forms of the Menger property have been discussed in [1, 13, 14, 28, 31, 32, 35, 39]. Recently, Kočinac et al. [21, 32] studied weak versions of the classical Menger covering property by using several other forms of open sets. In selection principles theory, authors study mainly in two directions: (1). The closure operator is used in the definition of selection principle [1, 7, 13, 15, 16, 27, 28, 31, 34] and (2). Sequences of open covers are replaced by sequences of covers by some weak or strong form of open sets [19, 20, 21, 32, 33]. Recently, Luthra et al. [26] worked in first direction and studied various selective version of separability in (a)topological spaces which is more general than bitopological spaces [12], (ω)topological spaces [2, 3, 5] and $(\aleph_0)$topological spaces [4]. (a)topological space, introduced by Choudhury et. al. [6], is a non empty set on which a sequence of topologies are imposed. Here, we concerned in the second direction by using the notion of (a)-θ-open sets in (a)topological spaces. We introduced (a)-θ-open sets and studied various types of continuity. We investigated the classical Menger covering property and the compactness property by using (a)-θ-open sets.

In 1966, Velicko [42] introduced θ-closed sets in topological spaces. In [24, 25, 30], the different types of continuity via θ-open sets are introduced and studied in detail. By using θ-open sets, some covering properties are discussed by Kohli et. al. in [22]. In this paper, we define (a)-θ-open sets in (a)topological spaces. We studied the notion of (a)-θ-open set in (a)subspaces, product (a)spaces and discussed its relationship with (a)-open sets [26]. We discussed various types of continuity and some covering properties in detail.

In Section 2, we discussed various properties of (a)-θ-open sets. We characterize (a)-θ-open sets in bi(a)spaces. Section 3 deals with different types of continuity in (a)topological spaces and inter-relationships between them. We characterize (a)-θ-open sets in (a)subspaces. Further, we characterize (a)-strongly-θ-map in product (a)topological spaces. In Section 4, we discussed the classical Menger covering property and the compactness property via (a)-θ-open sets. We construct many counterexamples that shows the inter-relationships between various covering properties. It is shown that the property of (a)-θ-Menger is hereditary under (a)-clopen subspaces. The product of two (a)-θ-Menger (resp. (a)-θ-compact) spaces is (a)-θ-Menger (resp. (a)-θ-compact) if one of them is $(a)^s$-compact. If $\mathcal{X}^i$ (a)-θ-Menger for each finite i, then $\mathcal{X}$ satisfies $S_{fin}(\Theta\text{-}\Omega(\mathcal{X}), \Theta\text{-}\Omega(\mathcal{X}))$. Further, we show that the (a)-θ-Menger covering property is preserved under (a)-θ-continuous and (a)-strongly-θ-continuous map.

Let $(\mathcal{X}, \{\tau_n\})$ be an (a)topological space and $\mathcal{Y} \subseteq \mathcal{X}$. Then we say $(\mathcal{Y}, \{\tau_{n\mathcal{Y}}\})$ is an (a)subspace of $\mathcal{X}$, where $\tau_{n\mathcal{Y}}$ is the induced subspace topology on $\mathcal{Y}$ inherited from τ_n for each $n \in \mathbf{N}$. If $\mathcal{Y}$ is (a)-open (resp. (a)-closed, (a)-clopen) in $\mathcal{X}$, we say $\mathcal{Y}$ is an (a)-open (resp. (a)-closed, (a)-clopen) subspace of $\mathcal{X}$.

Throughout the paper, $(\mathcal{X}, \{\tau_n\})$ denotes an (a)topological space and if there is no scope of confusion, we will write $\mathcal{X}$ instead of $(\mathcal{X}, \{\tau_n\})$. For $A \subseteq \mathcal{X}$, the (τ_n)interior (resp. (τ_n)closure) of A in $\mathcal{X}$ is denoted by τ_n-int(A) (resp. τ_n-cl(A)). In (a)subspace $(\mathcal{Y}, \{\tau_{n\mathcal{Y}}\})$ of $(\mathcal{X}, \{\tau_n\})$, τ_n-int$_{\mathcal{Y}}(A)$ (resp. τ_n-cl$_{\mathcal{Y}}(A)$) denotes the τ_n-interior (resp. τ_n-closure) of A in $\mathcal{Y}$. By $\mathcal{X}^k$, we mean the cartesian product of k-copies of $\mathcal{X}$. Following are the standard notions used in this paper.

- τ_d - Discrete Topology
- τ_u - Usual Topology on $\mathbb{R}$
- τ_c - Cocountable Topology
- τ_f - Cofinite Topology
- τ_l - Lower limit Topology on $\mathbb{R}$

For general notion of topology, we follow [8] and one can see [17, 18, 35, 41] for other basic notions regarding selection principles.

2 (a)-θ-open sets

Definition 2.1. [6] If $< \tau_n : n \in \mathbb{N} >$ is a sequence of topologies on a non empty set $\mathcal{X}$, then $(\mathcal{X}, \{\tau_n\}_{n\in\mathbb{N}})$ is called an (a)topological space (in short, (a)space).

Definition 2.2. [26] A set $G \subseteq \mathcal{X}$ is said to be:

(1). τ_n-open if $G \in \tau_n$.

(2). (a)-open if G is τ_n-open for all $n \in \mathbb{N}$.

(3). (a)-closed if $\mathcal{X} - G$ is (a)-open.

We denote the family of all (a)-open sets and (a)-closed sets by $\mathcal{O}(\mathcal{X})$ and $\mathcal{C}(\mathcal{X})$, respectively. If there is no scope of confusion, we will write $\mathcal{O}$ and $\mathcal{C}$ instead of $\mathcal{O}(\mathcal{X})$ and $\mathcal{C}(\mathcal{X})$, respectively.

Remark 2.3. $\mathcal{O}(\mathcal{X})$ forms a topology on $\mathcal{X}$.

Definition 2.4. Let $(\mathcal{X}, \{\tau_n\})$ be an (a)space. A point $g \in \mathcal{X}$ is said to be (m,n)-θ-cluster point of $G \subseteq \mathcal{X}$ if for every $O \in \tau_n$ containing g, τ_m-cl$(O) \cap G \neq \emptyset$. The set $\{g \in \mathcal{X} : g$ is (m,n)-θ-cluster point of $G\}$, denoted by $\tau_{(m,n)}$-cl$_\theta(G)$, is the (m,n)-θ-closure of G. If $\tau_{(m,n)}$-cl$_\theta(G) = G$, then G is called (m,n)-θ-closed. G is (m,n)-θ-open if $\mathcal{X} - G$ is (m,n)-θ-closed. In case, G is (m,n)-θ-open (resp. (m,n)-θ-closed) for all $m \neq n$, we say G is (a)-θ-open (resp. (a)-θ-closed).

We denote the family of all (m,n)-θ-open sets, (a)-θ-open sets and (a)-θ-closed sets by $\mathcal{O}_\theta(m,n)(\mathcal{X})$, $\mathcal{O}_\theta(\mathcal{X})$ and $\mathcal{C}_\theta(\mathcal{X})$, respectively. If there is no scope of confusion, we will write $\mathcal{O}_\theta(m,n)$, $\mathcal{O}_\theta$ and $\mathcal{C}_\theta$ instead of $\mathcal{O}_\theta(m,n)(\mathcal{X})$, $\mathcal{O}_\theta(\mathcal{X})$ and $\mathcal{C}_\theta(\mathcal{X})$, respectively.

Remark 2.5. $\mathcal{O}_\theta(\mathcal{X})$ forms a topology on $\mathcal{X}$.

Proposition 2.6. $\mathcal{O}_\theta(\mathcal{X}) \subseteq \mathcal{O}(\mathcal{X})$ for any (a)space $\mathcal{X}$.

Proof. Let $G \in \mathcal{O}_\theta$. By definition, G is (m,n)-θ-open for all $m \neq n$. Then $\tau_{(m,n)}$-$\mathrm{cl}_\theta(\mathcal{X}-G) = \mathcal{X}-G$ for all $m \neq n$. But τ_n-$\mathrm{cl}(A) \subseteq \tau_{(m,n)}$-$\mathrm{cl}_\theta(A)$ for every set A, so τ_n-$\mathrm{cl}(\mathcal{X}-G) = \mathcal{X}-G$. Thus, G is τ_n-open for all $n \in \mathbb{N}$. □

The converse of the Proposition 2.6 is not true.

Example 2.7. Let τ_n be the digital topology on $\mathbb{Z}$ generated by $\{\{2k-1, 2k, 2k+1\} \colon k \in \mathbb{Z}\}$ for each odd n and τ_n be the topology on $\mathbb{Z}$ generated by $\{..., \{-8,-7,-6\}, \{-5,-4,-3\}, \{-2,-1, 0\}, \{1\}, \{2,3,4\}, \{5,6,7\}, \{8,9,10\}, ...\}$ for each even n. Let $G = \{1\}$. It is obvious that $G \in \mathcal{O}$. We will show that $G \notin \mathcal{O}_\theta$. Let $U \in \tau_3$ and $1 \in U$. Then $\{0,1,2\} \subseteq \tau_1$-$\mathrm{cl}(U)$. This implies that τ_1-$\mathrm{cl}(U)) \cap (\mathbb{Z}-G) \neq \emptyset$. So $1 \in \tau_{(1,3)}$-$\mathrm{cl}_\theta(\mathbb{Z}-G)$ and thus, $\tau_{(1,3)}$-$\mathrm{cl}_\theta(\mathbb{Z}-G) \neq \mathbb{Z}-G$. Therefore, G is not $(1,3)$-θ-open and hence, $G \notin \mathcal{O}_\theta$.

In following results, we characterize (m,n)-θ-open sets in (a)spaces.

Theorem 2.8. A set $G \in \mathcal{O}_\theta(m,n)(\mathcal{X})$ if and only if for each $g \in G$ there exists $O \in \tau_n$ satisfying $g \in O \subseteq \tau_m$-$\mathrm{cl}(O) \subseteq G$.

Proof. Let $G \in \mathcal{O}_\theta(m,n)(\mathcal{X})$. Then $\tau_{(m,n)}$-$\mathrm{cl}_\theta(\mathcal{X}-G) = \mathcal{X}-G$. So for each $g \in G$, there exists $O \in \tau_n$ such that $g \in O$ and τ_m-$\mathrm{cl}(O) \cap (\mathcal{X}-G) = \emptyset$. Therefore, $g \in O \subseteq \tau_m$-$\mathrm{cl}(O) \subseteq G$. Conversely, let for each $g \in G$ there exists $O \in \tau_n$ such that $g \in O \subseteq \tau_m$-$\mathrm{cl}(O) \subseteq G$. Therefore, τ_m-$\mathrm{cl}(O) \cap (\mathcal{X}-G) = \emptyset$. So $\tau_{(m,n)}$-$\mathrm{cl}_\theta(\mathcal{X}-G) = \mathcal{X}-G$ and thus, $G \in \mathcal{O}_\theta(m,n)(\mathcal{X})$. □

Corollary 2.9. A set $G \in \mathcal{O}_\theta(\mathcal{X})$ if and only if for each $g \in G$ there exists $O_n \in \tau_n$ (for each $n \in \mathbb{N}$) such that $g \in O_n \subseteq \tau_m$-$\mathrm{cl}(O_n) \subseteq G$ for all $m \neq n$.

Definition 2.10. A set $G \subseteq \mathcal{X}$ is (m,n)-regular-open if $G = \tau_n$-$\mathrm{int}(\tau_m$-$\mathrm{cl}(G))$.

Proposition 2.11. A set $G \in \mathcal{O}_\theta(m,n)(\mathcal{X})$ if and only if for each $g \in G$ there exists an (m,n)-regular-open set R such that $g \in R \subseteq \tau_m$-$\mathrm{cl}(R) \subseteq G$.

Proof. Let $G \in \mathcal{O}_\theta(m,n)(\mathcal{X})$. For each $g \in G$, there exists $O \in \tau_n$ satisfying $g \in O \subseteq \tau_m$-$\mathrm{cl}(O) \subseteq G$. Let $R = \tau_n$-$\mathrm{int}(\tau_m$-$\mathrm{cl}(O))$. Since R is τ_n-open, $R \subseteq \tau_n$-$\mathrm{int}(\tau_m$-$\mathrm{cl}(R))$. Also τ_n-$\mathrm{int}(\tau_m$-$\mathrm{cl}(R)) = \tau_n$-$\mathrm{int}(\tau_m$-$\mathrm{cl}(\tau_n$-$\mathrm{int}(\tau_m$-$\mathrm{cl}(O)))) \subseteq \tau_n$-$\mathrm{int}(\tau_m$-$\mathrm{cl}(O)) = R$. Thus, R is an (m,n)-regular-open set such that $g \in R \subseteq \tau_m$-$\mathrm{cl}(R) \subseteq G$. Conversely, every (m,n)-regular-open set is τ_n-open, so the proof follows from Theorem 2.8. □

Proposition 2.12. If $A \subseteq B \subseteq \mathcal{X}$, then $A \in \mathcal{O}_\theta(\mathcal{X})$ imply that $A \in \mathcal{O}_\theta(B)$.

Proof. Let $A \in \mathcal{O}_\theta(\mathcal{X})$ with $A \subseteq B \subseteq \mathcal{X}$ and $a \in A$. For each $m \neq n$, there exists $O \in \tau_n$ such that $a \in O$ and τ_m-$\mathrm{cl}(O) \subseteq A$. Then $a \in O \cap B$ and $O \cap B \subseteq \tau_m$-$\mathrm{cl}_B(O \cap B) = \tau_m$-$\mathrm{cl}(O \cap B) \cap B \subseteq \tau_m$-$\mathrm{cl}(O)$. Therefore, $G = O \cap B \in \tau_{nB}$ satisfying $a \in G \subseteq \tau_m$-$\mathrm{cl}_B(G) \subseteq A$ for all $m \neq n$. Thus, $A \in \mathcal{O}_\theta(B)$. □

Let $\{(\mathcal{X}_\alpha, \{\tau_{n\alpha}\}_{n\in\mathbb{N}}) \colon \alpha \in \wedge\}$ be an arbitrary family of (a)spaces and $\mathcal{X} = \prod_{\alpha\in\wedge} \mathcal{X}_\alpha$. We define an (a)topology structure $(\mathcal{X}, \{\tau_n\})$ on $\mathcal{X}$ by considering τ_n as the product topology on $\mathcal{X}$ generated by the continuous projections $p_{n\alpha} \colon (\mathcal{X}, \tau_n) \to (\mathcal{X}_\alpha, \tau_{n\alpha})$ for every $\alpha \in \wedge$. $(\mathcal{X}, \{\tau_n\})$ is called the product (a)space. In particular, the product of two (a)spaces is called the bi(a)space. In the following two results, we characterize (a)-θ-open sets in bi(a)spaces.

Theorem 2.13. Let $(\mathcal{X}, \{\tau_n\})$ and $(\mathcal{Y}, \{\sigma_n\})$ be two (a)spaces. If $A \in \mathcal{O}_\theta(\mathcal{X})$ and $B \in \mathcal{O}_\theta(\mathcal{Y})$, then $A \times B \in \mathcal{O}_\theta(\mathcal{X} \times \mathcal{Y})$, where $(\mathcal{X} \times \mathcal{Y}, \{\gamma_n\})$ is a bi(a)space.

Proof. Let $A \in \mathcal{O}_\theta(\mathcal{X})$, $B \in \mathcal{O}_\theta(\mathcal{Y})$ and $(a,b) \in A \times B$. For each $m \neq n$, there exists $A_n \in \tau_n$ and $B_n \in \sigma_n$ such that $a \in A_n \subseteq \tau_m$-cl$(A_n) \subseteq A$ and $b \in B_n \subseteq \tau_m$-cl$(B_n) \subseteq B$. It follows that $(a,b) \in A_n \times B_n \subseteq \tau_m$-cl$(A_n) \times \sigma_m$-cl$(B_n) \subseteq A \times B$. But τ_m-cl$(A_n) \times \sigma_m$-cl$(B_n) = \gamma_m$-cl$(A_n \times B_n)$. Thus, $A \times B \in \mathcal{O}_\theta(m,n)(\mathcal{X} \times \mathcal{Y})$ for all $m \neq n$ and hence, $A \times B \in \mathcal{O}_\theta(\mathcal{X} \times \mathcal{Y})$. □

Theorem 2.14. Let $(\mathcal{X}, \{\tau_n\})$ and $(\mathcal{Y}, \{\sigma_n\})$ be two (a)spaces. Let $p_\mathcal{X} \colon (\mathcal{X} \times \mathcal{Y}, \{\gamma_n\}) \to (\mathcal{X}, \{\tau_n\})$ be a map defined by $p_\mathcal{X}(x,y) = x$. If $A \in \mathcal{O}_\theta(\mathcal{X} \times \mathcal{Y})$, then $p_\mathcal{X}(A) \in \mathcal{O}_\theta(\mathcal{X})$.

Proof. Let $A \in \mathcal{O}_\theta(\mathcal{X} \times \mathcal{Y})$ and $a \in p_\mathcal{X}(A)$. Then there exists $b \in \mathcal{Y}$ such that $(a,b) \in A$. Since $A \in \mathcal{O}_\theta(m,n)(\mathcal{X} \times \mathcal{Y})$ for all $m \neq n$, there exists $U_n \in \gamma_n$ (for each n) such that $(a,b) \in U_n \subseteq \gamma_m$-cl$(U_n) \subseteq A$ for all $m \neq n$. But $U_n = A_n \times B_n$ where $A_n \in \tau_n$ and $B_n \in \sigma_n$, so $(a,b) \in A_n \times B_n \subseteq \tau_m$-cl$(A_n) \times \sigma_m$-cl$(B_n) = \gamma_m$-cl$(A_n \times B_n) \subseteq A$. Thus, $a \in A_n \subseteq \tau_m$-cl$(A_n) \subseteq p_\mathcal{X}(A)$ and hence, $p_\mathcal{X}(A) \in \mathcal{O}_\theta(\mathcal{X})$. □

We conclude from Theorem 2.13 and Theorem 2.14 that $A \in \mathcal{O}_\theta(\mathcal{X} \times \mathcal{Y})$ if and only if $A = G \times H$, where $G \in \mathcal{O}_\theta(\mathcal{X})$ and $H \in \mathcal{O}_\theta(\mathcal{Y})$.

Theorem 2.15. Let $(\mathcal{X}, \{\tau_n\})$ and $(\mathcal{Y}, \{\sigma_n\})$ be two (a)spaces. A set $A \in \mathcal{O}_\theta(\mathcal{X} \times \mathcal{Y})$ if and only if $A = G \times H$, where $G \in \mathcal{O}_\theta(\mathcal{X})$ and $H \in \mathcal{O}_\theta(\mathcal{Y})$.

Proposition 2.16. If a point $x \in \tau_{(m,n)}$-cl$_\theta(G)$ for $G \subseteq \mathcal{X}$, then every (m,n)-θ-open set containing x intersects G.

Proof. Let $G \subseteq \mathcal{X}$ and $x \in \tau_{(m,n)}$-cl$_\theta(G)$. Let $O \in \mathcal{O}_\theta(m,n)$ with $x \in O$. Then there exists $U \in \tau_n$ such that $x \in U \subseteq \tau_m$-cl$(U) \subseteq O$. Since $x \in \tau_{(m,n)}$-cl$_\theta(G)$, τ_m-cl$(U) \cap G \neq \emptyset$. It follows that $O \cap G \neq \emptyset$. □

3 Mappings and (a)-θ-open sets

Definition 3.1. A function $\psi \colon (\mathcal{X}, \{\tau_k\}) \to (\mathcal{Y}, \{\sigma_k\})$ is said to be:

(1). $(a)^s$-continuous (resp. $(a)^s$-weakly-continuous) if for each $x \in \mathcal{X}$ and each $O \in \sigma_n$ with $\psi(x) \in O$, there exists $V \in \tau_n$ with $x \in V$ satisfying $\psi(V) \subseteq O$ (resp. $\psi(V) \subseteq \sigma_m$-cl$(O)$) for all $m \neq n$.

(2). $(a)^s$-faintly-continuous if for each $x \in \mathcal{X}$ and each $O \in \mathcal{O}_\theta(m,n)(\mathcal{Y})$ with $\psi(x) \in O$, there exists $V \in \tau_n$ with $x \in V$ satisfying $\psi(V) \subseteq O$ for all $m \neq n$.

(3). (a)-θ-continuous (resp. (a)-strongly-θ-continuous) if for each $x \in \mathcal{X}$ and each $O \in \sigma_n$ with $\psi(x) \in O$, there exists $V \in \tau_n$ with $x \in V$ satisfying $\psi(\tau_m$-cl$(V)) \subseteq \sigma_m$-cl(O) (resp. $\psi(\tau_m$-cl$(V)) \subseteq O$) for all $m \neq n$.

(4). (a)-θ-open (resp. (a)-θ-closed) if $\psi(A) \in \mathcal{O}_\theta(\mathcal{Y})$ (resp. $\psi(A) \in \mathcal{C}_\theta(\mathcal{Y})$) for each $A \in \mathcal{O}_\theta(\mathcal{X})$ (resp. $A \in \mathcal{C}_\theta(\mathcal{X})$).

(5). $(a)^s$-open (resp. $(a)^s$-closed) if $\psi \colon (\mathcal{X}, \tau_k) \to (\mathcal{Y}, \sigma_k)$ is open (resp. closed) for all $k \in \mathbb{N}$.

Proposition 3.2. Let $\psi \colon (\mathcal{X}, \{\tau_k\}) \to (\mathcal{Y}, \{\sigma_k\})$ be an (a)-θ-continuous map. If $G \in \mathcal{O}_\theta(\mathcal{Y})$, then $\psi^{-1}(G) \in \mathcal{O}_\theta(\mathcal{X})$.

Proof. Let $G \in \mathcal{O}_\theta(\mathcal{Y})$ and $g \in \psi^{-1}(G)$. Then $\psi(g) \in G$. So there exists $V_n \in \sigma_n$ (for each n) such that $\psi(g) \in V_n \subseteq \sigma_m$-cl$(V_n) \subseteq G$ for all $m \neq n$. Since ψ is (a)-θ-continuous, there exists $U_n \in \tau_n$ (for each n) with $g \in U_n$ satisfying $\psi(\tau_m$-cl$(U_n)) \subseteq \sigma_m$-cl$(V_n)$ for all $m \neq n$. It follows that $g \in U_n \subseteq \tau_m$-cl$(U_n) \subseteq \psi^{-1}(\sigma_m$-cl$(V_n)) \subseteq \psi^{-1}(G)$ for all $m \neq n$. Thus, $\psi^{-1}(G) \in \mathcal{O}_\theta(\mathcal{X})$. □

From Definition 3.1, it is clear that every (a)-strongly-θ-continuous function is (a)-θ-continuous, so the following corollary is immediate.

Corollary 3.3. Let $\psi \colon (\mathcal{X}, \{\tau_k\}) \to (\mathcal{Y}, \{\sigma_k\})$ be an (a)-strongly-θ-continuous map. If $G \in \mathcal{O}_\theta(\mathcal{Y})$, then $\psi^{-1}(G) \in \mathcal{O}_\theta(\mathcal{X})$.

Proposition 3.4. Let $\psi\colon (\mathcal{X}, \{\tau_k\}) \to (\mathcal{Y}, \{\sigma_k\})$ be an (a)-strongly-θ-continuous map. If $G \in \mathcal{O}(\mathcal{Y})$, then $\psi^{-1}(G) \in \mathcal{O}_\theta(\mathcal{X})$.

Proof. Let $G \in \mathcal{O}(\mathcal{Y})$ and $g \in \psi^{-1}(G)$. Then $\psi(g) \in G$. Since $G \in \mathcal{O}(\mathcal{Y})$ and ψ is (a)-strongly-θ-continuous, there exists $U_n \in \tau_n$ (for each n) with $g \in U_n$ satisfying $\psi(\tau_m\text{-cl}(U_n)) \subseteq G$ for all $m \neq n$. Therefore, $g \in U_n \subseteq \tau_m\text{-cl}(U_n) \subseteq \psi^{-1}(G)$ for all $m \neq n$. Thus, $\psi^{-1}(G) \in \mathcal{O}_\theta(\mathcal{X})$. □

Proposition 3.5. Let $\psi\colon (\mathcal{X}, \{\tau_k\}) \to (\mathcal{Y}, \{\sigma_k\})$ be an (a)-strongly-θ-continuous map. If $G \in \mathcal{C}(\mathcal{Y})$, then $\psi^{-1}(G) \in \mathcal{C}_\theta(\mathcal{X})$.

Proof. It can be easily proved from the Proposition 3.4. □

Theorem 3.6. Let $\psi\colon (\mathcal{X}, \{\tau_k\}) \to (\mathcal{Y}, \{\sigma_k\})$ be an $(a)^s$-open and $(a)^s$-closed map. Then ψ preserve (a)-θ-open sets.

Proof. Let $G \in \mathcal{O}_\theta(\mathcal{X})$ and $y \in \psi(G)$. Then $y = \psi(x)$ for some $x \in G$. Since $G \in \mathcal{O}_\theta(m,n)(\mathcal{X})$ for all $m \neq n$, there exists $U_n \in \tau_n$ (for each n) satisfying $x \in U_n \subseteq \tau_m\text{-cl}(U_n) \subseteq G$ for all $m \neq n$. Therefore, $y \in \psi(U_n) \subseteq \psi(\tau_m\text{-cl}(U_n)) \subseteq \psi(G)$. Since ψ is $(a)^s$-open and $(a)^s$-closed, so $\psi(U_n) \in \sigma_n$ and $\sigma_m\text{-cl}(\psi(U_n)) \subseteq \psi(\tau_m\text{-cl}(U_n))$. Therefore, $y \in \psi(U_n) \subseteq \sigma_m\text{-cl}(\psi(U_n)) \subseteq \psi(\tau_m\text{-cl}(U_n)) \subseteq \psi(G)$ for all $m \neq n$. Thus, $\psi(G) \in \mathcal{O}_\theta(\mathcal{Y})$. □

As an application of Theorem 3.6 we are able to characterize (a)-θ-open sets in (a)-clopen subspaces.

Theorem 3.7. Let $(B, \{\tau_{nB}\})$ be an (a)-clopen subspace of (a)space $(\mathcal{X}, \{\tau_n\})$. For $G \subseteq B$, $G \in \mathcal{O}_\theta(B)$ if and only if $G = O \cap B$, where $O \in \mathcal{O}_\theta(\mathcal{X})$.

Proof. Let $(B, \{\tau_{nB}\})$ be an (a)-clopen subspace of $(\mathcal{X}, \{\tau_n\})$. Let $O \in \mathcal{O}_\theta(\mathcal{X})$. Since every (a)-clopen set is (a)-θ-open and finite intersection of (a)-θ-open sets is (a)-θ-open, so $O \cap B \in \mathcal{O}_\theta(\mathcal{X})$. But $O \cap B \subseteq B \subseteq \mathcal{X}$, so by Proposition 2.12, $O \cap B \in \mathcal{O}_\theta(B)$. Conversely, let $G \subseteq B$ such that $G \in \mathcal{O}_\theta(B)$. It is enough to show that $G \in \mathcal{O}_\theta(\mathcal{X})$. Consider the inclusion map $\psi\colon (B, \{\tau_{nB}\}) \to (\mathcal{X}, \{\tau_n\})$. Since B is both (a)-open and (a)-closed, ψ is $(a)^s$-open and $(a)^s$-closed. In view of Theorem 3.6, $\psi(G) = G \in \mathcal{O}_\theta(\mathcal{X})$. □

Theorem 3.8. Let $(B, \{\tau_{nB}\})$ be an (a)-clopen subspace of (a)space $(\mathcal{X}, \{\tau_n\})$. For $G \subseteq B$, $G \in \mathcal{O}_\theta(B)$ if and only if $G \in \mathcal{O}_\theta(\mathcal{X})$.

Proof. It can be easily proved from the Proposition 2.12 and the Theorem 3.7. □

Theorem 3.9. Let $\psi\colon (\mathcal{X}, \{\tau_k\}) \to (\mathcal{Y}, \{\sigma_k\})$ be an (a)-strongly-θ-continuous map and $\phi\colon (\mathcal{Y}, \{\sigma_k\}) \to (\mathcal{Z}, \{\gamma_k\})$ be an $(a)^s$-continuous map. Then $\phi \circ \psi\colon (\mathcal{X}, \{\tau_k\}) \to (\mathcal{Z}, \{\gamma_k\})$ is (a)-strongly-θ-continuous.

Proof. Let $g \in \mathcal{X}$ and $G \in \gamma_n$ with $(\phi \circ \psi)(g) \in G$. Since ϕ is $(a)^s$-continuous, there exists $S \in \sigma_n$ with $\psi(g) \in S$ satisfying $\phi(S) \subseteq G$. Also ψ is (a)-strongly-θ-continuous, so there exists $O \in \tau_n$ with $g \in O$ satisfying $\psi(\tau_m\text{-cl}(O)) \subseteq S$ for all $m \neq n$. It follows that $(\phi \circ \psi)(\tau_m\text{-cl}(O)) \subseteq \phi(S) \subseteq G$ for all $m \neq n$. Hence, $\phi \circ \psi$ is (a)-strongly-θ-continuous. □

Theorem 3.10. Let $\{(\mathcal{Y}_\alpha, \{\tau_{n\alpha}\}_{n\in\mathbb{N}})\colon \alpha \in \wedge\}$ be an arbitrary family of (a)spaces and consider the product (a)space $(\mathcal{Y}, \{\tau_n\})$ where $\mathcal{Y} = \prod_{\alpha\in\wedge} \mathcal{Y}_\alpha$. For each $\alpha \in \wedge$, let $q_\alpha\colon (\mathcal{Y}, \{\tau_n\}) \to (\mathcal{Y}_\alpha, \{\tau_{n\alpha}\})$ be defined by $q_\alpha(y) = y_\alpha$, where $y = (y_\alpha)_{\alpha\in\wedge}$. A mapping $\psi\colon (\mathcal{X}, \{\sigma_k\}) \to (\mathcal{Y}, \{\tau_k\})$ is (a)-strongly-θ-continuous if and only if each composition $(q_\alpha \circ \psi)\colon (\mathcal{X}, \{\sigma_k\}) \to (\mathcal{Y}_\alpha, \{\tau_{k\alpha}\})$ is (a)-strongly-θ-continuous for all $\alpha \in \wedge$.

Proof. Let $g \in \mathcal{X}$ and $O \in \tau_n$ with $\psi(g) \in O$. Then there exists a basic τ_n-open set G satisfying $\psi(g) \in G \subseteq O$. Let $G = p^{-1}_{n\alpha_1}(W_1) \cap p^{-1}_{n\alpha_2}(W_2) \cap \cdots \cap p^{-1}_{n\alpha_k}(W_k)$, where $W_i \in \tau_{n\alpha_i}$ for all $i = 1, 2, \ldots, k$. Clearly $G = q^{-1}_{\alpha_1}(W_1) \cap q^{-1}_{\alpha_2}(W_2) \cap \cdots \cap q^{-1}_{\alpha_k}(W_k)$. Since $(q_{\alpha_i} \circ \psi)$ is (a)-strongly-θ-continuous and $(q_{\alpha_i} \circ \psi)(g) \in W_i$ for all $i = 1, 2, \ldots, k$, there exist $A_i \in \sigma_n$ with $g \in A_i$ and $(q_{\alpha_i} \circ \psi)(\sigma_m\text{-cl}(A_i)) \subseteq W_i$ for all $m \neq n$. It follows that $\cap_{i=1}^k \psi(\sigma_m\text{-cl}(A_i)) \subseteq \cap_{i=1}^k q^{-1}_{\alpha_i}(W_i) = G \subseteq O$. It can be seen that $\psi[\sigma_m\text{-cl}(A_1 \cap A_2 \cap \cdots \cap A_k)] \subseteq O$. Thus, for each $g \in \mathcal{X}$ and $O \in \tau_n$ with $\psi(g) \in O$, there exists $A = A_1 \cap A_2 \cap \cdots \cap A_k \in \sigma_n$ with $g \in A$ satisfying $\psi(\sigma_m\text{-cl}(A)) \subseteq O$ for all $m \neq n$. Hence, ψ is (a)-strongly-θ-continuous. The converse follows from Theorem 3.9 as the map q_α is $(a)^s$-continuous for all $\alpha \in \wedge$. □

Proposition 3.11. Every $(a)^s$-continuous function is (a)-θ-continuous.

Proof. Let $\psi\colon (\mathcal{X}, \{\tau_k\}) \to (\mathcal{Y}, \{\sigma_k\})$ be an $(a)^s$-continuous function. Let $g \in \mathcal{X}$ and $G \in \sigma_n$ with $\psi(g) \in G$. Since ψ is $(a)^s$-continuous, there exists $O \in \tau_n$ with $g \in O$ satisfying $\psi(O) \subseteq G$. Then $O \subseteq \psi^{-1}(G)$ which readily follows that τ_m-cl$(O) \subseteq \tau_m$-cl$(\psi^{-1}(G))$ for all $m \in \mathbb{N}$. Since ψ is $(a)^s$-continuous, $\psi(\tau_m\text{-cl}(O)) \subseteq \psi(\tau_m\text{-cl}(\psi^{-1}(G))) \subseteq \sigma_m\text{-cl}[\psi(\psi^{-1}(G))] \subseteq \sigma_m\text{-cl}(G)$. Thus, $\psi(\tau_m\text{-cl}(O)) \subseteq \sigma_m\text{-cl}(G)$ for all $m \neq n$. Hence, ψ is (a)-θ-continuous. □

Corollary 3.12. Let $\psi\colon (\mathcal{X}, \{\tau_k\}) \to (\mathcal{Y}, \{\sigma_k\})$ be an $(a)^s$-continuous function. If $G \in \mathcal{O}_\theta(\mathcal{Y})$, then $\psi^{-1}(G) \in \mathcal{O}_\theta(\mathcal{X})$.

Proposition 3.13. Every $(a)^s$-weakly-continuous function is $(a)^s$-faintly-continuous.

Proof. Let $\psi\colon (\mathcal{X}, \{\tau_k\}) \to (\mathcal{Y}, \{\sigma_k\})$ be an $(a)^s$-weakly-continuous function. Let $g \in \mathcal{X}$ and $A \in \mathcal{O}_\theta(m,n)(\mathcal{Y})$ with $\psi(g) \in A$. Then there exists $G \in \sigma_n$ such that $\psi(g) \in G \subseteq \sigma_m$-cl$(G) \subseteq A$. Since ψ is $(a)^s$-weakly-continuous, there exists $O \in \tau_n$ with $g \in O$ satisfying $\psi(O) \subseteq \sigma_m$-cl$(G)$. Thus, $\psi(O) \subseteq A$ and hence, ψ is $(a)^s$-faintly-continuous. □

Proposition 3.14. A mapping $\psi\colon (\mathcal{X}, \{\tau_k\}) \to (\mathcal{Y}, \{\sigma_k\})$ is $(a)^s$-faintly-continuous if and only if for every $G \in \mathcal{O}_\theta(m,n)(\mathcal{Y})$, $\psi^{-1}(G) \in \tau_n$.

Proof. Let $G \in \mathcal{O}_\theta(m,n)(\mathcal{Y})$ and $g \in \psi^{-1}(G)$. Then $\psi(g) \in G$. Since ψ is $(a)^s$-faintly-continuous, there exists $U_n \in \tau_n$ with $g \in U_n$ satisfying $\psi(U_n) \subseteq G$. Therefore, $g \in U_n \subseteq \psi^{-1}(G)$. Thus, $\psi^{-1}(G) \in \tau_n$. Conversely, let $g \in \mathcal{X}$ and $G \in \mathcal{O}_\theta(m,n)(\mathcal{Y})$ with $\psi(g) \in G$. Then $O = \psi^{-1}(G) \in \tau_n$ such that $\psi(O) \subseteq G$. Therefore, ψ is $(a)^s$-faintly-continuous. □

Theorem 3.15. Let $\psi\colon (\mathcal{X}, \{\tau_k\}) \to (\mathcal{Y}, \{\sigma_k\})$ be an $(a)^s$-faintly-continuous map. Then the following results hold:

1. $\psi^{-1}(O) \in \mathcal{O}(\mathcal{X})$ for every $O \in \mathcal{O}_\theta(\mathcal{Y})$.
2. $\psi^{-1}(F) \in \mathcal{C}(\mathcal{X})$ for every $F \in \mathcal{C}_\theta(\mathcal{Y})$.

Proof. It can be easily proved from the Proposition 3.14. □

4 (a)-θ-Menger and (a)-θ-compact covering properties

In this section, we disussed and studied various covering properties. We say $\mathcal{X}$ is (a)-P if $(\mathcal{X}, \mathcal{O})$ satisfies property P. Like, if $(\mathcal{X}, \mathcal{O})$ is compact (resp. Lindelöf, Menger), we say $\mathcal{X}$ is (a)-compact (resp. (a)-Lindelöf, (a)-Menger). If $(\mathcal{X}, \mathcal{O})$ satisfies T_0 (resp. T_1, T_2, T_3) separation axiom, we say $\mathcal{X}$ is (a)-T_0 (resp. (a)-T_1, (a)-T_2, (a)-T_3) space. In case, $(\mathcal{X}, \tau_n)$ is compact (resp. Lindelöf, Menger) for all $n \in \mathbb{N}$, $\mathcal{X}$ is $(a)^s$-compact (resp. $(a)^s$-Lindelöf, $(a)^s$-Menger). A cover $\mathcal{G}$ of $\mathcal{X}$ is called an (a)-θ-cover (resp. (a)-cover, (m,n)-θ-cover, τ_n-cover) if each $G \in \mathcal{G}$ belongs to $\mathcal{O}_\theta$ (resp. $\mathcal{O}$, $\mathcal{O}_\theta(m,n)$, τ_n). Call $\mathcal{G}$ an (a)-θ-ω-cover of $\mathcal{X}$ if $\mathcal{X} \notin \mathcal{G}$ and for each finite set $F \subseteq \mathcal{X}$ there is a $G \in \mathcal{G}$ such that $F \subseteq G$. We denote the family of all (a)-θ-ω-covers (resp. (a)-θ-covers, (m,n)-θ-covers) of $\mathcal{X}$ by Θ-$\Omega(\mathcal{X})$ (resp. $\Theta(\mathcal{X})$, $\Theta(m,n)(\mathcal{X})$). We begin this section with some definitions we will do with.

Definition 4.1. An (a)space $(\mathcal{X}, \{\tau_n\})$ is said to be:

1. (a)-θ-compact if for every $\mathcal{P} \in \Theta(\mathcal{X})$ there exists finite $\mathcal{Q} \subseteq \mathcal{P}$ such that $\bigcup \mathcal{Q} = \mathcal{X}$.
2. (a)-θ-Menger if for every sequence $< \mathcal{P}_k\colon k \in \mathbb{N} >$ of elements of $\Theta(\mathcal{X})$ there exists finite $\mathcal{Q}_k \subseteq \mathcal{P}_k$ (for each $k \in \mathbb{N}$) such that $\bigcup(\bigcup_{k\in\mathbb{N}} \mathcal{Q}_k) = \mathcal{X}$, or equivalently, $\mathcal{X}$ satisfies the property $S_{fin}(\Theta(\mathcal{X}), \Theta(\mathcal{X}))$.
3. (m,n)-θ-compact if for every $\mathcal{P} \in \Theta(m,n)(\mathcal{X})$ there exists finite $\mathcal{Q} \subseteq \mathcal{P}$ such that $\bigcup \mathcal{Q} = \mathcal{X}$.
4. (m,n)-θ-Menger if for every sequence $< \mathcal{P}_k\colon k \in \mathbb{N} >$ of elements of $\Theta(m,n)(\mathcal{X})$ there exists finite $\mathcal{Q}_k \subseteq \mathcal{P}_k$ (for each $k \in \mathbb{N}$) such that $\bigcup(\bigcup_{k\in\mathbb{N}} \mathcal{Q}_k) = \mathcal{X}$, or equivalently, $\mathcal{X}$ satisfies the property $S_{fin}(\Theta(m,n)(\mathcal{X}), \Theta(m,n)(\mathcal{X}))$.

5. $(a)^s$-θ-compact (resp. $(a)^s$-θ-Menger) if $\mathcal{X}$ is (m,n)-θ-compact (resp. (m,n)-θ-Menger) for all $m \neq n$.

Every (a)-compact (resp. (a)-Menger) space is (a)-θ-compact (resp. (a)-θ-Menger) but the converse need not be true.

Example 4.2. Let $\mathcal{X} = \mathbb{R}$ and $\tau_m = \tau_c$ if m is odd and $\tau_m = \{A \subseteq \mathcal{X} : A = \emptyset \text{ or } 2 \in A\}$ if m is even. Let $F \in \tau_c$ be such that $\mathcal{X} - F$ is not finite. It is observed that $\mathcal{G} = \{F \cup \{2\} \cup \{g\} : g \in \mathcal{X} - F\}$ is an (a)-cover of $\mathcal{X}$ which does not satisfy the condition of (a)-compactness. Also $\emptyset$ and $\mathcal{X}$ are only (a)-θ-open sets, so $\mathcal{X}$ is (a)-θ-compact.

Example 4.3. Let $\mathcal{X} = \mathbb{R}$ and $\tau_m = \{A \subseteq \mathcal{X} : A = \emptyset \text{ or } 2 \in A\}$ if m is odd and $\tau_m = \{A \subseteq \mathcal{X} : A = \emptyset \text{ or } 3 \in A\}$ if m is even. It is observed that $\mathcal{G} = \{\{2,3\} \cup \{g\} : g \in \mathcal{X} - \{2,3\}\}$ is an (a)-cover of $\mathcal{X}$ which is not reducible to a countable subcover. Therefore, $\mathcal{X}$ is not (a)-Lindelöf and hence, not (a)-Menger. But $\mathcal{X}$ is (a)-θ-Menger as $\emptyset$ and $\mathcal{X}$ are only (a)-θ-open sets. Thus, $\mathcal{X}$ is (a)-θ-Menger but not (a)-Menger.

Note that

$$\begin{array}{ccccc} (a)^s\text{-compact} & \Rightarrow & (a)\text{-compact} & \Rightarrow & (a)\text{-}\theta\text{-compact} \\ \Downarrow & & \Downarrow & & \Downarrow \\ (a)^s\text{-Menger} & \Rightarrow & (a)\text{-Menger} & \Rightarrow & (a)\text{-}\theta\text{-Menger} \end{array}$$

Diagram 1.

Example 4.4. Let $\mathcal{X} = \mathbb{R}$ and $\tau_m = \tau_f$ if m is odd and $\tau_m = \{A \subseteq \mathcal{X} : A = \emptyset \text{ or } 2 \in A\}$ if m is even. $\mathcal{X}$ is (a)-compact as $(\mathcal{X}, \tau_f)$ is compact. But $\mathcal{X}$ is not $(a)^s$-compact as $(\mathcal{X}, \tau_2)$ is not compact.

Example 4.5. Consider Example 4.4, since $(\mathcal{X}, \tau_f)$ is Menger, $\mathcal{X}$ is (a)-Menger. But $(\mathcal{X}, \tau_2)$ is not Menger as $(\mathcal{X}, \tau_2)$ is not Lindelöf. Thus, $\mathcal{X}$ is not $(a)^s$-Menger.

Example 4.6. Let $\mathcal{X} = \mathbb{R}$ and $\tau_m = \tau_u$ if m is odd and $\tau_m = \tau_d$ if m is even. It is observed that $\mathcal{O}_\theta = \mathcal{O} = \tau_u$. Also $(\mathcal{X}, \tau_u)$ is Menger, so $\mathcal{X}$ is (a)-Menger as well as (a)-θ-Menger but neither (a)-compact nor (a)-θ-compact.

Example 4.7. Let $\mathcal{X} = \mathbb{R}$ and $\tau_m = \tau_u$ if m is odd and $\tau_m = \tau_f$ if m is even. Since $(\mathcal{X}, \tau_u)$ and $(\mathcal{X}, \tau_f)$ are Menger, $\mathcal{X}$ is $(a)^s$-Menger. But $(\mathcal{X}, \tau_u)$ is not compact, therefore $\mathcal{X}$ is not $(a)^s$-compact.

We conclude that all the implications in the Diagram 1. are not reversible in general.

Definition 4.8. A set $\mathcal{Y} \subseteq \mathcal{X}$ is called:

1. (a)-θ-compact if $\mathcal{Y}$ is (a)-θ-compact under (a)subspace topology.
2. (a)-θ-compact with respect to $\mathcal{X}$ if for every cover $\mathcal{P}$ of $\mathcal{Y}$ by (a)-θ-open subsets of $\mathcal{X}$ there exists finite $\mathcal{Q} \subseteq \mathcal{P}$ such that $\bigcup \mathcal{Q} = \mathcal{Y}$.

By the following example it is shown that the property of (a)-θ-Menger is not heriditary.

Example 4.9. Let $(\mathcal{X}, \tau^*)$ be as in [40, Example 78] and $\tau_m = \tau^*$ if m is odd and $\tau_m = \tau_d$ if m is even. It is known that $(\mathcal{X}, \tau_1)$ is θ-Menger (see [21]). Therefore, $\mathcal{X}$ is (a)-θ-Menger. But the real axis L is an uncountable subspace of $(\mathcal{X}, \{\tau_n\})$ with subspace topology $\sigma_n = \tau_d$ for all $n \in \mathbb{N}$. Thus, $(L, \{\sigma_n\})$ is not (a)-θ-Menger.

In fact, the previous example shows that the property of (a)-θ-Menger is not hereditary under (a)closed subspaces. But this is not the case with (a)-clopen subspaces.

Theorem 4.10. Every (a)-clopen subspace of an (a)-θ-Menger space is (a)-θ-Menger.

Proof. Let $\mathcal{Y}$ be an (a)-clopen subspace of an (a)-θ-Menger space $(\mathcal{X}, \{\tau_n\})$. Let $< \mathcal{A}_k : k \in \mathbb{N} >$ be a sequence of covers of $\mathcal{Y}$ by (a)-θ-open subsets of $\mathcal{Y}$. By Theorem 3.8, each $A \in \mathcal{A}_k$ is (a)-θ-open in $\mathcal{X}$ for all $k \in \mathbb{N}$, so each $\mathcal{P}_k = \{A : A \in \mathcal{A}_k\} \cup \{\mathcal{X} - \mathcal{Y}\}$ is an (a)-θ-cover of $\mathcal{X}$. By given hypothesis, there exists finite $\mathcal{Q}_k \subseteq \mathcal{P}_k$ such that $\mathcal{X} = \bigcup(\bigcup_{k\in\mathbb{N}} \mathcal{Q}_k)$. Let $\mathcal{W}_k = \{A : A \in \mathcal{Q}_k, A \neq \mathcal{X} - \mathcal{Y}\}$. Then $\mathcal{Y} = \bigcup(\bigcup_{k\in\mathbb{N}} \mathcal{W}_k)$ which witnesses for $< \mathcal{A}_k : k \in \mathbb{N} >$ that $\mathcal{Y}$ is (a)-θ-Menger. □

Theorem 4.11. Every (a)-clopen subspace of an (a)-θ-compact space is (a)-θ-compact.

It is to be noted that the class of all (a)-θ-Menger spaces is not closed under finite product. In fact, we show that the square of an (a)-θ-Menger space need not be (a)-θ-Menger.

Example 4.12. Let $\psi : (\mathbb{R}, \tau_l) \to (\mathbb{R}, \tau_u)$ be the identity map. It is known that if L is a Lusin set in $\mathbb{R}$, then $\psi^{-1}(L) = \mathcal{L}$ is Menger (see [23]). It is observed that every open set in $\mathcal{L}$ is θ-open in $\mathcal{L}$. Indeed, for any open set A in $\mathcal{L}$, there is some $G \in \tau_l$ such that $A = G \cap \mathcal{L}$. For each $a \in A$, there is a basic open set $[u, v)$ such that $a \in [u, v) \subseteq \tau_l$-cl$([u, v)) \subseteq G$. Therefore, $a \in [u, v) \cap \mathcal{L} \subseteq \tau_l$-cl$_{\mathcal{L}}([u, v) \cap \mathcal{L})$. But τ_l-cl$_{\mathcal{L}}([u, v) \cap \mathcal{L}) = \tau_l$-cl$([u, v) \cap \mathcal{L}) \cap \mathcal{L} \subseteq \tau_l$-cl$([u, v)) \cap \mathcal{L} \subseteq G \cap \mathcal{L} = A$. Thus, A is θ-open in $\mathcal{L}$. Now, let $\mathcal{X} = \mathcal{L}$ and τ_m be the subspace topology on $\mathcal{X}$ induced by τ_l if m is odd and $\tau_m = \tau_d$ if m is even. It is observed that $\mathcal{O}_\theta = \mathcal{O} = \tau_1$. Kocev showed in [14] that $\mathcal{X}$ with topology τ_1 is Menger but its square is not Menger. Thus, (a)space $\mathcal{X}$ is (a)-θ-Menger but the bi(a)space $\mathcal{X}^2$ is not (a)-θ-Menger.

Theorem 4.13. A mapping $\psi : (\mathcal{X}, \{\tau_k\}) \to (\mathcal{Y}, \{\sigma_k\})$ is (a)-θ-closed if and only if for each $H \subseteq \mathcal{Y}$ and each $G \in \mathcal{O}_\theta(\mathcal{X})$ with $\psi^{-1}(H) \subseteq G$, there exists $U \in \mathcal{O}_\theta(\mathcal{Y})$ with $H \subseteq U$ such that $\psi^{-1}(U) \subseteq G$.

Proof. Let $H \subseteq \mathcal{Y}$ and $G \in \mathcal{O}_\theta(\mathcal{X})$ such that $\psi^{-1}(H) \subseteq G$. Since ψ is (a)-θ-closed, $\psi(\mathcal{X} - G) \in \mathcal{C}_\theta(\mathcal{Y})$. Put $\mathcal{Y} - \psi(\mathcal{X} - G) = U$. Clearly $U \in \mathcal{O}_\theta(\mathcal{Y})$ with $H \subseteq U$. Now $\psi^{-1}(U) = \psi^{-1}(\mathcal{Y} - \psi(\mathcal{X} - G)) \subseteq \mathcal{X} - (\mathcal{X} - G) = G$. Conversely, let $G \in \mathcal{C}_\theta(\mathcal{X})$ and $y \in \mathcal{Y} - \psi(G)$. Then $\psi^{-1}(y) \subseteq \mathcal{X} - \psi^{-1}(\psi(G)) \subseteq \mathcal{X} - G$, which is (a)-θ-open in $\mathcal{X}$. So by given hypothesis, there exists $V_y \in \mathcal{O}_\theta(\mathcal{Y})$ with $y \in V_y$ such that $\psi^{-1}(V_y) \subseteq \mathcal{X} - G$. Then $G \subseteq \mathcal{X} - \psi^{-1}(V_y) = \psi^{-1}(\mathcal{Y}) - \psi^{-1}(V_y) = \psi^{-1}(\mathcal{Y} - V_y)$ and therefore $\psi(G) \subseteq \mathcal{Y} - V_y$. Thus, $V_y \subseteq \mathcal{Y} - \psi(G)$ and hence, $\mathcal{Y} - \psi(G) = \cup_{y\in\mathcal{Y}-\psi(G)} V_y$. By Remark 2.5, $\mathcal{Y} - \psi(G)$ is (a)-θ-open. □

Theorem 4.14. Let $\psi : (\mathcal{X}, \{\tau_k\}) \to (\mathcal{Y}, \{\sigma_k\})$ be an (a)-θ-closed and onto map. If $\mathcal{Y}$ is (a)-θ-compact and $\psi^{-1}(t)$ is (a)-θ-compact with respect to $\mathcal{X}$ for each $t \in \mathcal{Y}$, then $\mathcal{X}$ is (a)-θ-compact.

Proof. Let $\mathcal{P} = \{P_\alpha : \alpha \in I\} \in \Theta(\mathcal{X})$. For each $t \in \mathcal{Y}$, $\psi^{-1}(t) \subseteq \mathcal{X} = \cup_{\alpha\in I} P_\alpha$. Since for each $t \in \mathcal{Y}$, $\psi^{-1}(t)$ is (a)-θ-compact with respect to $\mathcal{X}$, there is a finite set $I_t \subseteq I$ (depending upon t) such that $\psi^{-1}(t) \subseteq \cup_{\alpha\in I_t} P_\alpha$. Since each $P_\alpha \in \mathcal{O}_\theta(\mathcal{X})$ and $\mathcal{O}_\theta(\mathcal{X})$ is a topology, so by Theorem 4.13, there exists $V_t \in \mathcal{O}_\theta(\mathcal{Y})$ with $t \in V_t$ satisfying $\psi^{-1}(V_t) \subseteq \cup_{\alpha\in I_t} P_\alpha$. Also $\mathcal{Y} = \cup_{t\in\mathcal{Y}} V_t$ and $\mathcal{Y}$ is (a)-θ-compact, so $\mathcal{Y} = \cup_{i=1}^m V_{t_i}$, $m \in \mathbb{N}$. Therefore $\mathcal{X} = \psi^{-1}(\mathcal{Y}) = \cup_{i=1}^m \psi^{-1}(V_{t_i}) \subseteq \cup_{i=1}^m (\cup_{\alpha\in I_{t_i}} P_\alpha)$. Hence, $\mathcal{X}$ is (a)-θ-compact. □

Theorem 4.15. Let $(\mathcal{X}, \{\tau_k\})$ and $(\mathcal{Y}, \{\sigma_k\})$ be two (a)spaces. If $\mathcal{X}$ is $(a)^s$-compact, then $p_{\mathcal{Y}} : (\mathcal{X} \times \mathcal{Y}, \{\gamma_k\}) \to (\mathcal{Y}, \{\sigma_k\})$ is an (a)-θ-closed surjection.

Proof. Let $G \in \mathcal{C}_\theta(\mathcal{X} \times \mathcal{Y})$ and $y \in \mathcal{Y} - p_{\mathcal{Y}}(G)$. Then for all $x \in \mathcal{X}$, $(x, y) \notin G$ and therefore, $\mathcal{X} \times \{y\} \subseteq (\mathcal{X} \times \mathcal{Y}) - G$. But $(\mathcal{X} \times \mathcal{Y}) - G \in \mathcal{O}_\theta(\mathcal{X} \times \mathcal{Y})$, so for each $n \in \mathbb{N}$, there exists a set $O(x) \in \gamma_n$ such that $(x, y) \in O(x) \subseteq \gamma_m$-cl$(O(x)) \subseteq (\mathcal{X} \times \mathcal{Y}) - G$. Let $O(x) = O_1(x) \times O_2(x)$, where $O_1(x) \in \tau_n$ and $O_2(x) \in \sigma_n$. It follows that $\{O_1(x) \times O_2(x) : x \in \mathcal{X}\}$ is a γ_n-cover of $\mathcal{X} \times \{y\}$ such that $O_1(x) \times O_2(x) \subseteq \tau_m$-cl$(O_1(x)) \times \sigma_m$-cl$(O_2(x)) = \gamma_m$-cl$(O(x)) \subseteq (\mathcal{X} \times \mathcal{Y}) - G$. Also $\mathcal{X} \times \{y\}$ is $(a)^s$-compact as $\mathcal{X}$ is $(a)^s$-compact. Therefore, $\mathcal{X} \times \{y\} \subseteq \cup_{i=1}^l \{O_1(x_i) \times O_2(x_i) : x_i \in \mathcal{X}\}$ for some $l \in \mathbb{N}$. Put $\cap_{i=1}^l O_2(x_i) = W$. Then $W \in \sigma_n$ such that $y \in W \subseteq \sigma_m$-cl$(W) \subseteq \mathcal{Y} - p_{\mathcal{Y}}(G)$ for all $m \neq n$. Thus, $\mathcal{Y} - p_{\mathcal{Y}}(G)$ is (a)-θ-open. □

Theorem 4.16. Let $(\mathcal{X}, \{\tau_k\})$ and $(\mathcal{Y}, \{\sigma_k\})$ be two (a)spaces. If $\mathcal{X}$ is $(a)^s$-compact and $\mathcal{Y}$ is (a)-θ-compact, then $\mathcal{X} \times \mathcal{Y}$ is (a)-θ-compact.

Proof. Let $(\mathcal{X}, \{\tau_k\})$ be an $(a)^s$-compact space and $(\mathcal{Y}, \{\sigma_k\})$ be an (a)-θ-compact space. By Theorem 4.15, $p_{\mathcal{Y}}\colon \mathcal{X}\times\mathcal{Y} \to \mathcal{Y}$ is an (a)-θ-closed surjection. For each $t \in \mathcal{Y}$, $p_{\mathcal{Y}}^{-1}(t) = \mathcal{X}\times\{t\}$. We claim that each $\mathcal{X}\times\{t\}$ is (a)-θ-compact with respect to $\mathcal{X}\times\mathcal{Y}$. Let $\mathcal{P} = \{P_\alpha\colon \alpha \in I\}$ be any cover of $\mathcal{X}\times\{t\}$ by (a)-θ-open subsets of $\mathcal{X}\times\mathcal{Y}$. For each $\alpha \in I$, let $P_\alpha = Q_\alpha \times R_\alpha$, where $Q_\alpha \in \mathcal{O}_\theta(\mathcal{X})$ and $R_\alpha \in \mathcal{O}_\theta(\mathcal{Y})$. But $\mathcal{X}$ is $(a)^s$-compact and hence, (a)-θ-compact, So there exists a finite set $J \subseteq I$ such that $\mathcal{X} = \bigcup_{\alpha\in J} Q_\alpha$. Let $t \in R_\beta$ for some $\beta \in I$. Then $\mathcal{X}\times\{t\} \subseteq \bigcup_{\alpha\in J}(Q_\alpha \times R_\beta)$. Thus, $p_{\mathcal{Y}}^{-1}(t)$ is (a)-θ-compact with respect to $\mathcal{X}\times\mathcal{Y}$ for each $t \in Y$. By Theorem 4.14, the proof follows. □

Theorem 4.17. Let $\psi\colon (\mathcal{X}, \{\tau_k\}) \to (\mathcal{Y}, \{\sigma_k\})$ be an (a)-θ-closed and onto map. If $\mathcal{Y}$ is (a)-θ-Menger and $\psi^{-1}(t)$ is (a)-θ-compact with respect to $\mathcal{X}$ for each $t \in \mathcal{Y}$, then $\mathcal{X}$ is (a)-θ-Menger.

Proof. Let $< \mathcal{G}_k\colon k \in \mathbb{N} >$ be a sequence of elements of $\Theta(\mathcal{X})$. For each $t \in \mathcal{Y}$ and for each $k \in \mathbb{N}$, $\psi^{-1}(t) \subseteq \mathcal{X} = \cup\{G\colon G \in \mathcal{G}_k\}$. Since $\psi^{-1}(t)$ is (a)-θ-compact with respect to $\mathcal{X}$, there is a finite subcovering, say $\mathcal{U}_k^t$ (depending on $t \in \mathcal{Y}$), for each covering $\mathcal{G}_k$ of $\psi^{-1}(t)$ such that $\psi^{-1}(t) \subseteq \cup\{G\colon G \in \mathcal{U}_k^t\}$ for each $k \in \mathbb{N}$. Since $\cup\{G\colon G \in \mathcal{U}_k^t\} \in \mathcal{O}_\theta(\mathcal{X})$, by Theorem 4.13, there exists $V_t^k \in \mathcal{O}_\theta(\mathcal{Y})$ with $t \in V_t^k$ satisfying $\psi^{-1}(V_t^k) \subseteq \cup\{G\colon G \in \mathcal{U}_k^t\}$ for each $k \in \mathbb{N}$. Clearly $\mathcal{Y} = \cup\, Y_k$ for each $k \in \mathbb{N}$, where $Y_k = \{V_t^k\colon t \in Y\}$. Since $\mathcal{Y}$ is (a)-θ-Menger, there exists finite $\mathcal{V}_k \subseteq Y_k$ such that $\mathcal{Y} = \cup(\cup_{k\in\mathbb{N}}\mathcal{V}_k)$. For each $k \in \mathbb{N}$, there exists finite set $A_k \subseteq \mathcal{Y}$ such that $\mathcal{V}_k = \{V_t^k\colon t \in A_k\}$ with the condition that $\psi^{-1}(V_t^k) \subseteq \cup\{G\colon G \in \mathcal{U}_k^t\}$ for each $t \in A_k$. Put $\cup_{t\in A_k} \mathcal{U}_k^t = \mathcal{U}_k$. It follows that $\mathcal{X} = \cup(\cup_{k\in\mathbb{N}} \mathcal{U}_k)$. Indeed, $\psi(x) \in \mathcal{Y} = \cup(\cup_{k\in\mathbb{N}} \mathcal{V}_k)$ for $x \in \mathcal{X}$ which implies that $\psi(x) \in V_t^k$ for some $k \in \mathbb{N}$ and for some $t \in A_k$. Therefore, $x \in \psi^{-1}(V_t^k)$ for some $k \in \mathbb{N}$ and for some $t \in A_k$. Thus, $x \in \cup\{G\colon G \in \mathcal{U}_k^t\}$ for some $k \in \mathbb{N}$ and for some $t \in A_k$. Hence, $x \in G$ for some $G \in \mathcal{U}_k$ and for some $k \in \mathbb{N}$. □

Theorem 4.18. Let $(\mathcal{X}, \{\tau_k\})$ and $(\mathcal{Y}, \{\sigma_k\})$ be two (a)spaces. If $\mathcal{X}$ is $(a)^s$-compact and $\mathcal{Y}$ is (a)-θ-Menger, then $\mathcal{X}\times\mathcal{Y}$ is (a)-θ-Menger.

Proof. Let $(\mathcal{X}, \{\tau_k\})$ be an $(a)^s$-compact space and $(\mathcal{Y}, \{\sigma_k\})$ be an (a)-θ-Menger space. By Theorem 4.15, $p_{\mathcal{Y}}\colon \mathcal{X}\times\mathcal{Y} \to \mathcal{Y}$ is an (a)-θ-closed surjection. For each $t \in \mathcal{Y}$, $p_{\mathcal{Y}}^{-1}(t) = \mathcal{X}\times\{t\}$ and $\mathcal{X}$ is $(a)^s$-compact, so $p_{\mathcal{Y}}^{-1}(t)$ is (a)-θ-compact with respect to $\mathcal{X}\times\mathcal{Y}$ for each $t \in \mathcal{Y}$ (see Theorem 4.16). From Theorem 4.17, $\mathcal{X}\times\mathcal{Y}$ is (a)-θ-Menger. □

Theorem 4.19. If $\mathcal{X}^k$ is (a)-θ-Menger for each finite natural k, then $\mathcal{X}$ satisfies $S_{fin}(\Theta\text{-}\Omega(\mathcal{X}), \Theta\text{-}\Omega(\mathcal{X}))$.

Proof. Let $< \mathcal{G}_k\colon k \in \mathbb{N} >$ be a sequence of elements of $\Theta\text{-}\Omega(\mathcal{X})$. Assume that each $\mathcal{G}_k$ is closed under finite union. Let $\mathbb{N} = N_1 \cup N_2 \cup \cdots \cup N_k \cup \ldots$ where each N_i is infinite and $N_i \cap N_j \neq \emptyset$ for all $i \neq j$. For each $m \in \mathbb{N}$ and for each $k \in N_m$, let $\mathcal{U}_k = \{G^m\colon G \in \mathcal{G}_k\}$. Since finite product of (a)-θ-open sets is again (a)-θ-open (see Theorem 2.13), $\mathcal{U}_k \in \Theta(\mathcal{X}^m)$ for all $k \in N_m$. In fact, $\mathcal{U}_k \in \Theta\text{-}\Omega(\mathcal{X}^m)$ for all $k \in N_m$. Let $A = \{a_1, a_2, \ldots, a_r\} \subseteq \mathcal{X}^m$ be finite. For each $i \in \{1, 2, \ldots, r\}$, $a_i = (a_{i1}, a_{i2}, \ldots, a_{im})$ where $a_{ij} \in \mathcal{X}$ for all $j = 1, 2, \ldots, m$. Since $B = \{a_{ij}\colon 1 \leq i \leq r \text{ and } 1 \leq j \leq m\} \subseteq \mathcal{X}$ is finite, so $B \subseteq G$ for some $G \in \mathcal{G}_k$. Thus, $A \subseteq G^m$. Also it is clear that $\mathcal{X}^m \notin \mathcal{U}_k$ as $\mathcal{X} \notin \mathcal{G}_k$. Thus, $< \mathcal{U}_k\colon k \in N_m >$ is a sequence of elements of $\Theta\text{-}\Omega(\mathcal{X}^m)$. As $\mathcal{X}^m$ is (a)-θ-Menger, so there exists finite $\mathcal{V}_k \subseteq \mathcal{U}_k$ (for each $k \in N_m$) such that $\mathcal{X}^m = \bigcup(\bigcup_{k\in N_m} \mathcal{V}_k)$. For each $k \in N_m$, let $\mathcal{H}_k = \{G_V\colon G_V^m = V, V \in \mathcal{V}_k\}$. It follows that $\bigcup(\bigcup_{k\in N_m} \mathcal{H}_k) = \mathcal{X}$. Thus, $\bigcup_{k\in\mathbb{N}} \mathcal{H}_k \in \Theta(\mathcal{X})$. Moreover, $\bigcup_{k\in\mathbb{N}} \mathcal{H}_k \in \Theta\text{-}\Omega(\mathcal{X})$. Indeed, for any finite set $D = \{d_1, d_2, \ldots, d_l\} \subseteq \mathcal{X}$, $(d_1, d_2, \ldots, d_l) \in \mathcal{X}^l$. So there exists a $t \in N_l$ such that $(d_1, d_2, \ldots, d_l) \in H^l$ for some $H \in \mathcal{H}_t$. Thus, $D \subseteq H$ for some $H \in \bigcup_{k\in\mathbb{N}} \mathcal{H}_k$. It is clear that $\mathcal{X} \notin \bigcup_{k\in\mathbb{N}} \mathcal{H}_k$ as $\mathcal{X}^m \notin \mathcal{U}_k$ for any $k \in \mathbb{N}$. Hence, $\mathcal{X}$ satisfies $S_{fin}(\Theta\text{-}\Omega(\mathcal{X}), \Theta\text{-}\Omega(\mathcal{X}))$. □

In the following couple of theorems, we shall discuss only the (a)-θ-Menger property; (a)-θ-compactness can be studied in a similar way.

Theorem 4.20. Let $\psi\colon (\mathcal{X}, \{\tau_k\}) \to (\mathcal{Y}, \{\sigma_k\})$ be an (a)-θ-continuous and onto map. If $\mathcal{X}$ is (a)-θ-Menger, then $\mathcal{Y}$ is (a)-θ-Menger.

Proof. Let $\mathcal{X}$ be an (a)-θ-Menger space and $< \mathcal{G}_k\colon k \in \mathbb{N} >$ be a sequence of elements of $\Theta(\mathcal{Y})$. Since ψ is (a)-θ-continuous, by Proposition 3.2, each $\mathcal{H}_k = \{\psi^{-1}(G)\colon G \in \mathcal{G}_k\} \in \Theta(\mathcal{X})$.

Since $\mathcal{X}$ is (a)-θ-Menger, there exists finite $\mathcal{V}_k \subseteq \mathcal{H}_k$ such that $\mathcal{X} = \bigcup(\bigcup_{k\in\mathbb{N}} \mathcal{V}_k)$. For each $k \in \mathbb{N}$, let $\mathcal{W}_k = \{G_V : V = \psi^{-1}(G_V),\ V \in \mathcal{V}_k\}$. It is easy to see that each $\mathcal{W}_k \subseteq \mathcal{G}_k$ is finite. Also, $\mathcal{X} = \bigcup(\bigcup_{k\in\mathbb{N}}\{V : V \in \mathcal{V}_k\})$. Therefore, $\mathcal{Y} = \psi[\bigcup(\bigcup_{k\in\mathbb{N}}\{V : V \in \mathcal{V}_k\})] = \bigcup(\bigcup_{k\in\mathbb{N}}\{\psi(V) : V \in \mathcal{V}_k\}) = \bigcup(\bigcup_{k\in\mathbb{N}}\{G_V : G_V \in \mathcal{W}_k\})$. Hence, $\mathcal{Y}$ is (a)-θ-Menger. □

Theorem 4.21. Let $\psi : (\mathcal{X}, \{\tau_k\}) \to (\mathcal{Y}, \{\sigma_k\})$ be an $(a)^s$-continuous and onto map. If $\mathcal{X}$ is (a)-θ-Menger, then $\mathcal{Y}$ is (a)-θ-Menger.

Proof. The proof follows by Proposition 3.11 and Theorem 4.20. □

Theorem 4.22. Let $\psi : (\mathcal{X}, \{\tau_k\}) \to (\mathcal{Y}, \{\sigma_k\})$ be an (a)-stongly-θ-continuous and onto map. If $\mathcal{X}$ is (a)-θ-Menger, then $\mathcal{Y}$ is (a)-Menger.

Proof. Let $\mathcal{X}$ be an (a)-θ-Menger space and $< \mathcal{G}_k : k \in \mathbb{N} >$ be a sequence of (a)-covers of $\mathcal{Y}$. Since ψ is (a)-strongly-θ-continuous, so by Proposition 3.4, $\psi^{-1}(G) \in \mathcal{O}_\theta(X)$ for each $G \in \mathcal{G}_k$. Therefore, $\mathcal{U}_k = \{\psi^{-1}(G) : G \in \mathcal{G}_k\} \in \Theta(X)$ for all $k \in \mathbb{N}$. Since $\mathcal{X}$ is (a)-θ-Menger, there exists finite $\mathcal{V}_k \subseteq \mathcal{U}_k$ such that $\bigcup(\bigcup_{k\in\mathbb{N}} \mathcal{V}_k) = \mathcal{X}$. For each $V \in \mathcal{V}_k$ there exists $G_V \in \mathcal{G}_k$ such that $V = \psi^{-1}(G_V)$. Let $\mathcal{W}_k = \{G_V : V = \psi^{-1}(G_V),\ V \in \mathcal{V}_k\}$. It is clear that each $\mathcal{W}_k \subseteq \mathcal{G}_k$ is finite and $\mathcal{Y} = \psi[\bigcup(\bigcup_{k\in\mathbb{N}}\{V : V \in \mathcal{V}_k\})] = \bigcup(\bigcup_{k\in\mathbb{N}}\{\psi(V) : V \in \mathcal{V}_k\}) = \bigcup(\bigcup_{k\in\mathbb{N}}\{G_V : G_V \in \mathcal{W}_k\})$. Hence, $\mathcal{Y}$ is (a)-Menger. □

Theorem 4.23. Let $\psi : (\mathcal{X}, \{\tau_k\}) \to (\mathcal{Y}, \{\sigma_k\})$ be an $(a)^s$-faintly-continuous and onto map. If $\mathcal{X}$ is (a)-Menger, then $\mathcal{Y}$ is (a)-θ-Menger.

Proof. It can be easily proved with the help of Theorem 3.15. □

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