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Commutativity of prime rings with generalized derivations involving Jordan involution

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Abstract Let R be a prime ring with characteristic different from two and F be a generalized derivation associated with a derivation d of R. In the present paper we investigate the commutativity of R with Jordan involution. Further, examples are given to demonstrate that the restrictions imposed on the hypothesis of our results are not superfluous.

1 Introduction

Recently, a new line of research has been initiated by several authors. A great interest has been given to the study of the structures of certain rings when their elements were subjected to certain types of algebraic conditions. In 1983, J. Bergen, I. N. Herstein and C. Lanski in [5] have studied the structure of a unit ring R having a non-trivial derivation d such that $d(x) = 0$ or $d(x)$ is invertible in R for every $x \in R$. They showed that R is a division ring D or R is of the form $M_2(D)$ or $D[x]/x^2$ were D is a division ring. Several authors have investigated the relationships between the commutativity of prime and semi-prime rings admitting suitably constrained additive mappings, as automorphisms, derivations, skew derivations and generalized derivations acting on appropriate subsets of the rings (see [1],[2],[7],[10] and [11] for further details). In this paper we continue the line of investigation regarding the study of commutativity for rings with Jordan involution satisfying certain differential identities involving generalized derivations.

For completeness, we recall some preliminaries which are useful for the development of this paper. Throughout this paper R will represent an associative ring with center $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ will denote the commutator $xy - yx$; while the symbol $x \circ y$ will stand for the anti-commutator $xy + yx$. R is prime if $aRb = 0$ implies $a = 0$ or $b = 0$. An additive mapping $d : R \longrightarrow R$ is a derivation on R if $d(xy) = d(x)y + xd(y)$ for all $x, y \in R$. Let $a \in R$ be a fixed element. A map $d : R \longrightarrow R$ defined by $d(x) = [a, x] = ax - xa$, $x \in R$, is a derivation on R, which is called an inner derivation defined by a. In [6], Brešar introduced the notion of generalized derivations in rings: an additive mapping $F : R \longrightarrow R$ is called generalized derivation if there exists a derivation d such that $F(xy) = F(x)y + xd(y)$ holds for all $x, y \in R$, and d is called the associated derivation of F. Obviously, the concept of generalized derivations covers both the concepts of a derivation and of a left multiplier (i.e., an additive mapping satisfying $f(xy) = f(x)y$ for all $x, y \in R$). Basic examples of generalized derivations are the following: (i) $F(x) = ax + xb$ for $a, b \in R$; (ii) $F(x) = ax$ for some $a \in R$.

An additive map $* : R \longrightarrow R$ is called an involution if $*$ is an anti-automorphism of order 2; that is $(x^*)^* = x$ for all $x \in R$. An element x in a ring with involution $(R, *)$ is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of R will be denote by $H(R)$ and $S(R)$, respectively. The involution is said to be of the first kind if $Z(R) \subseteq H(R)$, otherwise it is said to be of the second kind. In the later case $S(R) \cap Z(R) \neq (0)$. An element x is *normal* if $xx^* = x^*x$. If all elements in R are normal, then R is called a *normal ring* (or equivalently, $*$ is commuting). In the spirit of the definition of involution, Yood [12] introduced Jordan involution as : An additive map $\natural : R \to R$ is called Jordan involution if for any $x; y \in R$, $(x^\natural)^\natural = x$ and $(xy + yx)^\natural = x^\natural y^\natural + y^\natural x^\natural$. Obviously, every involution is a Jordan involution but the converse need not be true. For example:

Example.

Let us consider $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{C} \right\}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\sharp} = \begin{pmatrix} \bar{a} & -\bar{b} \\ -\bar{c} & \bar{d} \end{pmatrix}$. It is straightforward to check that $\sharp$ is a Jordan involution but not involution.

2 Main results

We first fix the following facts which shall be used frequently throughout the paper.

Fact 1 : Let $(R, \natural)$ be a ring with Jordan involution. If R is prime, and $d(h) = 0$ for all $h \in H(R) \cap Z(R)$ then $d(s) = 0$ for all $s \in S(R) \cap Z(R)$.

Fact 2 : Let $(R, \natural)$ be a 2-torsion free prime ring with Jordan involution of the second kind. Then the following assertions are equivalent:
(1) $[x, x^{\natural}] \in Z(R)$ for all $x \in R$;
(2) $x \circ x^{\natural} \in Z(R)$ for all $x \in R$;
(3) R is a commutative integral domain.

Theorem 2.1. *Let $(R, \natural)$ be a 2-torsion free prime ring with Jordan involution of the second kind. If R admits a generalized derivation F associated with a nonzero derivation d, then the following assertions are equivalent:*
(1) $[F(x), d(x^{\natural})] = [x, x^{\natural}]$ *for all* $x \in R$.
(2) $[F(x), d(x^{\natural})] = -[x, x^{\natural}]$ *for all* $x \in R$.
(3) $[F(x), d(x^{\natural})] = 0$ *for all* $x \in R$.
(4) *R is a commutative integral domain.*

Proof. We need only prove $(1) \Longrightarrow (4)$, $(2) \Longrightarrow (4)$ and $(3) \Longrightarrow (4)$.

$(1) \Longrightarrow (4)$ Assume that

$$[F(x), d(x^{\natural})] = [x, x^{\natural}] \quad \text{for all} \quad x \in R. \tag{2.1}$$

Linearization of (2.1) yields

$$[F(x), d(y^{\natural})] + [F(y), d(x^{\natural})] = [x, y^{\natural}] + [y, x^{\natural}] \quad \text{for all} \quad x, y \in R.$$

Substituting y for $y^{\natural}$, we obtain

$$[F(x), d(y)] + [F(y^{\natural}), d(x^{\natural})] = [x, y] + [y^{\natural}, x^{\natural}] \quad \text{for all} \quad x, y \in R. \tag{2.2}$$

Taking $y = yh$ for y in (2.2), where $h \in Z(R) \cap H(R) \backslash \{0\}$ and using (2.2), we get

$$\big([F(x), y] + [y^{\natural}, d(x^{\natural})]\big)d(h) = 0 \quad \text{for all} \quad x, y \in R. \tag{2.3}$$

In view of the primeness, the last equation assures that $[F(x), y] + [y^{\natural}, d(x^{\natural})] = 0$ or $d(h) = 0$. In the first case suppose that,

$$[F(x), y] + [y^{\natural}, d(x^{\natural})] = 0 \quad \text{for all} \quad x, y \in R. \tag{2.4}$$

Taking ys instead of y, where $0 \neq s \in Z(R) \cap S(R)$ we find that

$$[F(x), y] - [y^{\natural}, d(x^{\natural})] = 0 \quad \text{for all} \quad x, y \in R. \tag{2.5}$$

By adding the relations (2.4) and (2.5), we have

$$[F(x), y] = 0 \quad \text{for all} \quad x, y \in R$$

and therefore

$$[F(x), x] = 0 \quad \text{for all} \;\; x \in R$$

hence [9, Theorem 3] assures that R is commutative

If $d(h) = 0$, then by **Fact 1** we find that $d(Z(R) \cap S(R)) = \{0\}$.
Replacing y by ys in (2.2), with $s \in Z(R) \cap S(R) \backslash \{0\}$, we arrive at

$$[F(x), d(y)] - [F(y^{\natural}), d(x^{\natural})] = [x, y] - [y^{\natural}, x^{\natural}] \quad \text{for all} \;\; x, y \in R. \tag{2.6}$$

Using (2.2) together with (2.6) we conclude that

$$[F(x), d(y)] = [x, y] \quad \text{for all} \;\; x, y \in R \tag{2.7}$$

and [3, Theorem 2.11] gives that R is commutative ring.

(2) $\Longrightarrow$ (4) Arguing as above, with slight modifications, it is obvious to show that $[F(x), d(x^{\natural})] = -[x, x^{\natural}]$ for all $x \in R$ implies that R is commutative.

(3) $\Longrightarrow$ (4) We are assuming that

$$[F(x), d(x^{\natural})] = 0 \quad \text{for all} \;\; x \in R. \tag{2.8}$$

A linearization of (2.8) implies that

$$[F(x), d(y)] + [F(y^{\natural}), d(x^{\natural})] = 0 \quad \text{for all} \;\; x, y \in R. \tag{2.9}$$

Replacing y with yh in the above equation with $0 \neq h \in Z(R) \cap H(R)$ and using (2.9) we may write

$$\big([F(x), y] + [y^{\natural}, d(x^{\natural})]\big)d(h) = 0 \quad \text{for all} \;\; x, y \in R. \tag{2.10}$$

Using the primeness of R, we get either $[F(x), y] + [y^{\natural}, d(x^{\natural})] = 0$ or $d(h) = 0$.
If

$$[F(x), y] + [y^{\natural}, d(x^{\natural})] = 0 \quad \text{for all} \;\; x, y \in R. \tag{2.11}$$

Taking ys instead of y in (2.11), where $0 \neq s \in Z(R) \cap S(R)$, one can see that

$$[F(x), y] - [y^{\natural}, d(x^{\natural})] = 0 \quad \text{for all} \;\; x, y \in R. \tag{2.12}$$

Adding (2.11) and (2.12) we conclude that

$$[F(x), y] = 0 \quad \text{for all} \;\; x, y \in R$$

thereby obtaining

$$[F(x), x] = 0 \quad \text{for all} \;\; x \in R$$

and R is commutative by [9, Theorem 3].
Now suppose that $d(h) = 0$ for all $h \in Z(R) \cap H(R)$. From **Fact 1**, we get $d(s) = 0$ for all $s \in Z(R) \cap S(R)$. Putting $y = ys$ in (2.9), we arrive at

$$[F(x), d(y)] - [F(y^{\natural}), d(x^{\natural})] = 0 \quad \text{for all} \;\; x, y \in R. \tag{2.13}$$

From the last relation and (2.9) it follows that

$$[F(x), d(y)] = 0 \quad \text{for all} \;\; x, y \in R \tag{2.14}$$

hence R is commutative by [3, Theorem 2.6].

□

Remark 2.2. To prove (1) $\Longleftrightarrow$ (4) and (2) $\Longleftrightarrow$ (4) the condition $d \neq 0$ is not necessary. Indeed, if $d = 0$ then R is normal and the required result follows from **Fact 2**.

Corollary 2.3. *Let $(R,\natural)$ be a 2-torsion free prime ring with Jordan involution of the second kind. If R admits a generalized derivation F associated with a nonzero derivation d, then the following assertions are equivalent:*
(1) $[F(x),d(y)]=[x,y]$ *for all* $x,y\in R$.
(2) $[F(x),d(y)]=-[x,y]$ *for all* $x,y\in R$.
(3) $[F(x),d(y)]=0$ *for all* $x,y\in R$.
(4) R *is a commutative integral domain.*

A natural question is what can we say about the commutativity of R if the commutator in the preceding theorem is replaced by anti-commutator. We have studied this problem and proved that the commutativity cannot be characterized by the same conditions on anti-commutator.

Theorem 2.4. *Let $(R,\natural)$ be a 2-torsion free prime ring with Jordan involution of the second kind. There is no generalized derivation F associated with a derivation d satisfying one of the following assertions :*
(1) $F(x)\circ d(x^\natural)=x\circ x^\natural$ *for all* $x\in R$.
(2) $F(x)\circ d(x^\natural)=-x\circ x^\natural$ *for all* $x\in R$.

Proof. For all $x\in R$, assume there exists a generalized derivation (F,d) such that

$$F(x)\circ d(x^\natural)=x\circ x^\natural. \tag{2.15}$$

If $d=0$, the last expression leads to

$$x\circ x^\natural=0 \text{ for all } x\in R$$

which reduces to

$$x\circ y^\natural+y\circ x^\natural=0 \text{ for all } x,y\in R.$$

Substituting h for y, where $h\in Z(R)\cap H(R)\backslash\{0\}$, we find that $x+x^\natural=0$.
Replacing y by s, where $s\in Z(R)\cap S(R)\backslash\{0\}$, we get $x-x^\natural=0$.
The last two expressions forces us to conclude that $R=0$, a contradiction.
Therefore, we can assume that $d\neq 0$.
By linearization of (2.15) gives

$$F(x)\circ d(y)+F(y^\natural)\circ d(x^\natural)=x\circ y+y^\natural\circ x^\natural \quad \text{for all } x,y\in R. \tag{2.16}$$

Substituting yh for y in the last expression, where $h\in Z(R)\cap H(R)\backslash\{0\}$, it follows that

$$\big(F(x)\circ y+y^\natural\circ d(x^\natural)\big)d(h)=0 \quad \text{for all } x,y\in R. \tag{2.17}$$

By primeness, we conclude that either $d(h)=0$ or $F(x)\circ y+y^\natural\circ d(x^\natural)=0$.
Assume that

$$F(x)\circ y+y^\natural\circ d(x^\natural)=0 \quad \text{for all } x,y\in R. \tag{2.18}$$

Setting $y=h$, with $h\in Z(R)\cap H(R)\backslash\{0\}$, we obviously get

$$F(x)+d(x^\natural)=0 \quad \text{for all } x\in R.$$

Writing s instead of y in (2.18), where $s\in Z(R)\cap S(R)\backslash\{0\}$, we find that

$$F(x)-d(x^\natural)=0 \quad \text{for all } x\in R$$

and thus $d(x)=0$ for all $x\in R$, a contradiction.
Now assume that $d(h)=0$ for all $h\in Z(R)\cap H(R)$. Hence **Fact 1** assures that $d(s)=0$ for all $s\in Z(R)\cap S(R)$.
Replacing y by ys in (2.16), where $0\neq s\in Z(R)\cap S(R)$ we obtain

$$F(x)\circ d(y)-F(y^\natural)\circ d(x^\natural)=x\circ y-y^\natural\circ x^\natural \quad \text{for all } x,y\in R. \tag{2.19}$$

Adding (2.19) and (2.16), we find that

$$F(x)\circ d(y)=x\circ y \quad \text{for all } x,y\in R \tag{2.20}$$

and R is commutative by [3, Theorem 2.7]. Then the last expression leads to

$$F(x)d(y) = xy \quad \text{for all} \;\; x, y \in R.$$

Putting $y = yt$ in the previous relation, we arrive at $F(x)yd(t) = 0$ which leads to

$$F(x)Rd(t) = \{0\} \quad \text{for all} \;\; x, t \in R$$

then $d = 0$, a contradiction.

Arguing as above, with similar arguments, it is obvious to show that there is no generalized derivation F associated with a derivation d satisfying $F(x) \circ d(x^\natural) = -(x \circ x^\natural)$ for all $x \in R$. □

Corollary 2.5. *Let $(R, \natural)$ be a 2-torsion free prime ring with Jordan involution of the second kind. There is no generalized derivation F associated with a derivation d satisfying one of the following assertions :*
(1) $F(x) \circ d(y) = x \circ y$ *for all* $x, y \in R$.
(2) $F(x) \circ d(y) = -x \circ y$ *for all* $x, y \in R$.

Now if we replace the commutator by anti-commutator in (3) of Theorem 2.1 then it does not constitute a commutativity criterion. However, we can classify generalized derivations satisfying the hypothesis $F(x) \circ d(x^\natural) = 0$ for all $x \in R$.

Theorem 2.6. *Let $(R, \natural)$ be a 2-torsion free prime ring with Jordan involution of the second kind and F a generalized derivation associated with a derivation d. Then the following assertions are equivalent:*
(1) $F(x) \circ d(x^\natural) = 0$ *for all* $x \in R$.
(2) *F is a left multiplier.*

Proof. For the nontrivial implication, assume that

$$F(x) \circ d(x^\natural) = 0 \quad \text{for all} \;\; x \in R. \tag{2.21}$$

A linearization of the last expression yields that

$$F(x) \circ d(y^\natural) + F(y) \circ d(x^\natural) = 0 \quad \text{for all} \;\; x, y \in R,$$

that is

$$F(x) \circ d(y) + F(y^\natural) \circ d(x^\natural) = 0 \quad \text{for all} \;\; x, y \in R. \tag{2.22}$$

Substituting yh for y, where $h \in Z(R) \cap H(R) \backslash \{0\}$, we get

$$\big(F(x) \circ y + y^\natural \circ d(x^\natural)\big)d(h) = 0 \quad \text{for all} \;\; x, y \in R. \tag{2.23}$$

By primeness we conclude that either $F(x) \circ y + y^\natural \circ d(x^\natural) = 0$ or $d(h) = 0$.
If

$$F(x) \circ y + y^\natural \circ d(x^\natural) = 0 \quad \text{for all} \;\; x, y \in R. \tag{2.24}$$

Putting $y = h$ where $h \in Z(R) \cap H(R) \backslash \{0\}$, then we have $2\big(F(x) + d(x^\natural)\big)h = 0$ therefore

$$F(x) + d(x^\natural) = 0 \quad \text{for all} \;\; x \in R.$$

Now substituting s for y in (2.24), where $s \in Z(R) \cap S(R) \backslash \{0\}$, we find that

$$F(x) - d(x^\natural) = 0 \;\; \text{for all} \;\; x \in R.$$

Combining the two previous relations, we obtain $F = d = 0$.
Then $d(h) = 0$ for all $h \in Z(R) \cap H(R)$ using **Fact 1**, we get $d(Z(R) \cap S(R)) = \{0\}$.
Replacing y by ys, in (2.22) where $0 \neq s \in Z(R) \cap S(R)$, we arrive at

$$F(x) \circ d(y) - F(y^\natural) \circ d(x^\natural) = 0 \quad \text{for all} \;\; x, y \in R. \tag{2.25}$$

This equation, when combined with (2.22) shows that

$$F(x) \circ d(y) = 0 \quad \text{for all} \quad x, y \in R. \tag{2.26}$$

and [3, Theorem 2.5] gives that R is commutative ring.
The last relation leads to

$$F(x)d(y) = 0 \quad \text{for all} \quad x, y \in R$$

which reduces to $d = 0$.
Finally $F(xy) = F(x)y$ for all $x, y \in R$. □

Corollary 2.7. *Let $(R, \natural)$ be a 2-torsion free prime ring with Jordan involution of the second kind and F a generalized derivation associated with a derivation d. Then the following assertions are equivalent:*
(1) $F(x) \circ d(y) = 0$ *for all* $x, y \in R$.
(2) F *is a left multiplier.*

Theorem 2.8. *Let $(R, \natural)$ be a 2-torsion free prime ring with Jordan involution of the second kind. There is no generalized derivation F associated with a derivation d satisfying one of the following assertions :*
(1) $[F(x), d(x^\natural)] = x \circ x^\natural$ *for all* $x \in R$.
(2) $[F(x), d(x^\natural)] = -x \circ x^\natural$ *for all* $x \in R$.

Proof. We assume that R is noncommutative ring and $d \neq 0$, otherwise we get $x \circ x^\natural = 0$ for all $x \in R$. Then arguing as above, we get $R = \{0\}$.
Suppose that

$$[F(x), d(x^\natural)] = x \circ x^\natural \quad \text{for all} \quad x \in R. \tag{2.27}$$

A linearization of the last equation, implies that

$$[F(x), d(y^\natural)] + [F(y), d(x^\natural)] = x \circ y^\natural + y \circ x^\natural$$

thereby

$$[F(x), d(y)] + [F(y^\natural), d(x^\natural)] = x \circ y + y^\natural \circ x^\natural \quad \text{for all} \quad x, y \in R. \tag{2.28}$$

Taking yh instead of y, where $0 \neq h \in Z(R) \cap H(R)$, we get

$$\big([F(x), y] + [y^\natural, d(x^\natural)]\big)d(h) = 0 \quad \text{for all} \quad x, y \in R. \tag{2.29}$$

In view of the primeness, the last equation assures that $d(h) = 0$ or $[F(x), y] + [y^\natural, d(x^\natural)] = 0$.
Suppose that

$$[F(x), y] + [y^\natural, d(x^\natural)] = 0 \quad \text{for all} \quad x, y \in R. \tag{2.30}$$

Since this equation is exactly (2.4) in the proof of Theorem (2.1), then we have R is commutative, a contradiction.
Now suppose that $d(h) = 0$ for all $h \in Z(R) \cap H(R)$, then by virtue of **Fact 1** we get $d(Z(R) \cap S(R)) = \{0\}$.
Substituting ys for y in (2.28), we find that

$$[F(x), d(y)] - [F(y^\natural), d(x^\natural)] = x \circ y - y^\natural \circ x^\natural \quad \text{for all} \quad x, y \in R. \tag{2.31}$$

Adding equations (2.28) and (2.31) we easily get

$$[F(x), d(y)] = x \circ y \quad \text{for all} \quad x, y \in R. \tag{2.32}$$

Applying [4, Theorem 2.2] we show that R is commutative, a contradiction.

Using a similar proof, with slight modifications, one can easily prove the case : $[F(x), d(x^\natural)] = -x \circ x^\natural$. □

Corollary 2.9. *Let $(R, \natural)$ be a 2-torsion free prime ring with Jordan involution of the second kind. There is no generalized derivation F associated with a derivation d satisfying one of the following assertions :*
(1) $[F(x), d(y)] = x \circ y$ *for all* $x, y \in R$.
(2) $[F(x), d(y)] = -x \circ y$ *for all* $x, y \in R$.

Theorem 2.10. *Let $(R, \natural)$ be a 2-torsion free prime ring with Jordan involution of the second kind. If R admits a generalized derivation F associated with a derivation d such that either $F(x) \circ d(x^\natural) = [x, x^\natural]$ for all $x \in R$ or $F(x) \circ d(x^\natural) = -[x, x^\natural]$ for all $x \in R$, then R is a commutative integral domain. Furthermore, F is a left multiplier.*

Proof. Suppose that

$$F(x) \circ d(x^\natural) = [x, x^\natural] \quad \text{for all} \quad x \in R. \tag{2.33}$$

If $F = 0$ then the last relation leads to $[x, x^\natural] = 0$ for all $x \in R$.
Then R is commutative by **Fact 2**. Thus, we may assume that $F \neq 0$.
A linearization of (2.33), implies that

$$F(x) \circ d(y) + F(y^\natural) \circ d(x^\natural) = [x, y] + [y^\natural, x^\natural] \quad \text{for all} \quad x, y \in R. \tag{2.34}$$

Substituting yh for y in the previous equation, where $h \in Z(R) \cap H(R) \backslash \{0\}$, and using (2.34), we get

$$\big(F(x) \circ y + y^\natural \circ d(x^\natural)\big) d(h) = 0 \quad \text{for all} \quad x, y \in R. \tag{2.35}$$

In light of primeness, it follows that $F(x) \circ y + y^\natural \circ d(x^\natural) = 0$ or $d(h) = 0$.
Assume that

$$F(x) \circ y + y^\natural \circ d(x^\natural) = 0 \quad \text{for all} \quad x, y \in R. \tag{2.36}$$

Taking h instead of y in (2.36), where $0 \neq h \in Z(R) \cap H(R)$, we get

$$F(x) + d(x^\natural) = 0 \quad \text{for all} \quad x \in R.$$

Now, replacing y by s in (2.36) where $0 \neq s \in Z(R) \cap S(R)$, one obtains

$$F(x) - d(x^\natural) = 0 \quad \text{for all} \quad x \in R.$$

Combining the two previous relations we get $F = 0$, a contradiction.
Then we need only consider the case $d(h) = 0$ for all $h \in Z(R) \cap H(R)$. Putting $y = ys$ in (2.34), where $0 \neq s \in Z(R) \cap S(R)$, because of $d(s) = 0$ it follows that

$$\big(F(x) \circ d(y) - F(y^\natural) \circ d(x^\natural)\big) s = \big([x, y] + [y^\natural, x^\natural]\big) s \quad \text{for all} \quad x, y \in R. \tag{2.37}$$

By primeness we conclude that

$$F(x) \circ d(y) - F(y^\natural) \circ d(x^\natural) = [x, y] - [y^\natural, x^\natural] \quad \text{for all} \quad x, y \in R. \tag{2.38}$$

Combining equations (2.34) and (2.38), we have

$$F(x) \circ d(y) = [x, y] \quad \text{for all} \quad x, y \in R. \tag{2.39}$$

Then R is commutative by [4, Theorem 2.5]. In consequence of which, equation (2.39) becomes $F(x)d(y) = 0$ then we have $d = 0$, implies that F is a left multiplier.

The case when $F(x) \circ d(x^\natural) = -[x, x^\natural]$ for all $x \in R$, we arguing as above. □

Corollary 2.11. *Let $(R, \natural)$ be a 2-torsion free prime ring with Jordan involution of the second kind. If R admits a generalized derivation F associated with a derivation d such that either $F(x) \circ d(y) = [x, y]$ for all $x, y \in R$ or $F(x) \circ d(y) = -[x, y]$ for all $x, y \in R$, then R is a commutative integral domain. Furthermore, F is a left multiplier.*

The following example proves that the primeness hypothesis in Theorems 2.1 and 2.6 is not superfluous.

Example 1.

Let us consider $R = M_2(\mathbb{Z})$ and define $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^\natural = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ and $F\begin{pmatrix} a & b \\ c & d \end{pmatrix} =$

$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$. Then F is a left multiplier and $(R, \natural)$ is a prime ring with involution of the first kind such that $[x, x^{\natural}] = 0$ for all $x \in R$.
Set $\mathcal{R} = R \times \mathbb{C}$, then it is obvious to verify that $(\mathcal{R}, \sigma)$ is a semi-prime ring with involution of the second kind where $\sigma(r, z) = (r^{\natural}, \bar{z})$.
Moreover, if we put

$$\mathcal{F}(r, z) = (F(r), 0)$$

then $\mathcal{F}$ is a left multiplier satisfying the condition (1) of Theorem 1 but $\mathcal{R}$ is not commutative.

The following example proves that the condition "$\natural$ is of the second kind" is necessary in Theorems 2.1, and 2.10.

Example 2.
Let us consider $R = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid a, b, c, d \in \mathbb{Z} \right\}$ and $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\natural} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. It is straightforward to check that $(R, \natural)$ is a prime ring with involution of the first kind such that

$$[x, x^{\natural}] = 0 \text{ for all } x \in R.$$

Furthermore, the mapping $F : R \longrightarrow R$ defined by

$$F \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}$$

is a left multiplier that satisfies conditions of Theorems 2.1 and 2.10 however R is not commutative.

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