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Nil-clean and weakly nil-clean properties in Bi-Amalgamated algebras along ideals

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Abstract This paper establishes necessary and sufficient conditions for a bi-amalgamated algebras along ideals to inherit the nil-clean (resp., weakly nil-clean) property. The new results compare to previous works carried on various settings of duplications and amalgamations, and capitalize on recent results on bi-amalgamations. All results are backed with new and illustrative examples arising as bi-amalgamations.

1 Introduction

All rings considered in this paper are commutative with identity. For a ring R, $U(R)$, $Nil(R)$ and $Id(R)$ denotes the groupe of all units of R, the nil-radical of R and the set of all idempotents of R respectively.
In [15], Nicholson introduced the notion of clean rings. A ring R is called clean if for all $r \in R$ there are $u \in U(R)$ and $e \in Id(R)$ such that $r = u + e$. If the presentation of r is unique, we said that R is uniquely clean.
Later in [10], Diesl modified the preveous definition and introduced an interesing class of rings called nil-clean rings. A ring R is called nil-clean if for all $r \in R$ there are $n \in Nil(R)$ and $e \in Id(R)$ such that $r = n + e$. If the representation of r is unique, we said that R is uniquely nil-clean. He proved that every nil-clean ring is clean [10, Proposition 3.4].
In [5], Peter. V. Danchev and W. W. McGovern generalized the notion of nil-clean rings, they introduced and studied a new class of rings called weakly nil-clean rings. A ring R is called a weakly nil-clean if for all $r \in R$ there are $n \in Nil(R)$ and $e \in Id(R)$ such that $r = n + e$ or $r = n - e$. If the representation of r is unique, we said that R is uniquely weakly nil-clean. They proved that every commutative nil-clean ring is uniquely nil-clean [5, Remark 1.5]. It is clear that every nil-clean ring is weakly nil-clean. In [5], the autors gave an example of a weakly nil-clean ring which is not nil-clean. They proved that a weakly nil-clean ring R is nil-clean if $2 \in Nil(R)$ (cf. [5, Proposition 1.10]). They showed also that every weakly nil-clean ring is clean[5, Proposition 1.9(iv)]. They gave an example of a clean ring which is not weakly nil-clean.

Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let J and J' be two ideals of B and C respectively such that $f^{-1}(J) = g^{-1}(J')$. The bi-amalgamated algebra of A with (B, C) along (J, J') with respect to (f, g) is the subring of $B \times C$ given by:

$$A \bowtie^{(f,g)} (J, J') := \{(f(a) + j, g(a) + j'/a \in A, (j, j') \in J \times J'\}$$

This construction is introduced and studied by Kabbaj, Louartiti and Tamekante in [13]. They established numerous results on the transfer of ring properties from $f(A) + J$ and $g(A) + J'$ to $A \bowtie^{(f,g)} (J, J')$. This new construction cover some basic constructions in commutative rings such as: pullback ([13, Section 3]) and amalgamated algebra along an ideal ([13, Example 2.1]). Moreover, other classical constructions such as: $f(A) + J$ ([13, Remark 2.2]) and the $A + J$ construction ([13, Example 2.4]) can be studied as particular case of bi-amalgamated algebra

along an ideal.
In this chapter, we give a characterization for $A \bowtie^{(f,g)} (J, J')$ to be nil-clean and weakly nil-clean. In section 2 we establishes necessary and sufficient conditions for $A \bowtie^{(f,g)} (J, J')$ to be nil-clean. The new results generalize well know results in [3]. Section 3 is devoted to the transfer of weakly nil-clean property in $A \bowtie^{(f,g)} (J, J')$. Our aim is to provide examples of new classes of commutative rings satisfying the above-mentioned properties.

In the rest of this paper unless otherwise stated, A, B, and C are rings, $f : A \to B$ and $g : A \to C$ are rings homomorphism, J and J' are ideals of B and C respectively such that $f^{-1}(J) = g^{-1}(J')$ and $A \bowtie^{(f,g)} (J, J')$ is the bi-amalgamated algebra of A with (B, C) along (J, J') with respect to (f, g).

2 Nil-clean property in bi-amalgamated algebras along ideals

Recall that a ring R is called nil-clean if for all $r \in R$, there are $n \in Nil(R)$ and $e \in Id(R)$ such that $r = n+e$. Our first main result gives necessary and sufficient conditions for $A \bowtie^{(f,g)} (J, J')$ to be nil-clean.

Theorem 2.1. *The following statements are equivalent.*

(1) $A \bowtie^{(f,g)} (J, J')$ *is a nil-clean ring.*

(2) $f(A) + J$ *and* $g(A) + J'$ *are nil-clean rings.*

Proof. (1) $\Rightarrow$ (2): If $A \bowtie^{(f,g)} (J, J')$ is a nil-clean ring, then (2) holds by [13, Proposition 4.1(2)], since every homomorphic image of a nil-clean ring is a nil-clean ring.
(2) $\Rightarrow$ (1): Assume that $f(A) + J$ and $g(A) + J'$ are nil-clean rings. Let $a \in A$, $(j, j') \in J \times J'$, then by [3, Lemma 2.2], $f(a) + j - (f(a) + j)^2$ and $g(a) + j' - (g(a) + j')^2$ are nilpotents. Therefore, it is easy now to show that again $(f(a) + j, g(a) + j') - ((f(a) + j, g(a) + j'))^2$ is a nilpotent element of $A \bowtie^{(f,g)} (J, J')$. Thus, $A \bowtie^{(f,g)} (J, J')$ is a nil-clean ring by [3, Lemma 2.2].

□

Remark 2.2. (1) If $J = (0)$ (respectively. $J' = (0)$) then, $A \bowtie^{(f,g)} (J, J')$ is a nil-clean ring if and only if $g(A) + J'$ (respectively. $f(A) + J$)) is a nil-clean ring.

(2) If $J \times J' = B \times C$ then, $A \bowtie^{(f,g)} (J, J')$ is a nil-clean ring if and only if B and C are nil-clean rings.

Proof. (1) If $J = (0)$ (respectively. $J' = (0)$), then $A \bowtie^{(f,g)} (J, J') \cong g(A) + J'$ (respectively. $A \bowtie^{(f,g)} (J, J') \cong f(A) + J$) by [13, Proposition 4.1(2)]. Thus, the conclusion is straightforward.
(2) If $J \times J' = B \times C$, then $f(A) + J = B$ and $g(A) + J' = C$. Thus, the conclusion follow directly from Theorem 2.1. □

The following corollaries are immediate applications of Theorem 2.1.

Corollary 2.3. *Let* A *be a nil-clean ring and let* $f : A \to B$ *and* $g : A \to C$ *be two surjective ring homomorphisms. Let* J *and* J' *be two ideals of* B *and* C *respectively such that* $f^{-1}(J) = g^{-1}(J')$*. Then* $A \bowtie^{(f,g)} (J, J')$ *is a nil-clean ring.*

Proof. It is clear that B and C are nil-clean rings. So, since $f(A) + J = B$ and $g(A) + J' = C$, we conclude that $A \bowtie^{(f,g)} (J, J')$ is a nil-clean ring by Theorem 2.1. □

Recall that the amalgamation of A with B along an ideal J of B with respect the ring homomorphism $f : A \to B$, is given by

$$A \bowtie^f J := \{(a, f(a) + j) \mid a \in A, j \in J\}.$$

Clearly, every amalgamation can be viewed as a special bi-amalgamation, since $A \bowtie^f J = A \bowtie^{id_A, f} (f^{-1}(J), J)$. The following result recovers [3, Theorem 2.2].

Corollary 2.4. *Let $f : A \to B$ be a ring homomorphism and let J be an ideal of B. Then, the following are equivalent:*

(1) $A \bowtie^f J$ is nil-clean.

(2) A and $f(A) + J$ are nil-clean.

Let I be an ideal of A. The amalgamated duplication of A along I is a special amalgamation given by

$$A \bowtie I := A \bowtie^{id_A} I = \{(a, a+i) \mid a \in A, i \in I\}.$$

The next corollary is an immediate consequence of Corollary 2.4 on the transfer of nil-clean property to duplications.

Corollary 2.5. *Let A be a ring and I be an ideal of A. Then $A \bowtie I$ is a nil-clean ring if and only if A is a nil-clean ring.*

Theorem 2.1 enriches the literature with new examples of nil-clean rings.

Example 2.6. Let $A := \mathbb{Z}_4$, $B := \mathbb{Z}_4 \times \mathbb{Z}_4$, $J := 0 \times \mathbb{Z}_4$ be an ideal of B, $C := \mathbb{Z}_2$ and $J' := \mathbb{Z}_2$. Consider $f : A \to B$ and $g : A \to C$ defined by $f(a) = (a, 0)$ for all $a \in A$ and $g(0) = g(2) = 0$ and $g(1) = g(3) = 1$. It is well know that A and C are nil-clean rings, then by [10, Proposition 3.13], so is $B = A \times A$. Therefore, $f(A) + J = B$ and $g(A) + J' = C$ are nil-clean rings. Then, by Theorem 2.1, $A \bowtie^{(f,g)} (J, J')$ is a nil-clean ring.

Example 2.7. Let A be a nil-clean ring, and I_1, I_2 and I be three ideals of A such that $I_1 \subseteq I$ and $I_2 \subseteq I$. Set $B := A/I_1, C := A/I_2, J := I/I_1$ and $J' = I/I_2$. Let $f : A \longrightarrow B$ and $g : A \longrightarrow C$ be the canonical surjections. Thus $A \bowtie^{(f,g)} (J, J')$ is a nil-clean ring by Corollary 2.3.

The next result is a partial result when a Bi-amalgamation is a nil-clean ring in case A is a nil-clean ring.

Theorem 2.8. *Assume that A is a nil-clean ring. Then, the following statements hold:*

(1) If ($J \subseteq Nil(B)$ or $J \subseteq Id(B)$) and ($J' \subseteq Nil(C)$ or $J' \subseteq Id(C)$), then $A \bowtie^{(f,g)} (J, J')$ is a nil-clean ring.

(2) Assume that ($J \cap Id(B) = 0$ or $J \cap Nil(B) = 0$) and ($J' \cap Id(C) = 0$ or $J' \cap Nil(C) = 0$). Then, $A \bowtie^{(f,g)} (J, J')$ is a nil-clean ring if and only if ($J \subseteq Nil(B)$ or $J \subseteq Id(B)$) and ($J' \subseteq Nil(C)$ or $J' \subseteq Id(C)$).

Before proving Theorem 2.8, we establish the following lemma.

Lemma 2.9. *If $A \bowtie^{(f,g)} (J, J')$ is a nil-clean ring, then for all $j \in J$ and $j' \in J'$, there are $k \in J \cap Nil(B)$, $t \in J \cap Id(B)$, $k' \in J' \cap Nil(C)$ and $t' \in J' \cap Id(C)$ such that $j = k + t$ and $j' = k' + t'$.*

Proof. Assume that $A \bowtie^{(f,g)} (J, J')$ is nil-clean. Let $j \in J$, without loss of generality, we may assume that $0 \neq j$. Therefore, there are a nilpotent element and an idempotent element $(f(n) + k_0, g(n) + k'_0)$, $(f(e) + k_1, g(e) + k'_1)$ of $A \bowtie^{(f,g)} (J, J')$ respectively such that,

$$(j, 0) = (f(n) + k_0, g(n) + k'_0) + (f(e) + k_1, g(e) + k'_1), \text{ then}$$
$$j = f(n) + k_0 + f(e) + k_1 \text{ and } 0 = g(n) + k'_0 + g(e) + k'_1$$

The fact that $(f(n) + k_0, g(n) + k'_0)$ is nilpotent and $(f(e) + k_1, g(e) + k'_1)$ is an idempotent element of $A \bowtie^{(f,g)} (J, J')$ yields that

$$(f(n) + k_0, f(e) + k_1) \in Nil(f(A) + J) \times Id(f(A) + J) \text{ and}$$
$$(g(n) + k'_0, g(e) + k'_1) \in Nil(g(A) + J') \times Id(g(A) + J')$$

On the other hand, since $0 = g(n) + k'_0 + g(e) + k'_1$, then $g(e) + k'_1 = -(g(n) + k_0)$. Therefore, $g(e) + k'_1 \in Id(g(A) + J') \cap Nil(g(A) + J') = 0$, then $g(e) + k'_1 = g(n) + k'_0 = 0$. Then, $(n, e) \in g^{-1}(J') \times g^{-1}(J') = f^{-1}(J) \times f^{-1}(J)$ and so $(f(n), f(e)) \in J \times J$. Which implies that $f(n) + k_0 \in J \cap Nil(B)$ and $f(e) + k_1 \in J \cap Id(B)$. Hence, $j = k + t$ where $k = f(n) + k_0$ and $t = f(e) + k_1$.

Similarly, let $j' \in J'$ and assume, without loss of generality, that $0 \neq j'$. Therefore, there are a nilpotent element and an idempotent element $(f(n) + k_0, g(n) + k'_0)$ $(f(e) + k_1, g(e) + k'_1)$ of $A \bowtie^{(f,g)} (J, J')$ respectively such that,

$$(0, j') = (f(n) + k_0, g(n) + k'_0) + (f(e) + k_1, g(e) + k'_1)$$

For the same reasoning, we shows that,

$j' = k' + t'$ where $k' = g(n) + k'_0 \in J' \cap Nil(C)$ and $t' = g(e) + k'_1 \in J' \cap Id(C)$, as desired.

□

Proof of Theorem 2.8. (1) Let $a \in A$, $j \in J$. Then, there are a nilpotent element n and an idempotent element e of A such that $a = n + e$ (since A is nil-clean). Therefore $f(n)$ is a nilpotent element and $f(e)$ is an idempoten element of $f(A)+J$. Suppose that $J \subseteq Nil(B)$ then it is easy to show that $f(n) + j \in Nil(f(A) + J)$. Hence, $f(a) + j = (f(n) + j) + f(e)$ is a sum of a nilpotent element $f(n) + j$ and an idempotent element $f(e)$ of $f(A) + J$. Now, assume that $J \subseteq Id(B)$. Since $2j = 0$ and $j^2 = j$, we can easily show that $f(e) + j \in Id(f(A) + J)$. Thus, $f(a) + j = f(n) + (f(e) + j)$ is a sum of a nilpotent element $f(n)$ and an idempotent element $f(e) + j$ of $f(A) + J$. In all cases, $f(A) + J$ is a nil-clean ring. If $J' \subseteq Nil(C)$ or $J' \subseteq Idem(C)$, by the same technique as the preveous part of proof by exchanging the role of J by J', we can then prove that $g(A) + J'$ is a nil-clean ring. Therefore $A \bowtie^{(f,g)} (J, J')$ is a nil-clean ring by Theorem 2.1.
(2) Suppose that $A \bowtie^{(f,g)} (J, J')$ is a nil-clean ring and let $j \in J$. Hence, Lemma 2.9 implies that $j = k + t$ for some $k \in J \cap Nil(B)$ and $t \in J \cap Id(B)$. Clearly, if $J \cap Id(B) = 0$ (or $J \cap Nil(B) = 0$) then we have $J \subseteq Nil(B)$ (or $J \subseteq Id(B)$). Now, using the same technique of the preveous by exchanging the role of J by J' and B by C, we can similarly show that $J' \subseteq Nil(C)$ (or $J' \subseteq Id(C)$). The converse follows directly by (1).

Theorem 2.8 recovers the special case of amalgamated algebra, as recorded in the following corollary.

Corollary 2.10. *Let $f : A \to B$ be a ring homomorphism and let J be an ideal of B. Then the following statements hold:*

(1) If $J \subseteq Nil(B)$ or $J \subseteq Id(B)$, then $A \bowtie^f J$ is nil-clean if and only if A is a nil-clean ring.

(2) If $J \cap Id(B) = 0$, then $A \bowtie^f J$ is nil-clean if and only if A so is and $J \subseteq Nil(B)$.

(3) If $J \cap Nil(B) = 0$, then $A \bowtie^f J$ is nil-clean if and only if A so is and $J \subseteq Id(B)$

Theorem 2.8 enriches the literature with new original examples of nil-clean rings. Recall that for a ring A and an A-module E, the *trivial ring extension of A by E* (also called *idealization of E over A*) is the ring $R := A \ltimes E$ whose underlying group is $A \times E$ with multiplication given by $(a, e)(a', e') = (aa', ae' + a'e)$ for all $a, a' \in A$ and $e, e' \in E$ (cf. [1, 11, 14]).

Example 2.11. Let $(A, m) := (A_1 \ltimes E_1, m_1 \ltimes E_1)$ be the trivial ring extension of a nil-clean ring A_1 by an A_1-module E_1, (for instance $(A_1, m_1) := (\mathbb{Z}_4, 2\mathbb{Z}_4)$) and E_1 is a nonzero $(A_1/m_1)-$vector space (for instance $E_1 = \mathbb{Z}_4/2\mathbb{Z}_4$). Let $B := A_1$. Consider

$$\begin{array}{cccc} f: & A & \to & B \\ & (a,e) & \to & f((a,e)) = a; \end{array}$$

Set $J = m_1$ the maximal ideal of B. Let $C := A \ltimes E$ be the trivial ring extension of A by a nonzero $A/m-$vector space E and let

$$\begin{array}{cccc} g: & A & \hookrightarrow & C \\ & (a,e) & \hookrightarrow & g((a,e)) = ((a,e),0); \end{array}$$

Set $J' := m \ltimes E = (m_1 \ltimes E_1) \ltimes E$ the maximal ideal of C. Clearly, $f^{-1}(J) = g^{-1}(J') = m_1 \ltimes E_1$. Then :
1) By Theorem 2.8 $A \bowtie^{f,g} (J, J')$ is a nil-clean ring since $J \subseteq Nil(B)$, $J' \subseteq Nil(C)$ and A is nil-clean by [3, Corollary 2.12].
2) $A \bowtie^{f,g} (J, J')$ is not a Von Neumann Regular ring since it is not reduced by [13, Proposition 4.7].

Proof. (1) [3, Corollary 2.6]. (2) [3, Corollary 2.7]. □

Peter V. Danchev and W. W. McGowen proved that a ring R is nil-clean if and only if $R/Nil(R)$ is a Boolean ring [5, Proposition 1.3]. That leads to the following result.

Proposition 2.12. *Let $f : A \to B$ and $g : A \to C$ be two ring homomorphisms and let J and J' be two ideals of B and C respectively such that $f^{-1}(J) = g^{-1}(J')$. Set $\overline{A} = A/Nil(A)$, $\overline{B} = B/Nil(B)$, $\overline{C} = C/Nil(C)$, $\pi_B : B \to \overline{B}$, $\pi_C : C \to \overline{C}$ be the canonical projection, $\overline{J} = \pi_B(J)$ and $\overline{J'} = \pi_C(J')$. Consider these two ring homomorphisms $\overline{f} : \overline{A} \to \overline{B}$ and $\overline{g} : \overline{A} \to \overline{C}$ defined by: $\overline{f}(\overline{a}) = \overline{f(a)}$ and $\overline{g}(\overline{a}) = \overline{g(a)}$. Then, $A \bowtie^{(f,g)} (J, J')$ is nil-clean if and only if $\overline{A} \bowtie^{(\overline{f},\overline{g})} (\overline{J}, \overline{J'})$ is Boolean.*

Proof. Consider the map:

$$\begin{array}{rcl} \phi : A \bowtie^{(f,g)} (J, J')/Nil(A \bowtie^{(f,g)} (J, J')) & \to & \overline{A} \bowtie^{(\overline{f},\overline{g})} (\overline{J}, \overline{J'}) \\ \overline{(f(a) + j, g(a) + j')} & \mapsto & (\overline{f}(\overline{a}) + \overline{j}, \overline{g}(\overline{a}) + \overline{j'}) \end{array}$$

It is easy to show that ϕ is well defined and is a ring homomorphism. By construction ϕ is surjective. Let $a \in A$ and $(j, j') \in J \times J'$ and assume that $(\overline{f}(\overline{a}) + \overline{j}, \overline{g}(\overline{a}) + \overline{j'}) = 0$. Then $(\overline{f(a) + j}, \overline{g(a) + j'}) = 0$ and so $(f(a) + j, g(a) + j') \in Nil(A \bowtie^{(f,g)} (J, J')$. Which implies that $\overline{(f(a) + j, g(a) + j')} = 0$ and hence ϕ is injective. Consequently, ϕ is a ring isomorphism. Assume that $A \bowtie^{(f,g)} (J, J')$ is nil-clean. Then, $A \bowtie^{(f,g)} (J, J')/Nil(A \bowtie^{(f,g)} (J, J'))$ is Boolean by [5, Proposition 1.3]. Therefore so is $\overline{A} \bowtie^{(\overline{f},\overline{g})} (\overline{J}, \overline{J'})$. Conversely, assume that $\overline{A} \bowtie^{(\overline{f},\overline{g})} (\overline{J}, \overline{J'})$ is a Boolean ring, then so is $A \bowtie^{(f,g)} (J, J')/Nil(A \bowtie^{(f,g)} (J, J'))$. Thus, by [5, Proposition 1.3], $A \bowtie^{(f,g)} (J, J')$ is a nil-clean ring. □

Proposition 2.12 recovers the special case of amalgamated algebra, as recorded in the following corollary.

Corollary 2.13. *[3, Theorem 2.9] Let $f : A \to B$ be a ring homomorphism and let J be an ideal of B. Set $\overline{A} = A/Nil(A)$, $\overline{B} = B/Nil(B)$, $\pi : B \to \overline{B}$, be the canonical projection, $\overline{J} = \pi(J)$. Consider the ring homomorphism $\overline{f} : \overline{A} \to \overline{B}$ such that: $x \to \overline{f}(\overline{x}) = \overline{f(x)}$. Then $A \bowtie^f J$ is nil-clean if and only if $\overline{A} \bowtie^{\overline{f}} \overline{J}$ is Boolean.*

3 Weakly nil-clean property in a bi-amalgamated algebras along ideals

We recall that a ring R is called weakly nil-clean if for all $r \in R$ there are $n \in Nil(R)$ and $e \in Id(R)$ such that $r = n + e$ or $r = n - e$. If this representation is unique, we say that R is uniquely weakly nil-clean. In [5], the autors proved that the class of weakly nil-clean rings is closed under homomorphic image but not closed under finite product (cf. [5, Proposition 1.9 (i), (ii)]).
In this section we study the transfer of weakly nil-clean property to the bi-amalgamated algebra of a ring along ideals $A \bowtie^{(f,g)} (J, J')$. We establishes necessary and sufficient conditions for $A \bowtie^{(f,g)} (J, J')$ to be weakly nil-clean.

The following studies the transfer of the weakly nil-clean property to $A \bowtie^{(f,g)} (J, J')$.

Theorem 3.1. *If $A \bowtie^{(f,g)} (J, J')$ is weakly nil-clean, then so are $f(A) + J$ and $g(A) + J'$. The converse is true provided that $J \subseteq Nil(B)$ or $J' \subseteq Nil(C)$.*

Proof. We recall that the weakly nil-clean property is closed under homomorphic image by [5, Proposition 1.9(i)]. Assume that $A \bowtie^{(f,g)} (J, J')$ is a weakly nil-clean ring, then so are $f(A) + J$ and $g(A) + J'$ by [13, Proposition 4.1(2)]). Conversely, suppose,without loss of generality, that $J \subseteq Nil(B)$. Thus, $J \times \{0\} \subseteq Nil(A \bowtie^{(f,g)} (J, J')$. Then [5, Proposition 1.9(i)] implies that $A \bowtie^{(f,g)} (J, J')$ is weakly nil-clean if and only if $A \bowtie^{(f,g)} (J, J')/(J \times \{0\})$ is weakly nil-clean. Now the conclusion follows directly from [13, Proposition 4.1(2)]. □

Remark 3.2. The following statements are true:

(1) If $J=(0)$ (respectively. $J'=(0)$). Then, $A\bowtie^{(f,g)}(J,J')$ is a weakly nil-clean ring if and only if $g(A)+J'$ is a weakly nil-clean ring (respectively. $f(A)+J$ is a weakly nil-clean ring).

(2) If $J=B$ and $J'=C$. Then, if $A\bowtie^{(f,g)}(J,J')$ is weakly nil-clean, then so are B and C. The converse is true provided that B or C is nil-clean.

Proof. (1) If $J=0$ (respectively. J'=0). Then, the conclusion follows directly from [13, Proposition 4.1(2)].
(2) Assume that $J=B$ and $J'=C$. In this case $f(A)+J=B$ and $g(A)+J'=C$ and so $A\bowtie^{(f,g)}(J,J')=B\times C$. Therefore, if $A\bowtie^{(f,g)}(J,J')$ is a weakly nil-clean ring, then so are B and C by Theorem 3.1. For the converse, suppose for example that B is nil-clean and C is weakly nil-clean. We will show that that $A\bowtie^{(f,g)}(J,J')$ is weakly nil-clean. Let $(b,c)\in B\times C$, then there are $n\in Nil(C)$ and $e\in Id(C)$ such that $c=n+e$ or $c=n-e$. If $c=n+e$, set $b=n_1+e_1$ where $(n_1,e_1)\in Nil(B)\times Id(B)$. Then, $(b,c)=(n,n_1)+(e,e_1)$ where $(n,n_1)\in Nil(B)\times Nil(C)\subseteq Nil(B\times C)$ and $(e,e_1)\in Id(B)\times Id(C)\subseteq Id(B\times C)$. If $c=n-e$, set $b=n_1-e_1$ with $(n_1,e_1)\in Nil(B)\times Id(B)$. Therefore, $(b,c)=(n,n_1)-(e,e_1)$. Hence, $(b,c)=(n,n_1)+(e,e_1)$ or $(b,c)=(n,n_1)-(e,e_1)$ where $(n,n_1)\in Nil(B\times C)$ and $(e,e_1)\in Id(B\times C)$, as desired. □

Theorem 3.1 recovers the special case of amalgamated algebra, as recorded in the following corollary.

Corollary 3.3. *Let $f:A\to B$ be a ring homomorphism and let J be an ideal of B such that $J\subseteq Nil(B)$. Then, $A\bowtie^f J$ is weakly nil-clean if and only if A is weakly nil-clean.*

Proof. This follows from the proof of Theorem 3.1 and [13, Example 2.1]. □

In the special case of amalgamated duplication of a ring along an ideal, we obtain the following result which is a direct consequence of Corollary 3.3.

Corollary 3.4. *Let A be a ring and I be an ideal of A such that $I\subseteq Nil(A)$. Then $A\bowtie I$ is weakly nil-clean if and only if A is weakly nil-clean.*

The next corollary studies when the trivial ring extension is a weakly nil-clean ring.

Corollary 3.5. *Let A be a ring and E an A-module. Then $A\ltimes E$ is a weakly nil-clean ring if and only if A is a weakly nil-clean ring.*

Proof. Consider a ring homomorphism

$$\begin{array}{rcccl} f & : & A & \hookrightarrow & A\ltimes E \\ & & a & \mapsto & f(a)=(a,0) \end{array}$$

and an ideal $J:=0\ltimes E$ of $A\ltimes E$. Then, we have $A\bowtie^f J\cong A\ltimes E$ and $J\subseteq Nil(A\ltimes E)$ since $J^2=0$. Thus, the conclusion follows directly by Corollary 3.3. □

The following result is a partial result when a bi-amalgamation is a weakly nil-clean ring.

Proposition 3.6. *With the notation of Theorem 3.1. Assume that $J\cap Id(B)=0$ (respectively. $J'\cap Id(C)=0$). Then, the following statements are equivalent:*

(1) $A\bowtie^{(f,g)}(J,J')$ is weakly nil-clean.

(2) $g(A)+J'$ is weakly nil-clean and $J\subseteq Nil(B)$ (respectively. $f(A)+J$ is weakly nil-clean and $J'\subseteq Nil(C)$).

Proof. (1) ⇒ (2): By Theorem 3.1, we only need prove that $J\subseteq Nil(B)$ (respectively. $J'\subseteq Nil(C)$) if $J\cap Id(B)=0$ (respectively. $J'\cap Id(C)=0$). Suppose that $J\cap Id(B)=0$ and let $j\in J$. Without loss of generality we may assume that $0\neq j$. Then, there are a nilpotent element $(f(n)+j_1,g(n)+j'_1)$ and an idempotent element $(f(e)+j_2,g(e)+j'_2)$ of $A\bowtie^{(f,g)}(J,J')$ such that $(j,0)=(f(n)+j_1,g(n)+j'_1)+(f(e)+j_2,g(e)+j'_2)$ or $(j,0)=(f(n)+j_1,g(n)+j'_1)-$

$(f(e)+j_2, g(e)+j'_2)$. Therefore, $j = (f(n)+j_1)+(f(e)+j_2)$ or $j = (f(n)+j_1)-(f(e)+j_2)$ and $0 = (g(n)+j'_1)+(g(e)+j'_2)$ or $0 = (g(n)+j'_1)-(g(e)+j'_2)$. The fact that $(f(n)+j_1, g(n)+j'_1)$ is nilpotent and $(f(e)+j_2, g(e)+j'_2)$ is idempotent of $A \bowtie^{(f,g)} (J, J')$ respectively implies that $(f(n)+j_1, f(e)+j_2) \in Nil(f(A)+J) \times Id(f(A)+J)$ and $(g(n)+j'_1, g(e)+j'_2) \in Nil(g(A)+J') \times Id(g(A)+J')$. Moreover, since $0 = (g(n)+j'_1)+(g(e)+j'_2)$ or $0 = (g(n)+j'_1)-(g(e)+j'_2)$, we get that $g(e)+j'_2 = -(g(n)+j'_1)$ or $g(e)+j'_2 = g(n)+j'_1$. Thus, $g(e)+j'_2 \in Nil(g(A)+J') \cap Id(g(A)+J') = 0$ and so $g(e)+j'_2 = g(n)+j'_1 = 0$. Then $(n, e) \in g^{-1}(J') \times g^{-1}(J') = f^{-1}(J) \times f^{-1}(J)$ which implies that $(f(n), f(e)) \in J^2$. Consequently, $f(e)+j_2 \in J \cap Id(f(A)+J) \subseteq J \cap Id(B) = 0$ and thus $f(e)+j_2 = 0$. Hence, $j = f(n)+j_1 \in Nil(f(A)+J) \subseteq Nil(B)$. Respectively, if $J' \cap Id(C) = 0$, with the same technique with the preveous by exchanging the role of J by J' and the role of B by C, we can easily proves that $J' \subseteq Nil(C)$, as wanted.
(2) ⇒ (1): Assume that $g(A)+J'$ is weakly nil-clean and $J \subseteq Nil(B)$ (respectively, $f(A)+J$ is weakly nil-clean and $J' \subseteq Nil(C)$). Then, $J \times \{0\} \subseteq Nil(A \bowtie^{(f,g)} (J, J')$ (respectively. $\{0\} \times J' \subseteq Nil(A \bowtie^{(f,g)} (J, J'))$. Thus, by [5, Proposition 1.9(i)], $A \bowtie^{(f,g)} (J, J')$ is weakly nil-clean if and only if $A \bowtie^{(f,g)} (J, J')/(J \times \{0\})$ (respectively. $A \bowtie^{(f,g)} (J, J')/\{0\} \times J'$) is weakly nil-clean. Therefore, the conclusion follows easily from [13, Proposition 4.1(2)]. □

Proposition 3.6 recovers the special case of amalgamated algebra, as recorded in the following corollary.

Corollary 3.7. *Let $f : A \to B$ be a ring homomorphism and let J be an ideal of B. Assume that $J \cap Id(B) = 0$. Then, the following are equivalent:*

(1) $A \bowtie^f J$ is weakly nil-clean.

(2) A is weakly nil-clean and $J \subseteq Nil(B)$.

Proof. Follows directely from Proposition 3.6 and [13, Example 2.1]. □

Example 3.8. Let $A := \mathbb{Z}_2$, $B := \mathbb{Z}_4$ and let $J := 2\mathbb{Z}_4 := \{0, 2\}$ be an ideal of B. Let $C := \mathbb{Z}_2 \times \mathbb{Z}_3$ and let $J' := 0 \times \mathbb{Z}_3$ be an ideal of C. Consider, the following ring homomorphisms $f : A \to B$ defined by $f(a) = a$ for all $a \in A$ and $g : A \to C$ given by: $g(a) = (a, 0)$ for all $a \in A$. It is well know that A and B are weakly nil-clean. Moreover, $g(A)+J' = C$ is weakly nil-clean since $\mathbb{Z}_2$ is nil-clean by Remark 3.2(2). It is easy to show that $J \cap Id(B) = 0$ and that $J \subseteq Nil(B)$. Then, $A \bowtie^{(f,g)} (J, J')$ is weakly nil-clean by Proposition 3.6.

In the special case of amalgamated duplication of a ring along an ideal, we obtain the following result which is a direct consequence of Corollary 3.7.

Corollary 3.9. *Let A be a ring and I be an ideal of A such that $I \cap Id(A) = 0$. Then $A \bowtie I$ is weakly nil-clean if and only if A is weakly nil-clean and $I \subseteq Nil(A)$.*

In what follows, we studies the transfer of weakly nil-clean property from A to $A \bowtie^{(f,g)} (J, J')$.

Proposition 3.10. *Assume that $J \times J' \subseteq Nil(B) \times Nil(C)$. If A is weakly nil-clean, then $A \bowtie^{(f,g)} (J, J')$ is weakly nil-clean.*

Proof. Assume that A is weakly nil-clean. Let $a \in A$ and $(j, j') \in J \times J'$. Then, there are a nilpotent element n and an idempotent element e of A such that $a = n+e$ or $a = n-e$. Then, $(f(a)+j, g(a)+j') = (f(n)+j, g(n)+j') + (f(e), g(e))$ or $(f(a)+j, g(a)+j') = (f(n)+j, g(n)+j') - (f(e), g(e))$. Since, by the assumption $(f(n)+j, g(n)+j')$ is a nilpotent of $A \bowtie^{(f,g)} (J, J')$ and $(f(e), g(e)) \in Id(A \bowtie^{(f,g)} (J, J'))$ because $e \in Id(A)$. Thus, $(f(a)+j, g(a)+j')$ is a sum of a nilpotent with an idempotent or a difference of a nilpotent with an idempotent of $A \bowtie^{(f,g)} (J, J')$, as desired. □

The next result is a partial result when a bi-amalgamation is a weakly nil-clean ring in case J and J' are not necessary nil ideals of $f(A)+J$ and $g(A)+J'$ respectively.

Theorem 3.11. *Assume that the following conditions hold:*

(1) A is weakly nil-clean and A/I_0 is uniquely weakly nil-clean.

(2) $f(A)+J$ and $g(A)+J'$ are weakly nil-clean rings and at most one of them is not nil-clean.

Then $A \bowtie^{(f,g)} (J, J')$ is a weakly nil-clean ring.

Proof. Without loss of generality, we may assume that $f(A)+J$ is weakly nil-clean and $g(A)+J'$ is nil-clean. Let $a \in A$ and $(j, j') \in J \times J'$, then there are nilpotents n and $f(n_1)+j_1$ of A and $f(A)+J$ respectively and idempotents e and $f(e_1)+j_2$ of A and $f(A)+J$ respectively such that $a = n+e$ or $a = n-e$ and $f(a)+j = (f(n_1)+j_1)+(f(e_1)+j_2)$ or $f(a)+j = (f(n_1)+j_1)-(f(e_1)+j_2)$. Therefore, $f(a) = f(n)+f(e)$ or $f(a) = f(n)-f(e)$ and $f(a)+j = (f(n_1)+j_1)+(f(e_1)+j_2)$ or $f(a)+j = (f(n_1)+j_1)-(f(e_1)+j_2)$. Then, in $(f(A)+J)/J$ we have: $\overline{f(a)} = \overline{f(n)} + \overline{f(e)}$ or $\overline{f(a)} = \overline{f(n)} - \overline{f(e)}$ and $\overline{f(a)+j} = \overline{f(a)} = \overline{f(n_1)} + \overline{f(e_1)}$ or $\overline{f(a)+j} = \overline{f(a)} = \overline{f(n_1)} - \overline{f(e_1)}$. It is clear that $\overline{f(n_1)}$ (respectively. $\overline{f(n)}$) and $\overline{f(e_1)}$ (respectively. $\overline{f(e)}$) are respectively nilpotent and idempotent elements of $(f(A)+J)/J$. On the other hand, since $(f(A)+J)/J \cong A/I_0$ is uniquely weakly nil-clean, then it is clear that $\overline{f(n_1)} = \overline{f(n)}$ and $\overline{f(e_1)} = \overline{f(e)}$ in $f(A)+J/J$. Therefore, there is $(k_1, k_2) \in J \times J$ such that $f(n_1) = f(n)+k_1$ and $f(e_1) = f(e)+k_2$. Hence, $f(a)+j = (f(n)+k_1+j_1)+(f(e)+k_2+j_2)$ or $f(a)+j = (f(n)+k_1+j_1)-(f(e)+k_2+j_2)$. If $f(a)+j = (f(n)+k_1+j_1)+(f(e)+k_2+j_2)$, write $g(a)+j' = (g(n_2)+j'_1)+(g(e_2)+j'_2)$, where $g(n_2)+j'_1$ is nilpotent and $g(e_2)+j'_2$ is idempotent of $g(A)+J'$. Thus, using the same technique of the preveous $g(a)+j' = (g(n)+k'_1+j'_1)+(g(e)+k'_2+j'_2)$ for some $(k'_1, k'_2) \in J' \times J'$ since $(g(A)+J')/J' \cong A/I_0$ is uniquely weakly nil-clean. Which implies that $(f(a)+j, g(a)+j') = (f(n)+k_1+j_1, g(n)+k'_1+j'_1)+(f(e)+k_2+j_2, g(e)+k'_2+j'_2)$ where, $(f(n)+k_1, g(n)+k'_1+j'_1) = (f(n_1)+j_1, g(n_2)+j'_1) \in Nil(A \bowtie^{(f,g)} (J, J'))$ and $(f(e)+k_2+j_2, g(e)+k'_2+j'_2) = (f(e_1)+j_2, g(e_2)+j'_2) \in Id(A \bowtie^{(f,g)} (J, J'))$. In the remaining case, $f(a)+j = (f(n)+k_1+j_1)-(f(e)+k_2+j_2)$. Let $g(a)+j' = (g(n_2)+j'_1)-(g(e_2)+j'_2)$. Thus, $g(a)+j' = (g(n)+k'_1+j'_1)-(g(e)+k'_2+j'_2)$ and so $(f(a)+j, g(a)+j') = (f(n)+k_1+j_1, g(n)+k'_1+j'_1)-(f(e)+k_2+j_2, g(e)+k'_2+j'_2)$. In all cases, $(f(a)+j, g(a)+j')$ is a sum of a nilpotent with an idempotent or a difference of a nilpotent with an idempotent of $A \bowtie^{(f,g)} (J, J')$, that completes our proof. □

Theorem 3.12. *Set $\overline{A} = A/Nil(A)$, $\overline{B} = B/Nil(B)$, $\overline{C} = C/Nil(C)$, $\pi_B : B \to \overline{B}$, $\pi_C : C \to \overline{C}$ be the canonical projections, set $\overline{J} = \pi_B(J)$ and $\overline{J'} = \pi_C(J')$. Consider $\overline{f} : \overline{A} \to \overline{B}$ and $\overline{g} : \overline{A} \to \overline{C}$ defined by: $\overline{f}(\overline{a}) = \overline{f(a)}$ and $\overline{g}(\overline{a}) = \overline{g(a)}$. Then, $A \bowtie^{(f,g)} (J, J')$ is weakly nil-clean if and only if $\overline{A} \bowtie^{(\overline{f},\overline{g})} (\overline{J}, \overline{J'})$ is weakly nil-clean.*

Proof. We saw preveously that the map:

$$\begin{array}{rcl} \phi : A \bowtie^{(f,g)} (J, J')/Nil(A \bowtie^{(f,g)} (J, J') & \to & \overline{A} \bowtie^{(\overline{f},\overline{g})} (\overline{J}, \overline{J'}) \\ \overline{(f(a)+j, g(a)+j')} & \mapsto & (\overline{f}(\overline{a})+\overline{j}, \overline{g}(\overline{a})+\overline{j'}) \end{array}$$

is a ring isomorphism (see the proof of Theorem 2.12). Therefore, according to [5, Proposition 1.9(i)], we have $A \bowtie^{(f,g)} (J, J')$ is weakly nil-clean if and only if so is $A \bowtie^{(f,g)} (J, J')/Nil(A \bowtie^{(f,g)} (J, J')$ if and only if $\overline{A} \bowtie^{(\overline{f},\overline{g})} (\overline{J}, \overline{J'})$ so is, as wanted. □

Theorem 3.12 recovers the special case of amalgamated algebra, as recorded in the following corollary.

Corollary 3.13. *We preserve the notation of Corollary 3.7, set $\overline{A} = A/Nil(A)$, $\overline{B} = B/Nil(B)$, $\pi : B \to \overline{B}$ be the canonical projection and set $\overline{J} = \pi(J)$. Consider $\overline{f} : \overline{A} \to \overline{B}$ defined by: $\overline{f}(\overline{a}) = \overline{f(a)}$. Then, $A \bowtie^f J$ is weakly nil-clean if and only if $\overline{A} \bowtie^{\overline{f}} \overline{J}$ is weakly nil-clean.*

In the special case of amalgamated duplication of a ring along an ideal, we obtain the following result which is a direct consequence of Corollary 3.13.

Corollary 3.14. *Let A be a ring and I be an ideal of A, set $\overline{A} = A/Nil(A)$, $\pi : A \to \overline{A}$ be the canonical projection and set $\overline{I} = \pi(I)$. Then $A \bowtie I$ is weakly nil-clean if and only if $\overline{A} \bowtie \overline{I}$ is weakly nil-clean.*

It is clear that every nil-clean ring is a weakly nil-clean ring but the converse is not true in general. In [5, Proposition 1.10], the autors proved that a ring R is nil-clean if and only if R is weakly nil-clean and $2 \in Nil(R)$. In what follows, we generalize this result in bi-amalgamated algebra along an ideal.

Proposition 3.15. *The following statements are equivalent:*

(1) $A \bowtie^{(f,g)} (J, J')$ *is nil-clean.*

(2) $A \bowtie^{(f,g)} (J, J')$ *is weakly nil-clean,* $2 \in Nil(f(A) + J)$ *and* $2 \in Nil(g(A) + J')$.

(3) $2 \in Nil(f(A) + J)$, $2 \in Nil(g(A) + J')$ *and* $f(A) + J$ *and* $g(A) + J'$ *are weakly nil-clean.*

Proof. (1) $\Rightarrow$ (2): Assume that $A \bowtie^{(f,g)} (J, J')$ is nil-clean. Then, Theorem 2.1 implies that $f(A) + J$ and $g(A) + J'$ are nil clean and thus $2 \in Nil(f(A) + J)$ and $2 \in Nil(g(A) + J')$ by [5, Proposition 1.10]. It is clear that $A \bowtie^{(f,g)} (J, J')$ is weakly nil-clean, as desired.
(2) $\Rightarrow$ (3) This is clear by Theorem 3.1.
(3) $\Rightarrow$ (1) This implication follows easily from Theorem 2.1 and [5, Proposition 1.10]. □

In the special case of amalgamation we obtain the following result:

Corollary 3.16. *The following are equivalent:*

(1) $A \bowtie^f J$ *is nil-clean.*

(2) $A \bowtie^f J$ *is weakly nil-clean,* $2 \in Nil(A)$ *and* $2 \in Nil(f(A) + J)$.

(3) A *and* $f(A) + J$ *are weakly nil-clean,* $2 \in Nil(A)$ *and* $2 \in Nil(f(A) + J)$.

In the special case of amalgamated duplication of a ring along an ideal, we obtain the following result which is a direct consequence of Corollary 3.16.

Corollary 3.17. *Let A be a ring and I be an ideal of A. The following are equivalent:*

(1) $A \bowtie I$ *is nil-clean.*

(2) $A \bowtie I$ *is weakly nil-clean and* $2 \in Nil(A)$.

(3) $2 \in Nil(A)$ *and A is weakly nil-clean.*

Our results of the transfer enriche the literature with new examples of weakly nil-clean rings which are not nil-clean rings issued from bi-amalgamated algebras along an ideal.

Example 3.18. Let $A := \mathbb{Z}_2$, $B := \mathbb{Z}_2 \times \mathbb{Z}_3$, $J := 0 \times \mathbb{Z}_3$, $C := \mathbb{Z}_2 \times \mathbb{Z}_4$ and $J' := 0 \times \mathbb{Z}_4$. Consider these following ring homomorphisms $f : A \to B$ and $g : A \to C$ defined by: $f(a) = (a, 0)$ and $g(a) = (a, 0)$ for all $a \in A$. Then, $A \bowtie^{(f,g)} (J, J')$ is a weakly nil-clean ring that is not a nil-clean ring.

Proof. It is easy to show that $f(A) + J = \mathbb{Z}_2 \times \mathbb{Z}_3 = B$ is weakly nil-clean by Remark 3.2(2), since $\mathbb{Z}_2$ is nil-clean and $\mathbb{Z}_3$ is weakly nil-clean. Also, $g(A) + J' = \mathbb{Z}_2 \times \mathbb{Z}_4 = C$ is nil-clean because that is a finite product of nil-clean rings. Moreover, $f^{-1}(J) = 0$ and $A = A/f^{-1}(J)$ is a uniquely weakly nil-clean ring. Then, Theorem 3.11 implies that $A \bowtie^{(f,g)} (J, J')$ is a weakly nil-clean ring. Now, $A \bowtie^{(f,g)} (J, J')$ is not nil-clean by Theorem 2.1 since $f(A) + J$ is not a nil-clean ring. □

Example 3.19. Let A be a weakly nil-clean ring that is not a nil-clean ring and E an A-module. Set $B := A \ltimes E$, $J := 0 \ltimes E$ and let $f : A \to B$ be a ring homomorphism defined by: $f(a) = (a, 0)$ for all $a \in A$. Let $C := A \bowtie Nil(A)$, $J' := 0 \bowtie I$ and let $g : A \to B$ be a ring homomorphism defined by: $g(a) = (a, a)$ for all $a \in A$. Then:

(1) $A \bowtie^{(f,g)} (J, J')$ is a weakly nil-clean ring.

(2) $A \bowtie^{(f,g)} (J, J')$ is not a nil-clean ring.

Proof. (1) It is easy to show that $f(A) + J = B$ and $g(A) + J' = C$ are weakly nil-clean by Corollaries 3.5 and 3.3. Moreover, we can see that $J \subseteq Nil(B)$ and $J' \subseteq Nil(C)$. Then, by Theorem 3.1, $A \bowtie^{(f,g)} (J, J')$ is a weakly nil-clean ring.
(2) By Corollary 2.5 $g(A)+J'$ is not a nil-clean ring since A is not nil-clean. Therefore, Theorem 2.1 implies that $A \bowtie^{(f,g)} (J, J')$ is not a nil-clean ring. □

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